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G. ROYER

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## Study of Dobrushin's critical coupling in rotator models

by

**G. ROYER**

Département de mathématiques appliquées,  
Université de Clermont II, B.P. 45, 63170 Aubière, France

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**ABSTRACT.** — We show that Dobrushin's uniqueness theorem yields an  $(N(N-3))^{1/2}$  bound for critical interaction in general  $N$  components rotator models; related calculations show that at least an  $(N(N-1))^{1/2}$  bound may be attainable in a similar way.

**RÉSUMÉ.** — Nous montrons que le théorème d'unicité de Dobrushin conduit à une majoration par  $(N(N-3))^{1/2}$  de l'interaction critique dans les modèles généraux de rotateurs à  $N$  composantes; des calculs voisins indiquent que la borne  $(N(N-1))^{1/2}$  devrait pouvoir être atteinte.

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### INTRODUCTION

In the theory of  $N$  components rotator models, the basic estimate of the critical interaction strength  $J_c$  is:  $J_c \geq N$  (Simon [16]. See also [17]). Although this result has been improved in particular models by means of hard analysis ([4]), it would be interesting to prove it via Dobrushin's uniqueness theorem, since this theorem holds in a more general setting and owns interesting corollaries ([8] [9]); the present article is an attempt in this direction: first, we obtain  $J_c \geq (N(N-3))^{1/2}$  for  $N \geq 3$ ; using a slightly different method, we also obtain

$$J_c \geq (N/2)^{1/2} \left( \sum_{k \geq N-1} k^{-2} \right)^{-1/2} \quad \text{for } N \geq 3;$$

this last estimate is better only for  $N = 3, 4$  and bad for high dimension. Nevertheless, we think its proof has a value to suggest how to obtain  $J_c \geq (N(N-1))^{1/2}$  and perhaps a little more. These estimates improve on that given by Levin [10], namely  $N/\sqrt{5}$ , for  $N \geq 3$ .

NOTATION: in this article it is more convenient to use  $n = N - 1$  as the dimension parameter.

§ 1. Let us recall Dobrushin's method. We denote by  $I$  a countable set (of sites) and by  $(S, r)$  a separable metric space; for each finite subset  $L$  of  $I$  and each configuration  $y \in S^{I-L}$ , a probability measure  $\mu_L(\cdot | y)$  on  $S^L$  is given and the associated Gibbs measures are those probability measures on  $S^I$  which admit, for each  $L$ , the kernel  $\mu_L$  as conditional distribution of the restricted configuration  $x_L$ , under the condition  $x_{I-L} = y$ .

Dobrushin's condition uses the natural distance between probability measures induced by the distance  $r$  (Fortet, Kantorovitch, Vasershtein, see [6]); strictly speaking,  $R$  is defined only on the convex set of probability measures  $\nu$  on  $S$  such that,  $0$  denoting some fixed point in  $S$ ,  $\int r(0, y)\nu(dy) < \infty$  (a condition which is irrelevant if  $S$  is compact) by:

$$R(\nu_1, \nu_2) = \sup \left\{ \int f d\nu_1 - \int f d\nu_2 ; \mathcal{L}(f) \leq 1 \right\}$$

where  $\mathcal{L}(f)$  is the Lipschitz norm of the function  $f$ :

$$\mathcal{L}(f) = \sup_{\tilde{z} \neq z} (f(\tilde{z}) - f(z))/r(\tilde{z}, z).$$

The dependence coefficient of site  $i$  under site  $j$ ,  $\rho(i, j)$ , is defined as the supremum of  $R(\mu_{i|\tilde{i}}(\cdot | y), \mu_{i|\tilde{i}}(\cdot | \tilde{y}))/r(y_j, \tilde{y}_j)$  when the configurations  $y, \tilde{y}$  coincide off  $j$ ; if we put  $\rho = \sup_{i \in I} \sum_j \rho(i, j)$ , Dobrushin's theorem formulates as follows:

THEOREM 1.1. — *If  $\rho < 1$ , there exists at most one Gibbs measure  $\gamma$  such that  $\sup_{i \in I} \int r(0, x_i)\gamma(dx) < \infty$ .*

Here we aim to apply this result to rotator models:  $S$  is the unit sphere  $S^n$  with its natural (geodesic) distance  $r$  and  $\mu_L$  takes on the following form:

$$\mu_L(dy | z) = Z^{-1} \exp(-\mathcal{H}_L(y, z)) \prod_{i \in L} \sigma_n(dy_i),$$

( $\sigma_n$  denotes the uniform measure on  $S^n$ ) normalized by  $Z$  into a probability measure and

$$\mathcal{H}_L(y, z) = (1/2) \sum_{i, j \in L} J(i, j) y_i \cdot y_j + \sum_{\substack{i \in L \\ k \notin L}} J(i, k) y_i \cdot z_k,$$

where  $J$  is a symmetric function on  $I \times I$  which vanishes on the diagonal (we may also add an « external field »  $\sum_{i \in L} H(i) \cdot y_i$  without altering our results). We put  $J = \sup_i \sum_j |J(i, j)|$  and define  $J_c(n)$  as the supremum

of numbers  $M$  such that unicity holds in every  $n$ -dimensional rotator models such that  $J < M$ .

For each vector  $h \in \mathbb{R}^{n+1}$  let  $\nu(h)$  be the probability measure on  $S^n$ , the density of which is proportional to  $x \rightarrow \exp(x \cdot h)$ ; let us denote by  $\text{var}(f, \nu)$  the variance of a function  $f$  with respect to a bounded measure  $\nu$ , i. e., with respect to the proportional probability measure; our starting point is the following:

LEMMA 1.2. — Let  $C(n, h)$  be the best constant in the inequality

$$\text{var}(f, \nu(h)) \leq C(n, h) (\mathcal{L}(f))^2, \tag{1}$$

where  $f$  is an arbitrary Lipschitz function on  $S^n$ , and  $C(n) = \sup_h C(n, h)$ . Then  $J_c \geq ((n + 1)/C(n))^{1/2}$ .

As indicated in [10], we shall use a basic result of Dyson, Lieb and Simon [15]:

THEOREM 1.3. — For any vectors  $h$  and  $z$ , the function  $z$  on  $S^n$ ,  $y \rightarrow z \cdot y$  satisfies

$$\text{var}(z \cdot, \nu(h)) \leq |z|^2 / (n + 1). \tag{2}$$

Proof of 1.2. — Let  $K, z_1, z_2, \zeta$  be respectively a real number, unit vectors and a vector. Let us prove the following inequality:

$$R(\nu(Kz_1 + \zeta), \nu(Kz_2 + \zeta)) \leq |K| (C(n)/(n + 1))^{1/2} r(z_1, z_2). \tag{3}$$

We consider a minimal geodesic  $z(t)$  from  $z_1$  to  $z_2$ , parametrized by arc-length; a straightforward calculation gives for an arbitrary Lipschitz function  $f$

$$\left| \frac{d}{dt} \int f(y) \nu(Kz(t) + \zeta, dy) \right| = |K \text{cov}(f(y), y \cdot z'(t))| \tag{4}$$

(this covariance being calculated with respect to  $\nu(Kz(t) + \zeta)$ ); from the Schwarz inequality, (1), (2) and  $|z'(t)| = 1$ , we may majorize by  $(C(n)/(n + 1))^{1/2} \mathcal{L}(f)$  and integration in  $t$  gives (3).

To finish the proof it suffices to remark that the « *a priori* spin distribution »  $\mu_{(i)}(\cdot | y)$  is, as a function of  $y_j$ , of the form  $v(Ky_j + \zeta)$  where  $K = -J(i, j)$  and  $\zeta = -\sum_{k \neq i, j} J(i, k)y_k$ ; therefore (3) yields

$$\rho(i, j) \leq (C(n)/(n+1))^{1/2} |J(i, j)| \quad \text{so} \quad \rho \leq (C(n)/(n+1))^{1/2} J$$

and from theorem 1.1 follows uniqueness when  $J < (n+1)^{1/2}(C(n))^{-1/2}$ . ■

## § 2. USE OF INEQUALITIES OF POINCARÉ'S TYPE

Let us consider the best constant  $\chi(n, h)$  in the following inequality:

$$\text{var}(f, v(h)) \leq \chi(n, h) \int_{S^n} |\text{grad}(f)|^2 dv(h),$$

say for  $f \in \mathcal{C}^1(S^n)$ ; it is well-known that this constant is the inverse of the least non-zero eigenvalue of the self-adjoint operator of  $L^2(v(h))$ ,  $f \rightarrow -\Delta f - \text{grad}(U) \cdot \text{grad}(f)$  with  $U(y) = h \cdot y$ ; in particular, equality in (4) is obtained with at least one non-zero  $\mathcal{C}^\infty$  function. For  $h = 0$  the operator reduces to Laplacian and  $\chi(n, 0) = 1/n$ . Theorem 3.1 below suggests that  $\chi(n, h) \leq \chi(n, 0)$ , an inequality which would yield  $J_c \geq (n(n+1))^{1/2}$ ; an easy way to approach this conjecture is to use the following estimate of Brascamp-Lieb ([3]):

**THEOREM 2.1.** — *If  $W$  is a strictly convex  $\mathcal{C}^2$  function on  $]a, b[$  such that  $\exp(-W(t))$  vanishes at  $a$  and  $b$ , then*

$$\text{var}(f, \exp(-W(t))) \leq Z^{-1} \int_a^b f'^2(t)(W''(t))^{-1} \exp(-W(t)) dt$$

where 
$$Z = \int_a^b \exp(-W(t)) dt.$$

Let us sketch a pedagogical slight variation of the original proof: we may always suppose  $Z = 1$ . We have:

$$\text{var}(f) = \int_{a \leq x \leq y \leq b} (f(y) - f(x))^2 \exp(-W(y) + W(x)) dx dy; \quad (5)$$

Schwarz's inequality gives us the following inequality

$$(f(y) - f(x))^2 \leq (W'(y) - W'(x)) \int_x^y f'^2(t)/W''(t) dt,$$

which we insert in (5); using Fubini's theorem, it suffices to show:

$$\int_{\substack{y \leq t \leq b \\ a \leq x \leq t}} (W'(y) - W'(x)) \exp - (W(y) + W(x)) dx dy = \exp (- W(t))$$

and this is immediate from integration by parts.  $\blacksquare$

For  $a \in \mathbb{R}$ , we denote by  $\rho_{a,n}$  or simply  $\rho_a$  the probability on  $(-\pi/2, \pi/2)$  the density of which is proportional to  $\cos^{n-1}(\theta) \exp(a \sin \theta)$ .

LEMMA 2.2. — For  $n > 2$ , and non constant  $g$ ,

$$\text{var}(g, \rho_a) < (n-2)^{-1} \int g'^2(\theta) \rho_a(d\theta).$$

*Proof.* — By the change  $t = \sin(\theta)$  we have to estimate the variance of  $t \rightarrow g(\arcsin(t))$  with respect to the measure

$$(1-t^2)^{n/2-1} \exp(at) dt \quad \text{on } ]-1, +1[;$$

elementary calculations show that 2.1 applies and yields:

$$\begin{aligned} \text{var}(g, \rho_a) &\leq (Z(n-2))^{-1} \int_{-1}^{+1} g'^2(\arcsin(t)) \frac{(1-t^2)^{n/2}}{1+t^2} \exp(at) dt = \\ &= (n-2)^{-1} \int g'^2(\theta) \cos^2(\theta) (1+\sin^2(\theta))^{-1} \rho_a(d\theta) < (n-2)^{-1} \int g'^2(\theta) \rho_a(d\theta). \quad \blacksquare \end{aligned} \tag{6}$$

THEOREM 2.3. — For any vector  $h$ , if  $n > 2$ ,  $\chi(n, h) < (n-2)^{-1}$ .

*Proof.* — Let us consider the axis defined by  $h$  as a south-north diameter; we denote by  $\theta$  and  $\varphi$  the corresponding latitude and longitude;  $\theta \in ]-\pi/2, \pi/2[$ ;  $\varphi \in S^{n-1}$  is defined as follows: we fix a bijection from the parallel  $S_\theta$  at latitude  $\theta$  onto  $S_0 = S^{n-1}$  by shift along meridians; the distribution law of  $\theta$  with respect to  $\nu(h)$  is  $\rho_a$  with  $a = |h|$  and the conditional law of  $\varphi$  when  $\theta$  is fixed is the uniform law  $\sigma_{n-1}$  on  $S^{n-1}$ ; hence, like in [3], we have:

$$\text{var}(f, \nu(h)) = \text{var}(g, \rho_a) + \int \text{var}(f(\theta, \cdot), \sigma_{n-1}) \rho_a(d\theta) \tag{7}$$

where

$$g(\theta) = \int f(\theta, \varphi) \sigma_{n-1}(d\varphi).$$

We choose for  $f$  a non-constant function which realizes equality in (4).

From lemma 2.2 and Poincaré's inequality on  $S^{n-1}$ , we get

$$\text{var}(f, \nu_h) \leq (n-2)^{-1} \int g'^2 d\rho_a + (n-1)^{-1} \int \left| \frac{\partial f}{\partial \varphi} \right|^2 \sigma_{n-1}(d\varphi) \rho_a(d\theta);$$

we use Jensen's inequality in first integral and get

$$\begin{aligned} < (n-2)^{-1} \int \left( \left( \frac{\partial f}{\partial \theta} \right)^2 + \left| \frac{\partial f}{\partial \varphi} \right|^2 \right) dv_h \leq (n-2)^{-1} \int |\text{grad}(f)|^2 dv_h \\ = (\chi(n, h))^{-1} (n-2)^{-1} \text{var}(f, v_h); \end{aligned}$$

we have used strict inequality in 2.2 (and in  $(n-2) < (n-1)$ ) and

$$|\text{grad}(f)|^2 = \left( \frac{\partial f}{\partial \theta} \right)^2 + \cos^{-2}(\theta) \left| \frac{\partial f}{\partial \varphi} \right|^2 \quad \blacksquare$$

**COROLLARY 2.3.** — *The critical interaction  $J_c$  is no less than  $((n+1)(n-2))^{1/2}$ .*

*Proof.* — The space of  $\mathcal{C}^1$ -Lipschitz functions is dense in the space of Lipschitz functions; so  $C(n, h) \leq \chi(n, h) < (n-2)^{-1}$ ; thus  $C(n) \leq (n-2)^{-1}$  and we return to lemma 1.2.  $\blacksquare$

*Remarks.* — 1)  $\chi(n, h)$  is a continuous function of  $h$  which tends likely to zero when  $|h| \rightarrow \infty$ ; the same proof should give strict inequality.

2) Roughly speaking, log-concavity estimates are accurate in high dimensions: this is not surprising since the measure  $\sigma_n$  behaves then like a Gaussian and the situation becomes similar to that of [14].

### § 3. USE OF LIPSCHITZ NORMS

In what follows we shall work with inequalities of the same type as (1). Let us shorten the notation  $\text{var}(\text{Identity}, v)$  into  $\text{var}(v)$ .

**THEOREM 3.1.** — *The function  $a \rightarrow \text{var}(\rho_{a,n})$  is even and decreasing for  $a \geq 0$ , for any real number  $n \geq 1$ .*

The parity statement is obvious; to prove the other let us recall a classical probabilistic inequality (Chebychev, generalized by Preston [13] for F. K. G. inequalities).

**LEMMA 3.2.** — *Let  $\rho_1$  and  $\rho_2$  be two probability measures on an interval I with positive densities  $g_1, g_2$ ; in order that  $\rho_2$  may be stochastically greater than  $\rho_1$  (i. e.  $\int f d\rho_1 < \int f d\rho_2$  for any increasing function  $f$ ) it is sufficient that:*

$$g_1(t)/g_1(s) < g_2(t)/g_2(s) \quad \text{for any } s < t.$$

*Proof of theorem 3.1.* — We have

$$\begin{aligned} \text{var}(\rho_a) &= 1/2 \int (x-y)^2 \rho_a(dx) \rho_a(dy) \\ &= (2Z_a)^{-2} \int (x-y)^2 \cos^{n-1}(x) \cos^{n-1}(y) \exp(a(\sin(x) \\ &\quad + \sin(y))) dx dy. \end{aligned}$$

Put  $(y-x)/2 = u$ ,  $(y+x)/2 = v$ ; then

$$\text{var}(\rho_a) = 4Z_a^{-2} \int_{-\pi/2}^{\pi/2} du \int_{|u|-\pi/2}^{\pi/2-|u|} u^2 (\cos(2u) + \cos(2v))^{n-1} \exp(2a \cos(u) \sin(v)) dv.$$

By using symmetries, if  $t = (\pi/2) - u$  for  $u > 0$ , we get

$$\begin{aligned} \text{var}(v_a) &= 16 Z_a^{-2} \int_0^{\pi/2} (t - \pi/2)^2 g_a(t) dt \quad \text{where:} \\ g_a(t) &= \int_0^t (\cos(2v) - \cos(2t))^{n-1} \cosh(2a \sin(t) \sin(v)) dv. \end{aligned}$$

By lemma 3.2, since  $(t - \pi/2)^2$  decreases we only need to show that for  $0 \leq a$ ,  $s \leq t$ ,  $a \rightarrow g_a(t)/g_a(s)$  is increasing; it is sufficient to prove

$$I := g'_a(t)g_a(s) - g'_a(s)g_a(t) > 0, \quad \text{for } a > 0.$$

Let us write I as a double integral:

$$I = 2 \int_{\substack{0 \leq v \leq t \\ 0 \leq w \leq s}} \varphi(v, w) \Psi(v, w) dv dw$$

where

$$\begin{aligned} \varphi(v, w) &= ((\cos(2v) - \cos(2t))(\cos(2w) - \cos(2s)))^{n-1}, \\ \Psi(v, w) &= \sin(v) \sin(t) \sinh(2a \sin(t) \sin(v)) \cosh(2a \sin(s) \sin(w)) \\ &\quad - \sin(w) \sin(s) \sinh(2a \sin(s) \sin(w)) \cosh(2a \sin(t) \sin(v)). \end{aligned}$$

We remark that  $\Psi(v, w) > 0$  if  $w < v$ , since  $\tanh$  is increasing; hence the part of I coming from the domain  $s \leq v \leq t$ ,  $0 \leq w \leq s$  is positive; the other part may be written:

$$\tilde{I} = 2 \int_{0 \leq w \leq v \leq s} (\varphi(v, w) \Psi(v, w) + \varphi(w, v) \Psi(w, v)) dv.$$



In the new domain of integration,  $\Psi(v, w) \geq 0$ ; on the other hand  $\varphi(v, w) \geq \varphi(w, v)$ ; indeed a simple derivative calculation implies:

$$\begin{aligned} (\cos(2v) - \cos(2t))/(\cos(2w) - \cos(2t)) &\geq \\ &\geq (\cos(2w) - \cos(2t))/(\cos(2w) - \cos(2s)) \end{aligned}$$

for  $0 \leq w \leq v \leq s$ ,  $0 \leq s \leq t \leq \pi/2$  and we recall that  $n - 1 \geq 0$ .

Therefore to finish the proof it is sufficient to show  $\Psi(v, w) + \Psi(w, v) \geq 0$ ; denoting  $2 \sin(s)$ ,  $2 \sin(t)$ ,  $\sin(w)$ ,  $\sin(v)$  respectively by  $x$ ,  $y$ ,  $X$ ,  $Y$ , we have  $0 \leq x \leq y$ ,  $0 \leq X \leq Y$  and we want:

$$\begin{aligned} &yY \sinh(ayY) \cosh(axX) - xX \sinh(axX) \cosh(ayY) + \\ &+ yX \sinh(ayX) \cosh(axY) - xY \sinh(axY) \cosh(ayX) \geq 0. \end{aligned}$$

We expand in powers of  $a$ ; the coefficient of  $a^m$  cancels for  $m$  even and, for  $m$  odd, we get up to a positive factor, owing to the formula

$$\begin{aligned} 2 \sinh(u) \cosh(v) &= \sinh(u+v) + \sinh(u-v): \\ (xX + yY)^m (yY - xX) &+ (yY - xX)^m (yY + xX) \\ + (yX + xY)^m (yX - xY) &+ (yX - xY)^m (yX + xY). \end{aligned}$$

This coefficient is non-negative: if we set  $A = Y/X$ ,  $\alpha = y/x$ , we get up to a positive factor:

$$\begin{aligned} &(A\alpha - 1)(A\alpha + 1)[(A\alpha - 1)^{m-1} + (A\alpha + 1)^{m-1}] \\ &- (A - \alpha)(A + \alpha)[(A - \alpha)^{m-1} + (A + \alpha)^{m-1}]; \end{aligned}$$

if  $A \leq \alpha$  all terms are non-negative; if not, we note that  $A\alpha + 1 \geq A + \alpha$  and  $A\alpha - 1 \geq A - \alpha$  from  $A \geq 1$ ,  $\alpha \geq 1$ . ■

*Remark.* — Theorem 3.1 is also valid for  $0 < n \leq 1$ ; similar calculations have to be done with the new integration variable  $z = v/t$ .

So the maximum of  $\text{var}(\rho_{a,n})$  is  $\text{var}(\rho_{0,n})$  and we call it  $v_n$ .

**COROLLARY 3.3.** — *For any Lipschitz function  $g$  on  $(-\pi/2, \pi/2)$   $\text{var}(g, \rho_{a,n}) \leq v_n(\mathcal{L}(g))^2$  and  $v_n$  is the best constant in this inequality.*

*Proof.* — We write one more time  $\text{var}(g, \rho_a)$  as a double integral to get

$$\text{var}(g, \rho_a) \leq \text{var}(\text{Id}, \rho_a)(\mathcal{L}(g))^2 \leq v_n(\mathcal{L}(g))^2. \quad \blacksquare$$

Now we return to the sphere; let us recall the definition of, say, « the re-arrangement  $\tilde{f}$  of a function  $f$ , increasing with respect to latitude »: once a north pole on  $S^n$  has been chosen,  $\tilde{f}$  is characterized by two properties (except sometimes on negligible sets):

- 1)  $\tilde{f}$  depends only on latitude  $\theta$  and is non-decreasing
- 2)  $\tilde{f}$  has the same distribution function as  $f$ :  $\sigma(\tilde{f} > t) = \sigma(f > t)$ .

**THEOREM 3.4.** — *If  $f$  is Lipschitzian so is  $\tilde{f}$  and  $\mathcal{L}(\tilde{f}) \leq \mathcal{L}(f)$ ; similarly  $f \rightarrow \tilde{f}$  is a contraction for Dirichlet's integral.*

The first statement is a Corollary to lemma 1 in [2]; to get the second we may use theorem 2 in [2] as in [12]; see also [7] and its bibliography.

**COROLLARY 3.5.** — *We have  $C(n, 0) = v_n$  and  $v_n \leq 1/n$ .*

*Proof.* — By re-arrangement,  $\text{var}(f, \sigma)$  is not modified and the Lipschitz norm decreases; we may consider  $\tilde{f}$  as a function on  $(-\pi/2, \pi/2)$  and Corollary 3.3 gives  $C(n, 0) = v_n$ . Poincaré's inequality on  $S^n$  yields  $v_n \leq 1/n$ . ■

Actually  $v_n$  can be calculated explicitly:

$$v_n = \left( \int_{-\pi/2}^{\pi/2} \theta^2 \cos^{n-1}(\theta) d\theta \right) \left( \int_{-\pi/2}^{\pi/2} \cos^{n-1}(\theta) d\theta \right)^{-1}$$

hence:

$$\begin{aligned} v_n &= v_{n-2} - 2(n-1)^2, \\ v_{2p+1} &= \pi^2/12 - 2 \sum_{n=1}^p (2n)^{-2} \\ v_{2p} &= \pi^2/4 - 2 \sum_{n=1}^p (2n-1)^{-2}; \end{aligned}$$

finally, we find

$$v_n + v_{n-1} = 2 \sum_{k \geq n} k^{-2}.$$

**THEOREM 3.6.** — *For  $n > 1$ ,  $C(n) < v_n + v_{n-1}$  and therefore*

$$J_c > (n+1)^{1/2} \left( 2 \sum_{k \geq n} k^{-2} \right)^{-1/2}$$

*Proof.* — From compactness with respect to point-wise convergence of  $\{f; \mathcal{L}(f) \leq 1 \text{ and } f(0) = 1\}$ , follows the existence of a function  $f$  which realizes equality in relation (1) of lemma 1.2 and such that  $\mathcal{L}(f) = 1$ . Again we use relation (7) above; obviously we also have  $\mathcal{L}(g) \leq 1$ ; on the other hand, the Lipschitz norm of  $\varphi \rightarrow f(\theta, \varphi)$  is not greater than  $\cos(\theta)$ . So from (7), 3.3, 3.5 we get

$$\text{var}(f, v(\mathbf{X})) \leq v_n + v_{n-1} \int_{-\pi/2}^{\pi/2} \cos^2(\theta) \rho_d(d\theta).$$

An application of lemma 3.2 shows that the last integral is maximum for  $a = 0$ , in which case its value is

$$\Gamma^2((n+1)/2)\Gamma^{-1}(n/2)\Gamma^{-1}(n/2+1)$$

which is less than 1 since  $\Gamma$  is log-convex. ■

Finally, by analogy with 3.1, one might think that  $C(n, h) \leq C(n, 0)$ ; this conjecture would give  $J_c \geq (n+1)^{1/2}v_n^{-1/2}$  and should be true if equality in (1) is realizable with a function depending only on latitude (defined with respect to  $h$ ).

$$\begin{aligned} \text{NUMERICAL VALUES} \quad & A = (n+1)^{1/2}(v_n + v_{n-1})^{-1/2} \\ & B = (n+1)^{1/2}(n-2)^{1/2} \end{aligned}$$

$n = N - 1$	2	3	4	5	10	$\rightarrow \infty$
A	1.52	2.25	2.96	3.68	7.23	$\sim n/\sqrt{2}$
B	0	2	3.16	4.24	9.38	$\sim n$
$v_n^{-1}$	1.21	2.13	4.07	5.06	10.03	$\sim n$

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#### REFERENCES

- [1] M. B. AIZENMAN, B. SIMON, *Commun. Math. Phys.*, t. **77**, 1980, p. 137-143.
- [2] A. BAERNSTEIN, B. A. TAYLOR, *Duke Math. Journal*, t. **43**<sup>2</sup>, 1976, p. 245-269.
- [3] H. BRASCAMP, E. H. LIEB, *J. Funct. Anal.*, t. **22**, 1976, p. 366-389.
- [4] O. BRYDGES, J. FRÖHLICH, T. SPENCER, *Commun. Math. Phys.*, t. **83**, 1982, p. 123-150.
- [5] R. L. DOBRUSHIN, *Theor. Prob. Appl.*, t. **15**<sup>3</sup>, p. 1970, p. 458.
- [6] R. M. DUDLEY, Probabilities and metrics, *Lecture Notes*, Series 45, Aarhus Universit at.
- [7] A. EHRHARD, *Ann. Scient. Ec. Norm. Sup.*, t. **17**, 1984, p. 317-322.
- [8] H. F OLLMER, *J. Funct. Anal.*, t. **46**, 1982, p. 387-395.
- [9] H. K UNSCH, *Commun. Math. Phys.*, t. **84**, 1982, p. 207-222.
- [10] S. L. LEVIN, *Commun. Math. Phys.*, t. **78**<sup>1</sup>, 1980, p. 65-74.
- [11] E. H. LIEB, *Commun. Math. Phys.*, t. **77**, 1980, p. 127-135.
- [12] C. MUELLER, E. WEISSLER, *J. Funct. Anal.*, t. **48**, 1982, p. 252-283.
- [13] R. HOLLEY, *Commun. Math. Phys.*, t. **38**, 1974, p. 227-231.
- [14] G. ROYER, *J. Math. Pures et Appl.*, t. **58**, 1977, p. 455-478.
- [15] F. J. DYSON, E. H. LIEB, B. SIMON, *J. Stat. Phys.*, t. **18**, 1978, p. 335-383.
- [16] B. SIMON, *J. Stat. Phys.*, t. **22**<sup>4</sup>, 1980, p. 491-492.
- [17] W. DRIESSLER, L. LANDAU, J. FERNANDO-PEREZ, *J. Stat. Phys.*, t. **20**, 1979, p. 123.

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