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G. F. DELL'ANTONIO
R. FIGARI
E. ORLANDI

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An approach through orthogonal projections to the study of inhomogeneous or random media with linear response

by

G. F. DELL'ANTONIO
Centre de Physique théorique, CNRS Luminy,
On leave from Dip. di Matematica, Univ. di Roma

R. FIGARI
Dip. di Fisica, Univ. di Napoli and Bibos, University of Bielefeld

and

E. ORLANDI
Dip. di Matematica, Univ. di Roma, CNR, GNFM

ABSTRACT. — We present, for the study of effective constants in an homogeneous or random medium, a formalism based on orthogonal projections which stresses the mathematical equivalence of different physical problems.

We develop also an approximation scheme and apply it to a case of physical interest.

RÉSUMÉ. — On introduit, dans l'étude des constants effectives pour des milieux inhomogènes ou aléatoires, un procédé soulignant l'équivalence mathématique de plusieurs problèmes physiques. On considère aussi un schéma d'approximation et on l'applique à un problème d'intérêt physique.

1. INTRODUCTION AND SUMMARY

Consider a dielectric medium in a region $D \subset R^3$. Denote by $\varepsilon(x)$ the local dielectric tensor. Choose a boundary condition Γ on ∂D and let E be the electric field in D. When only partial information on ε is available, or when ε is a rapidly varying function, one is interested in averaged quantities. One of them is the effective dielectric constant ε^*, defined roughly speaking as the proportionality ratio between the average of $| \varepsilon E |$ and of $| E |$.

This situation prevails also in a number of other physical problems, e. g. in the study of dissipative properties (conductivity, thermal conductivity, viscous flows) or dynamical properties (magnetic susceptibility, elasticity, velocity of sound). In fact, from the point of view considered here, all these problems can be considered as mathematically equivalent.

In each of them one considers a field S in D, with values in a vector space K; physically, the field S may describe the electric field, the temperature gradient, the fluid velocity, the elastic strain tensor, etc. The field S is in some sense a gradient (more generally, a closed form). It also satisfies an equation of the form

$$\operatorname{div} \sigma S = 0 \tag{1.1}$$

where σ is a field of maps from K to itself, describing the local properties of the medium. If σ does not depend on S, and is bounded and strictly positive, (1.1) is equivalent to an elliptic system and has therefore a unique solution under very mild conditions on D and Γ [*1*]. Denote by $\overline{F}$ the average of the quantity F over D. Let K be a unit vector in R^3. If Γ is homogeneous, as we shall always assume, $\overline{K . \sigma S}$ is proportional to $\overline{K . S}$. The effective constant σ^* is then defined by $\overline{K . \sigma S} = \sigma^* \overline{K . S}$. Almost all published results on approximations or bounds for σ^* refer to the linear case. A notable exception is [*2*].

In this paper, in which we consider only the linear case, we solve (1.1) by a construction which emphasizes the mathematical equivalence of the physical problems described above. We work entirely in a Hilbert-space setting, and exploit the fact that rot and div are (Hodge) adjoints [*6a*] [*6b*]. In the case in which $\sigma(x)$ takes only a finite number of values σ_i in domains D_i, and ∂D_i are sufficiently regular, the approach we follow is equivalent—but not identical—to the use of boundary-layer potentials.

The paper is organized as follows:

In section 2 we develop the formalism when S is the electric field (denoted by E) and σ the dielectric tensor (denoted by ε). We construct E as uniformly convergent series. When the system is made of N homogeneous components with dielectric tensors ε_k, we determine the analyticity domain of E as a function of the ε_k's .We deduce then the corresponding results for ε^*. We also touch briefly on the variational approach.

In section 3 we indicate how other physical problems can be solved by the same formalism; we outline the procedure for elastostatics and viscous flow.

We also outline the analysis for a stationary random medium [*5a*] [*5b*].

In section 4 we develop an approximation scheme, based on the introduction of approximate local fields σ' and/or a different boundary condition Γ'. In way of example, we compute to order 21 the (uniformly convergent) expansion of ε^* in powers of ρ for the case when homogeneous spherical inclusions of radius ρ are placed at the centers of a unit cubic lattice in a homogeneous host medium.

In an Appendix, we sketch the relation of our approach to potential theory.

In a separate paper we use the approach developed here to prove existence of solutions of (1.1) in case of non-linear response, under suitable assumptions on the non-linear terms. We shall also extend to the non-linear case the approximation scheme of Section 4.

Our interest in this subject was originated some time ago through discussions with G. Papanicolaou, which are gratefully acknowledged. We have especially profitted from several enlightening discussions with D. Bergman.

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2. ELECTROSTATICS, EXPLICIT DETERMINATION OF THE ELECTRIC FIELD AND OF THE EFFECTIVE DIELECTRIC CONSTANT

Let D be a bounded domain in R^3, $\varepsilon(x)$ the local dielectric tensor field. We study the static electric field $\mathrm{E}(x)$ for suitable boundary conditions on $\partial\mathrm{D}$. Maxwell's equations are

$$\mathrm{rot}\ \mathrm{E} = 0, \qquad \mathrm{div}\ \varepsilon\mathrm{E} = 0, \qquad x \in \mathrm{D} \tag{2.1}$$

or, in abstract form

$$d\mathrm{E} = 0, \qquad d^*\varepsilon\mathrm{E} = 0 \tag{2.2}$$

where * is the Hodge-duality.

On ε we assume that one can find positive numbers α, β such that

$$\alpha . \mathrm{I} \leq \varepsilon(x) \leq \beta . \mathrm{I} \qquad x \in \mathrm{D} \tag{2.3}$$

We shall use the notation

$$\varepsilon_{+} = \sup | \varepsilon(x) |, \qquad x \in \mathrm{D} \tag{2.4}$$

$$\varepsilon_{-} = \inf | \varepsilon(x) |, \qquad x \in \mathrm{D}$$

where $| \varepsilon |$ denotes the norm of the matrix ε.

The system (2.2) is strictly elliptic, and has a unique solution by the Lax-Milgram lemma.

We shall give an equivalent version of (2.2), exploiting the fact that the kernel of d^* is contained in the orthogonal complement of the range of d. The projection operator we use has an explicit form, when $\partial \mathrm{D}$ is smooth, as an integral operator; in this case, the approach we follow is related to potential theory. We shall give in the Appendix an outline of this relation. We denote by $\mathrm{L}^2(\mathrm{D}))^3$ the Hilbert-space of vector-valued functions on D, with scalar product

$$\langle f, g \rangle = \frac{1}{| \mathrm{D} |} \int_{\mathrm{D}} f(x) g(x) d^3x, \qquad \langle f, f \rangle = \| f \|^2$$

where $| \mathrm{D} |$ is the Lebesgue measure of D.

Let $\mathrm{H}^1(\mathrm{D})$ be the space of those distributions whose (generalized) gradient is in $(\mathrm{L}^2(\mathrm{D}))^3$. Let $\mathrm{H}^1_0(\mathrm{D})$ be the closure in $\mathrm{H}^1(\mathrm{D})$ of the function of class C^1 with support strictly contained in D, and let V be a homogeneous boundary-value space, i. e. a closed proper linear subspace of $\mathrm{H}^1(\mathrm{D})$ such that $\mathrm{V} \supseteq \mathrm{H}^1_0(\mathrm{D})$.

Let $\mathscr{H}_{\mathrm{V}}$ be the smallest closed subspace of $(\mathrm{L}^2(\mathrm{D}))^3$ which contains all $\xi \in (\mathrm{L}^2(\mathrm{D}))^3$ which have the following property: for every $x \in \mathrm{D}$, one can find a neighbourhood n and a function $\psi \in \mathrm{V}$ such that $\xi = \nabla \psi$ in n. Let A be the orthogonal projection of $(\mathrm{L}^2(\mathrm{D}))^3$ onto $\mathscr{H}_{\mathrm{V}}$. In analogy with hydrodynamics, we call A a Hodge projection. We shall not indicate explicitely its dependence on V.

Let $\mathrm{E}_0 \in \mathscr{H}_{\mathrm{H}^1(\mathrm{D})}$, $\mathrm{E}_0 \notin \mathscr{H}_{\mathrm{V}}$.

We shall say that E is an irrotational field in D satisfying the boundary conditions $(\mathrm{V}, \mathrm{E}_0)$ if $\mathrm{E} - \mathrm{E}_0 \in \mathscr{H}_{\mathrm{V}}$ [*6*]. By definition therefore

$$\mathrm{A}(\mathrm{E} - \mathrm{E}_0) = \mathrm{E} - \mathrm{E}_0 \tag{2.5}$$

Define

$$\mathrm{B} = \mathrm{E}_0 - \mathrm{A}\mathrm{E}_0 \tag{2.6}$$

so that

$$\mathrm{A}\mathrm{B} = 0 \tag{2.6'}$$

One has then

$$\mathrm{A}\mathrm{E} - \mathrm{E} = - \mathrm{B} \tag{2.7}$$

In many practical cases, e. g. parallel-plate condenser or periodic boundary conditions, E_0 is a constant vector, and $AE_0 = 0$. This relation however does not hold in general, even if E_0 is a constant vector.

For practical purposes, it is convenient to know the explicit action of A when $\xi \in (C^1)^3$. If ∂D is such that a Green function G exists for the homogeneous boundary condition (V, 0), one has

$$(A\xi)_i(x) = \frac{\partial}{\partial x_i} \int_D G(x, y)\partial_k \xi_k(y) d^3y \tag{2.8}$$

Here and in the following summation over repeated indices is understood.

To find the action of A on a generic element of $(L^2(D))^3$ one can use (2.8) on approximating sequences.

For periodic boundary conditions over a regular lattice, the action of A is easily described using dual lattice variables. For example, for a cubic lattice of side one, one has

$$(\tilde{A} f)_i(k) = k_i \frac{1}{k^2} k_j \tilde{f}_j(k), \qquad k \in Z^3 \tag{2.9}$$

where, for each $i = 1, 2, 3$, $\tilde{f}_i(k)$ is the Fourier transform of $f_i(x)$.

Since $\nabla\phi \in \mathscr{H}_v$ for all V if $\phi \in C_0^1$, integration by part shows that if $\eta \in \mathscr{H}_V$ then $\frac{\partial \eta_i}{\partial x_i} = 0$, $x \in D$.

Therefore, for all boundary conditions (V, Φ_0) we interpret div $\varepsilon . E = 0$ as

$$A\varepsilon . E = 0 \tag{2.10}$$

Notice that (2.10) provides also information on the behaviour of εE at the boundary.

The system (2.7), (2.10) is equivalent to Maxwell's equations with boundary conditions (V, E_0). In particular, any solution E of (2.7), (2.10) of class C^1 is also a strong solution of Maxwell's equation.

Let $\delta > 0$ be any real number.

Using (2.6′) and $A^2 = A$ one verifies that (2.7), (2.10) are equivalent to

$$A(1 - \delta\varepsilon)E = E - B \tag{2.11}$$

Let $\varepsilon_0 \geq \varepsilon_+$, and define the positive operator Q_{ε_0} by

$$Q_{\varepsilon_0} \doteq \left(I - \frac{\varepsilon}{\varepsilon_0}\right)^{1/2} \tag{2.12}$$

(we do not distinguish notationally between the function Q_{ε_0} and the operator of multiplication by Q_{ε_0}).

Since Q_{ε_0} is invertible, one obtains from (2.11), with $\delta = \varepsilon_0^{-1}$ the equivalent form

$$Q_{\varepsilon_0} A Q_{\varepsilon_0} . Q_{\varepsilon_0} E = Q_{\varepsilon_0} E - Q_{\varepsilon_0} B \tag{2.13}$$

which can be written

$$E = B + AQ_{\varepsilon_0}(I - Q_{\varepsilon_0}AQ_{\varepsilon_0})^{-1}Q_{\varepsilon_0}B, \qquad \varepsilon_0 \geq \varepsilon_+ \tag{2.14}$$

Similarly, for $0 < \varepsilon_0 < \varepsilon_-$, let

$$Q'_{\varepsilon_0} \doteq \left(\frac{\varepsilon}{\varepsilon_0} - I\right)^{1/2} \tag{2.12'}$$

One has then

$$E = B - AQ'_{\varepsilon_0}(I + Q'_{\varepsilon_0}AQ'_{\varepsilon_0})^{-1}Q'_{\varepsilon_0}B, \qquad 0 < \varepsilon_0 \leq \varepsilon_- \tag{2.15}$$

Remark that $\| Q_{\varepsilon_0} \| < 1$, where $\| \ \|$ denotes the operator norm. Therefore (2.14) provides a uniformly convergent series which determines E.

Consider now the important special case when $\varepsilon(x)$ takes only a finite number of values ε_i, $i = 1 \dots N$, and let χ_i be the characteristic function of the domain D_i in which $\varepsilon(x) = \varepsilon_i$. Then

$$Q_{\varepsilon_0} = \sum_{k=1}^{N} \left(1 - \frac{\varepsilon_k}{\varepsilon_0}\right)^{1/2} \chi_k \tag{2.12''}$$

To study the response to external fields which depend slowly on time, it is often convenient to consider Maxwell's equations when the tensors ε_i are allowed to be complex-valued. We consider only the case in which the complex extensions, still denoted by ε_i, can be diagonalized. Denote by $\varepsilon_{i,\alpha}$, $\alpha = 1, 2, 3$ the eigenvalues of ε_i, and let $\varepsilon = \{ \varepsilon_{i,\alpha} \} \in C^{3N}$. For a fixed geometry (i. e. choice of $\{ D_i \}$), let $E(\varepsilon)$ be the (complex-valued) solution of Maxwell's equation when $\varepsilon(x) = \Sigma \varepsilon_i \chi_i$. Let $\Theta \subset C^{3N}$ be defined by

$$\Theta \doteq \bigcup_{a \in R} \Theta_a \tag{2.16}$$

$$\Theta_a \doteq \left\{ \varepsilon \in C^{3N} \backslash \{ 0 \} \;\middle|\; -\frac{\pi}{2} + a < \arg \varepsilon_{i,\alpha} < \frac{\pi}{2} + a,\ \alpha = 1, 2, 3,\ i = 1 \dots N \right\}$$

One has then

PROPOSITION 2.1. — For all $\varepsilon \in \Theta$, $E(\varepsilon)$ exists and is unique. Moreover $E(\varepsilon)$ is analytic in Θ (as an $(L^2(D))^3$-valued function), and for each $\varepsilon^{(1)} \in \Theta$ one can find $\varepsilon_0 \in C$ such that,

$$E(\varepsilon) = B + \sum_{n=0}^{\infty} AQ_{\varepsilon_0}(Q_{\varepsilon_0}AQ_{\varepsilon_0})^n Q_{\varepsilon_0}B$$

The series is uniformly convergent in a neighbourhood of $\varepsilon^{(1)}$ and Q_{ε_0} is defined in (2.12). ■

Proof. — The first statement, and also analyticity, follows from the coercivity of the operator $e^{i\phi_a} \frac{\partial}{\partial x_\alpha} \varepsilon_{\alpha\beta}(x) \frac{\partial}{\partial x_\beta}$ in each region Θ_a, for a suitable choice of ϕ_a.

For the representation through a convergent series, notice that (2.13), (2.15) hold also when ε takes complex values, provided one can find $\varepsilon_0 \in \mathrm{C}$ such that

$$\max_{i,\alpha} \left| 1 - \frac{\varepsilon_{i,\alpha}}{\varepsilon_0} \right| < 1 \tag{2.17}$$

For $\varepsilon^{(0)} \in \Theta_a$, let $\varepsilon_0 = \mathrm{C}e^{ia}$, $\mathrm{C} > 0$. Then

$$\left| 1 - \frac{\varepsilon_{i,\alpha}^{(0)}}{\varepsilon_0} \right|^2 = \left(1 - Re\left(e^{-ia}\varepsilon_{i,\alpha}^{(0)}\mathrm{C}^{-1}\right)\right)^2 + \left(\mathrm{Im}\left(e^{-ia}\varepsilon_{i,\alpha}^{(0)}\mathrm{C}^{-1}\right)\right)^2$$

and by construction $Re\left(e^{-ia}\varepsilon^{(0)}\right) > 0$.

One can then find a constant C_0, depending only on $\max_{i,\alpha} | \varepsilon_{i,\alpha}^{(0)} |$, such that (2.17) holds in a full neighbourhood of $\varepsilon^{(0)}$ if $\mathrm{C} > \mathrm{C}_0$. ■

REMARK 2.2. — One has

$$\partial\Theta = \bigcup_{(\mathrm{P},\mathrm{Q})} \bigcup_{a\in\mathrm{R}} \partial\Theta_a(\mathrm{P}, \mathrm{Q})$$

where P, Q = 1 ... 3 N stand for pair of indices (i, α) and

$$\partial\Theta_a^{(\mathrm{P},\mathrm{Q})} \doteq \left\{ \varepsilon \in \mathrm{C}^{3\mathrm{N}}, \frac{-\pi}{2} + a = \arg \varepsilon_{\mathrm{P}} \leq \arg \varepsilon_{\mathrm{R}} \leq \arg \varepsilon_{\mathrm{Q}} = \frac{\pi}{2} + a \right\} \tag{2.18}$$

$\mathrm{R} \neq \mathrm{P}, \mathrm{Q}$

It is easy to verify that Θ is the (inductive) limit of analytic polyhedra, in the sense of Weyl [7].

This allows to write integral representations for E (and for ε^*) where the integration is on a measure supported by $\partial\Theta$ and depending only on the domains D_i, D, while the integrand is a suitable function of the ε_k's.

REMARK 2.3. — If $\varepsilon_{ij}(x) \in \mathrm{C}^k$, multiplication by ε is a bounded operator on $(\mathrm{H}^k)^3$. If $k \geq 3$, the same is true if $\varepsilon_{ij} \in \mathrm{H}^k$. If ε_0 is chosen sufficiently large (2.12) defines then a bounded positive operator on $(\mathrm{H}^k)^3$, with norm smaller than one. The Hodge projection A can be defined on $(\mathrm{H}^k)^3$ in much the same way as on $(\mathrm{L}^2)^3$. Therefore under the assumptions indicated here, (2.14), (2.15) and Proposition 2.1 hold also in $(\mathrm{H}^k)^3$. ■

We now turn to the description of the effective dielectric constant ε^*, which we define, for boundary conditions (V, E_0), by

$$\langle \varepsilon \mathrm{E}, \hat{\mathrm{B}} \rangle = \varepsilon^* \| \mathrm{B} \| \tag{2.19}$$

where $\hat{\mathrm{B}} = \mathrm{B} . \| \mathrm{B} \|^{-1}$.

From (2.7), (2.6) one has

$$\langle \mathrm{E}, \hat{\mathrm{B}} \rangle = \| \mathrm{B} \| \tag{2.20}$$

and therefore ε^* satisfies

$$\langle \varepsilon \mathrm{E}, \hat{\mathrm{B}} \rangle = \varepsilon^* \langle \mathrm{E}.\mathrm{B} \rangle \tag{2.21}$$

In (2.21) the role of ε^* as effective parameter in linear response theory is clearly exhibited.

Let

$$\bar{\varepsilon}_{\mathrm{B}} \doteq \langle \hat{\mathrm{B}}, \varepsilon \hat{\mathrm{B}} \rangle \tag{2.22}$$

In most cases of interest $\hat{\mathrm{B}}$ is constant so that $\bar{\varepsilon}_{\mathrm{B}}$ is the average of $(\hat{\mathrm{B}}, \varepsilon(x)\hat{\mathrm{B}}) \doteq \varepsilon_{\mathrm{B}}(x)$. From (2.14), (2.15) one derives

$$\varepsilon^* - \bar{\varepsilon}_{\mathrm{B}} = -\varepsilon_0 \int_0^{1-\frac{\varepsilon_-}{\varepsilon_0}} \frac{\lambda}{1-\lambda} \mu_{\varepsilon_0}(d\lambda) \qquad \varepsilon_0 \geq \varepsilon_+ \tag{2.23'}$$

$$\varepsilon^* - \bar{\varepsilon}_{\mathrm{B}} = -\varepsilon'_0 \int_0^{\infty} \frac{\lambda}{1+\lambda} \mu'_{\varepsilon_0}(d\lambda) \qquad 0 < \varepsilon'_0 < \varepsilon_- \tag{2.23*}$$

where μ_{ε_0} is the spectral measure of $\mathrm{Q}_{\varepsilon_0}\mathrm{A}\mathrm{Q}_{\varepsilon_0}$ in the state $\mathrm{Q}_{\varepsilon_0}\mathrm{B}$ (i. e. the *pth* moment of μ_{ε_0} is ($\langle \mathrm{Q}_{\varepsilon_0}\mathrm{B}, (\mathrm{Q}_{\varepsilon_0}\mathrm{A}\mathrm{Q}_{\varepsilon_0})^p\mathrm{Q}_{\varepsilon_0}\mathrm{B} \rangle$) and μ'_0 is the spectral measure of $\mathrm{Q}'_{\varepsilon'_0}\mathrm{A}\mathrm{Q}'_{\varepsilon'_0}$ in $\mathrm{Q}'_{\varepsilon'_0}\mathbb{B}$.

From (2.19) and Proposition 1 one has

PROPOSITION 2.4. — With the notations of Proposition 2.1, $\varepsilon^*(\varepsilon)$ is analytic in Θ; for each $\varepsilon \in \Theta$ one can find $\varepsilon_0 \in \mathrm{C}$ such that

$$\varepsilon^*(\varepsilon) = \bar{\varepsilon} - \sum_{n \geq 1} \varepsilon_n(\varepsilon) \tag{2.24}$$

and the series is uniformly convergent in a neighbourhood of ε. ■

In (2.24) we have used the notation

$$\varepsilon_n(\varepsilon) = \langle \mathrm{Q}_{\varepsilon_0}\hat{\mathrm{B}}, (\mathrm{Q}_{\varepsilon_0}\mathrm{A}\mathrm{Q}_{\varepsilon_0})^n\mathrm{Q}_{\varepsilon_0}\hat{\mathrm{B}} \rangle$$

Remark that, when B is constant, one has

$$\bar{\varepsilon}(\varepsilon) = \sum_{i=1}^{\mathrm{N}} \mathrm{P}_i \varepsilon_i \tag{2.25}$$

where p_i is the relative volume of the domain D_i. We point out that the measures μ_{ε_0}, μ''_{ε_0} in (2.23', ") depend both on the geometry and on the ε_i's; (2.23) and (2.23') do not give a representation in which the analyticity of ε^* can be used directly.

The only exception is the case $N = 2$, $\varepsilon_i = c_i I$, $c_i \in R_+$. In this case the dependence of μ_{ε_0}, $\mu'_{\varepsilon'_0}$ on ε_1, ε_2 is only through their total weight, and, choosing $\varepsilon_0 = \max(\varepsilon_1, \varepsilon_2)$, one can write

$$\varepsilon^* - \bar{\varepsilon} = (\varepsilon_1 - \varepsilon_2)^2 \int \frac{\lambda \hat{\mu}(d\lambda)}{(1-\lambda)\varepsilon_1 + \lambda \varepsilon_2} \tag{2.26}$$

where μ is the spectral measure of $\chi_2 A \chi_2$ in the state $\chi_2 B$. In all other cases, this factorization does not occur in (2.23′, ″).

We point out, for later use, that one has the explicit formulae

$$\varepsilon^* - \bar{\varepsilon} = -(Q_{\varepsilon_0}\hat{B}, Q_{\varepsilon_0}AQ_{\varepsilon_0}(1 - Q_{\varepsilon_0}AQ_{\varepsilon_0})^{-1}Q_{\varepsilon_0}\hat{B}) \qquad (2.27)_1$$

$$\varepsilon^* - \bar{\varepsilon} = -(Q'_{\varepsilon_0}\hat{B}, Q'_{\varepsilon'_0}AQ'_{\varepsilon'_0}(1 + Q'_{\varepsilon'_0}AQ'_{\varepsilon'_0})^{-1}Q'_{\varepsilon'_0}\hat{B}) \qquad (2.27)_2$$

We conclude this chapter with a brief comment on the variational formulation.

It is evident that (2.13) is the Euler-Lagrange equation associated to the quadratic functional

$$P_{\varepsilon_0}(T) = \frac{1}{2}\langle T, Q_{\varepsilon_0}AQ_{\varepsilon_0}T \rangle - \frac{1}{2}\langle T, T \rangle + \langle T, Q_{\varepsilon_0}B \rangle \tag{2.28}$$

Since $Q_{\varepsilon_0}AQ_{\varepsilon_0} < I$, P_{ε_0} is strictly convex so that

$$P_{\varepsilon_0}(T) \leq {}^{(}P_{\varepsilon_0}(Q_{\varepsilon_0}E) \qquad \forall T \in (L_2(D))^3 \tag{2.29}$$

Using (2.28), (2.19), (2.20) one concludes

$$\varepsilon^* \leq \varepsilon_0\left(1 - \frac{P_{\varepsilon_0}(T)}{\| B \|^2}\right) \qquad \forall T \in (L_2(D))^3, \qquad \varepsilon_0 \geq \varepsilon_+ \tag{2.30}$$

We remark that P_{ε_0} does not coincide with the quadratic functionals which intervene in the direct or in the dual variational principles associated to (2.1). Therefore inequality (2.30) is different from the ones considered e. g. in [*3*].

Similarly, for $0 < \varepsilon_0 \leq \varepsilon_-$, one considers the quadratic functional on $(L^2(D))^3$ defined by

$$P'_{\varepsilon'_0}(T) \doteq \frac{1}{2}\langle T, Q'_{\varepsilon'_0}AQ'_{\varepsilon'_0}T \rangle + \frac{1}{2}\langle T, T \rangle - \langle T, Q'_{\varepsilon'_0}B \rangle \tag{2.31}$$

Since $Q'_{\varepsilon'_0}AQ'_{\varepsilon'_0} > 0$, P is strictly concave, so that

$$P'_{\varepsilon'_0}(T) \geq \frac{1}{2}\| B \|^2 - \frac{1}{2}\langle B, \varepsilon E \rangle$$

from which one concludes

$$\varepsilon^* \geq \varepsilon_0\left(1 - \frac{P'_{\varepsilon_0}(T)}{\| B \|^2}\right) \qquad \forall T \in (L^2(D))^3, \qquad 0 < \varepsilon'_0 \leq \varepsilon_- \tag{2.32}$$

3. ELASTOSTATICS, VISCOUS FLOW, RANDOM MEDIA

In this section we shall outline the steps by which the equations for elastostatics and viscous flows can be placed in the general frame described in § 2. We shall also indicate, using electrostatics as an example, how the scheme of § 2 extends to random media.

a) *Elastostatics* (see e. g. [*8*]).

In an orthogonal local coordinate system we denote by u the elastic displacement vector, and by v the strain tensor, with components

$$v_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.1}$$

Let τ, with components τ_{ij}, denote the stress tensor. Then Hooke's law reads

$$\tau = \mathrm{C}v \text{ i. e. } \tau_{ij} = \mathrm{C}_{ij,km}v_{km} \tag{3.2}$$

where C is the elastic stiffness tensor.

We shall only treat the case in which there are no body forces. In this case, τ satisfies

$$\frac{\partial \tau_{ik}}{\partial x_k} = 0, \qquad x \in \mathrm{D} \tag{3.3}$$

where D is the region occupied by the elastic medium.

We consider the problem described by equations (3.1), (3.2), (3.3) in a bounded simply connected domain D, with prescribed boundary conditions on the strain tensor

$$v_{ij}(x) = v_{ij}^{(0)}(x), \qquad x \in \partial\mathrm{D} \tag{3.4}$$

It is known (cf. [*6*], ch. 7) that, if ∂D is a Lyapunov surface, given $v(x)$ one can solve (3.1) for u, uniquely modulo rigid translations. We write therefore (3.1) together with the boundary condition (3.4) as

$$v - v^{(0)} \in \mathscr{V} \tag{3.5}$$

where $\mathscr{V}$ is the closure in $(\mathrm{L}^2(\mathrm{D}))^6$ of the vector fields of the form

$$\frac{\partial \phi_i}{\partial x_k} + \frac{\partial \phi_k}{\partial x_i}$$

with $\phi_i \in \mathrm{C}_0^1(\mathrm{D})$.

Let now A be the orthogonal projection of $(\mathrm{L}^2(\mathrm{D}))^3$ onto $\mathscr{V}$. Then (3.1), (3.3) read

$$\mathrm{A}(v - v^0) = v - v^0 \tag{3.6}$$

$$\mathrm{A}cv = \mathrm{A}\tau = 0 \tag{3.7}$$

as one easily verifies through integration by parts. For computational purpose, it is worth remarking that A in (3.6), (3.7) is the closure of the operator $\underset{\sim}{A}$ defined on $(C^1(D))^3$ by

$$(\underset{\sim}{A}f)_{ik} = \int_D \frac{\partial}{\partial x_i} G_{km}(x, y) \frac{\partial}{\partial y_n} f_{mn}(y)dy + i \leftrightarrow k \tag{3.8}$$

Here G_{km} is the Green function of the elastostatic problem (double-layer potential operator), i. e. the solution of $\Delta G_{km} + \partial_k \partial_i G_{im} = \delta_{km}$ with boundary conditions defined by $\mathscr{V}$.

Typical choices for v^0 are

$$v^0 = I \tag{3.9}$$

(isotropic strain), or

$$v^0_{ij} = 1 \qquad \forall i, j \tag{3.9'}$$

For the choices (3.9), (3.9'), one has $Av^0 = 0$.

We remark that (3.6), (3.7) are identical in form to (2.7), (2.10), so that the formalism of § 2 applies. To define the effective elastostatic structure, notice first that, as in Section 2,

$$\langle v, v^0 \rangle = \langle v^0, v^0 \rangle \tag{3.10}$$

where $(v_1, v_2) = \overline{T_2(v_1^t v_2)}$.

For simplicity, we assume that v^0 is a constant symmetric tensor, and that the boundary conditions are such that $Av^0 = 0$.

Then (2.14) implies for every symmetric tensor η

$$\langle v, \eta \rangle = \langle v^0, \eta \rangle \tag{3.11}$$

From (3.10), (3.11) one derives

$$\frac{1}{|D|} \int v(x)dx \doteq \overline{v} = v^0 \tag{3.12}$$

The effective elastic stiffness C* associated to the given boundary conditions is then defined by

$$\overline{\tau} = c^* v^0 \tag{3.13}$$

Of particular interest is the case when the microscopic elastic structure is represented by two Lamé scalar functions, i. e.

$$C(x)\, v(x) = \lambda(x) \mathrm{Tr}\,(v(x)) \,.\, I + 2\mu(x) v(x) \tag{3.14}$$

for every symmetric tensor v.

From (3.13) and (3.14) one derives easily that there are constants λ^*, μ^* such that

$$\overline{\tau} = \lambda^* (\mathrm{Tr}\, v^0) \,.\, I + 2\mu^* v^0 \tag{3.15}$$

This defines the effective Lamé coefficients.

We refer to [3c] [4c] (and the references given there) for a detailed study of bounds for λ^* and μ^*, which are obtained in [4c] by exploiting the structure as analytic functions of λ, μ, and in [3c] by a combination of perturbation theory and variational techniques.

Here we limit ourselves to a simple remark. Consider, for example, a two-component system in which each component D_α $\alpha = 1, 2$, is characterized by the Lamé coefficients λ_α, μ_α.

Comparing with Section 2, we see that the problem is mathematically equivalent to electrostatics for a two-phase system in which each phase is dielectrically anisotropic.

Define Π_0 on $(L^2(R^3))^3$ by

$$3\Pi_0 v(x) = \operatorname{Tr} v(x).I \tag{3.16}$$

Let A_p be the Hodge projection for periodic boundary conditions on a cubic lattice, and $A^{(0)}$ the Hodge projection in $(L^2(R^3))^3$. Then

$$A_p\Pi_0 = \Pi_0 A_p, \qquad A^{(0)}\Pi_0 = \Pi_0 A^{(0)}, \tag{3.17}$$

Taking $v^0 = I$ or v^0 a traceless symmetric tensor, one verifies then, for the cubic lattice or if one neglects the effect of the boundary, that $\langle v^0, \sigma \rangle$ is represented as in (2.23)', the measure $\mu^*_{\varepsilon_0}$ depending on λ_α, μ_α only through its total mass.

On the other hand, in view of (3.15), (3.16), for every choice of v^0 the expression $\langle v^0, \sigma \rangle$ is a linear combination of λ^* and μ^*.

We conclude that for $N = 2$ and neglecting boundary effects, there are two linear combinations of λ^* and μ^* which admit a representation of the form (2.23)' such that the dependence on the geometry is separated from the dependence on λ_α, μ_α. Bounds on these two linear combinations can therefore be obtained by the techniques of [4a, b] (see [4c]).

b) *Viscous flow* (see e. g. [9]).

We consider the Stokes flow in a region D filled with an incompressible fluid of viscosity tensor $\eta(x)$. Denote by u the velocity field, and by $p(x)$ the pressure field.

As in elastostatics, we introduce the symmetric tensor V defined by

$$V_{ij} \doteq \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{3.18}$$

and remark that, if ∂D is sufficiently regular, v defines u modulo rigid translations, so that v can be considered as the relevant physical variable. Define σ through

$$\sigma = \eta v \tag{3.19}$$

and let

$$\pi \doteq \sigma - p.I \tag{3.19$_1$}$$

Then π is a field of symmetric tensors and Stokes equation reads

$$\frac{\partial \pi_{ij}}{\partial x_j} = 0 \tag{3.20}$$

To write (3.18)(-3.20) in the form (2.7), (2.10), denote by $\mathscr{V}$ the closure in $(L^2(D))^6$ of the vector fields of the form

$$\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}$$

with $\phi_i \in C_0^1(D)$ and $\dfrac{\partial \phi_i}{\partial x_i} = 0$.

Take boundary condition of the form

$$u_i(x) = u_i^0(x) \qquad x \in \partial D \tag{3.21}$$

where u^0 is a divergenceless vector field. Let v^0 be defined by

$$v_{ik}^0 = \frac{\partial u_i^0}{\partial x_k} + \frac{\partial u_k^0}{\partial x_i} \qquad x \in \overline{D} \tag{3.22}$$

so that $v(x) = v^0(x)$, $x \in \partial D$. Assume that $v^0 \notin \mathscr{V}$.

PROPOSITION 3.1. — Let ∂D be of class C^1. Equations (3.18)-(3.20) with the boundary condition (3.21) are equivalent to

$$A(v - v^0) = v - v^0 \tag{3.23}$$

$$A\sigma \doteq A\eta v = 0 \quad \blacksquare \tag{3.24}$$

Proof. — From the definition of A it follows that (3.23) is equivalent to the fact that v can be written uniquely as in (3.18) where u satisfies (3.21). Indeed if ∂D is of class C^1, u is defined by v in (3.18) modulo rigid translations, and (3.21) determines u completely. Let now ζ be any field of symmetric tensors. Then $A\zeta = 0$ is equivalent to

$$\int_D \zeta_{ik}(\partial_i w_k + \partial_k(w_i)dx = 0 \qquad \forall w \in (C_0^1(D))^3, \qquad \text{div } w = 0$$

Integrating by parts this leads to

$$\int_D \frac{\partial}{\partial x_i} \zeta_{ik} w_k dx = 0$$

which implies that there exists a scalar valued field $p(x)$, uniquely defined modulo an additive constant, such that

$$\frac{\partial}{\partial x_i}(\xi_{ik} - p\delta_{ik}) = 0 \qquad x \in D \quad \blacksquare$$

We will not pursue here further the study of the viscous flow. We only remark that the flow through porous media can be regarded as the limit of a viscous flow when the viscosity tensor has the form

$$\eta^{(n)}_{ij,kl}(x) = \frac{1}{2} c_n(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad x \in D_1 \subset D \tag{3.25}$$

and $\lim_{n \to \infty} c_n = +\infty$.

In the limit the flow occupies only the region $D - D_1$, and $u = 0$ at ∂D_1. If for all $n \in Z^+$

$$\eta_{ij,kl}(x) = \frac{1}{2} \eta^0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}), \qquad x \in D_2 \doteq D - D_1 \tag{3.26}$$

one can compute the effective viscosity η^* for the viscous flow as $\lim_{n\to\infty} \eta^*_n$; it will depend linearly on η^0.

Therefore one can define a constant K (Darcy's constant) through $\eta^* = K^{-1}\eta^0$.

The formalism described in section 2 can then be used directly to study η^*_n in the limit $c_n \to \infty$, and therefore to give estimates on K.

c) *Random media.*

We denote by Ω the underlying probability space, which carries a measure P and a norm-preserving continuous ergodic representation $\underline{a} \to T_{\underline{a}}$ of the translation group $\mathbb{R}^3$. Ω can be regarded as the set of possible configurations of the system.

The dielectric properties of a stationary random medium are then described by a stationary random dielectric tensor $\varepsilon(x)$.

As in § 1, on ε we assume that one can find $0 < a < b < \infty$ such that

$$a.I \le \sup_{\omega \in \Omega} \varepsilon(\omega) \le b.I \tag{3.27}$$

We have used the notation

$$\varepsilon(\omega) \doteq \varepsilon(0, \omega)$$

We denote again by ε_+, ε_- the largest (resp. smallest) real number such that

$$\varepsilon(\omega) \le \varepsilon_+ .I, \qquad \varepsilon(\omega) \ge \varepsilon.I \qquad \forall \omega \in \Omega \tag{3.28}$$

Since $\varepsilon(x, .)$ is stationary, one has then

$$\varepsilon_+ .I \ge \varepsilon(x, \omega), \qquad \varepsilon_- .I \le \varepsilon(x, \omega) \qquad \forall x \in R^3, \qquad \omega \in \Omega$$

Maxwell's equations become

$$\text{rot } E(x, \omega) = 0 \qquad a.a.\omega \tag{3.29$_1$}$$

$$\text{div } (\varepsilon(x, \omega)E(x, \omega)) = 0 \qquad a.a.\omega \tag{3.29$_2$}$$

The derivatives must be understood in a weak sense, as usual. We look for solutions of $(3.29)_{1,2}$ which are stationary random fields. The equivalent of a boundary condition is now the assignment of the average value of E(x) with respect to P. This average is independent of x in view of (3.27) and will be denoted by B.

Let l_k, $k = 1, 2, 3$, be the anti-self-adjoint operators defined on $L^2(\Omega, dP)$ by $T_{\underline{a}} = \exp(a_k l_k)$.

The l_i's have a dense common core S, by Stone's theorem. On S define $\underset{\sim}{A}$ by

$$(\underset{\sim}{A} f)_k(\omega) \doteq l_k l^{-2}(l_j f_j)(\omega) \tag{3.30}$$

where $l^2 = \sum_k l_k l_k$.

It is straightforward to verify that $\underset{\sim}{A}$ is closable and bounded by I on S, and that its extension to $(L^2(\Omega, dP))^3$, which we denote by A, is a projection operator.

If the periodic case is given a probabilistic description, the operator K here is precisely the Hodge projection. We therefore call A the probabilistic Hodge projection. Notice that $[A, T_a] = 0 \; \forall \underline{a} \in R^3$, and moreover $T_{\underline{a}}$ leaves S invariant.

We use the notation

$$\langle \phi, \psi \rangle = \int P(d\omega)(\phi(\omega) . \psi(\omega)), \; \psi, \phi \in [L^2(\Omega, dP)]^3 .$$

One has (see also [5a, b]).

LEMMA 3.2. — Equations $(3.29)_{1,2'}$ supplemented by the condition

$$\int P(d\omega) E(x, \omega) = B \tag{3.31}$$

are equivalent to

$$AE(\omega) = E(\omega) - B \qquad (3.32)_1$$

$$A\varepsilon E(\omega) = 0 \qquad (3.32)_2$$

where all equalities are understood in the $L_2(\Omega, dP)$ sense. ■

Proof. — Define $H^1(\Omega)$ through

$$H^1(\Omega) \doteq \{ \phi \in L^2(\Omega), \phi \in \mathrm{Dom}(l_k), k = 1, 2, 3 \}^- \tag{3.33}$$

where the closure is taken in the (Hilbert) norm

$$\| \phi \|_1 = \left(\int \phi^2(\omega) P(d\omega) + \sum_{k=1}^{3} \int (l_k \phi)^2(\omega) P(d\omega) \right)^{1/2}$$

We denote by H^{-1} the dual of $H^1(\Omega)$.

One verifies that the following limit exists in $H^{-1}(\Omega)$

$$\lim_{a\to 0} a^{-1}[\varepsilon_{jm}E_m(x+a\hat{j}) - \varepsilon_{jm}E_m(x)] \doteq \frac{\partial}{\partial x_j}\varepsilon_{jm}E_m(x)$$

and moreover $(3.29)_2$ is equivalent to

$$\int (l_j\phi)(\omega)(\varepsilon_{jm}E_m)(\omega)P(d\omega) = 0 \qquad \forall\phi\in H^1(\Omega) \tag{3.34}$$

which is equivalent to $(3.32)_{1,2}$.

In the same way one proves that for all $x \in R_3$,

$$\text{rot}_x\, E(x,\omega) \in (H^{-1}(\Omega))^3$$

and that $(3.29)_1$ is equivalent to

$$AE(\omega) = E(\omega) - C \tag{3.35}$$

for some constant vector C. ■

Since A is self-adjoint, taking scalar products with a constant function, one derives from (3.35)

$$C = \int P(d\omega)E(\omega)$$

In view of Lemma 3.2, the formalism of section 2 applies without change to the case of stationary media.

For computational purpose, it is convenient to have an explicit form of expressions such as $\langle B, Q_{\varepsilon_0}(AQ_{\varepsilon_0})^pB\rangle$, $p \in Z^+$, in terms of the correlation functions of the random field ε. If $f \in L^1(\Omega)$ we write

$$\mathscr{E}(f) \doteq \int f(\omega)P(d\omega)$$

The correlation functions are defined by

$$W^{i_1,\ldots,i_n}_{k_1,\ldots,k_n}(x_1\ldots x_n) \doteq \mathscr{E}(\varepsilon_{i_1k_1}(x_1)\ldots\varepsilon_{i_nk_n}(x_n)) \tag{3.36}$$

Let $f \in L^2(\Omega)$ and $g(x)$ be a stationary random field such that $g(0) \in D(l^2)$. Then $g(x) \in D(\Delta)$ $\forall x \in R^3$, and

$$\mathscr{E}(f.l^2g(x)) = \Delta\mathscr{E}(fg(x)) \tag{3.37}$$

Therefore, if $a > 0$

$$(a-\Delta)\mathscr{E}(a-l^2)^{-1}.h.g(x)) = \mathscr{E}(hg(x)) \tag{3.38}$$

It follows that, if $E(h.g(x)) \in L^1(R^3)$, then

$$\mathscr{E}(g(x)(a-l^2)^{-1}h) = \int_{R^3} G_a(x-x')\mathscr{E}(hg(x'))dx' \tag{3.39}$$

where $G_a(.)$ is the integral Kernel of $(-\Delta+a)^{-1}$.

In the same way, if $f \in (L^2(\Omega))^3$ is a stationary vector-valued random field, such that $g_i(0) \in D(l_i)$ and $l_i g_i \in D(l^{-2})$, and if $E(fg_k(x)) \in L^2(R^3) \cap H^1(R^3)$, one has

$$\mathscr{E}(f_i(l_i(a - l^2)^{-1} l_k g_k)(x) = \frac{\partial}{\partial x_i} \int_{R^3} G_a(x - x') \mathscr{E}\left(f_i \frac{\partial g_k}{\partial x_k}(x')\right) dx' \quad (3.40)$$

Let A^0 be the Hodge projection associated to the Laplacian in R^3.

Since A is bounded on $L^2(\Omega)$ and A^0 is bounded on $L^2(R^3)$, the identity (3.40) can be extended to $a = 0$ to obtain

$$\langle f, Ag(x) \rangle = \sum_{i,k=1}^{3} A^0_{ik} \mathscr{E}(f_i g_k(0))(x) \quad (3.41)$$

From (3.41) it follows,

$$\mathscr{E}(\langle B, Q^0_{\varepsilon_0}(AQ^2_{\varepsilon_0})B \rangle) = \frac{1}{\varepsilon_0^2} \mathscr{E}(\langle B, \varepsilon(0)A\varepsilon(0)B \rangle) =$$

$$= \frac{1}{\varepsilon_0^2} \int d^3y (B, \nabla_y G(y))(B, \nabla_y W_2(y)) =$$

$$= \left(\frac{1}{2\pi}\right)^3 \frac{1}{\varepsilon_0^2} \int d^3k \frac{B_i B_m k_q k_p}{|k|^2} \tilde{W}^{mp}_{iq} \quad (3.42)$$

where $\tilde{W}$ is the Fourier transform of W.

In the special case in which $\varepsilon_{ij}(x) = \delta_{ij}\varepsilon(x)$, one has

$$\mathscr{E}(\langle B, Q^2_{\varepsilon_0}(AQ^2_{\varepsilon_0})B \rangle = \left(\frac{1}{2\pi}\right)^3 \frac{1}{\varepsilon_0^2} \int d^3k \frac{(k.B)^2}{|k|^2} \tilde{W}_2(k) \quad (3.43)$$

where $\tilde{W}_2$ is the Fourier transform of $\mathscr{E}(\varepsilon(x_1)\varepsilon(x_2))$. To the following order, one has

$$\mathscr{E}(\langle B, Q^2_{\varepsilon_0}(AQ^2_{\varepsilon_0})^2 B \rangle) = \frac{1}{\varepsilon_0^2} \mathscr{E}(\langle B, \varepsilon(0)A\varepsilon(0)B \rangle) - \frac{1}{\varepsilon_0^3} \mathscr{E}(\langle B, \varepsilon(0)(A\varepsilon(0))^2 B \rangle) \quad (3.44)$$

The first term has been evaluated in (3.42).

We give the second term only in the case in which $\varepsilon_{ij} = \delta_{ij}\varepsilon$.

Let $W_3(\xi, \eta) \doteq \mathscr{E}(\varepsilon(0)\varepsilon(\xi)\varepsilon(\eta))$, and let $\tilde{W}_3(p, p')$ be its Fourier transform.

Assume that the function W_3 can be written as

$$W_3(\xi, \eta) = C_1 + W^R_1(\xi) + W^R_1(\eta) + W^R_1(\xi + \eta) + W^R_2(\xi, \eta)$$

where $W^R_1 \in L^2(R^3)$, $W^R_2 \in L^2(R^3 \times R^3)$.

Then one has

$$\mathscr{E}(\langle B, \varepsilon(0)(A\varepsilon(0))^2 B\rangle) = \left(\frac{1}{2\pi}\right)^6 \int d_p^3 \int d^3p' B_i B_k \frac{p_i p_j p'_k p'_j}{|p|^2 |p'|^2} \tilde{W}_2^R(p, p') + \\ + \left(\frac{1}{2\pi}\right) \int d^3p B_i B_k \frac{p_i p_k}{|p|^2} \tilde{W}_1^R(p) \qquad (3.45)$$

All terms of the form $E(B, Q_{\varepsilon_0}(Q_{\varepsilon_0}AQ_{\varepsilon_0})^p Q_{\varepsilon_0}B)$ can be obtained in this way.

4. RESOLVENT EXPANSIONS, THE CLAUSIUS-MOSSOTTI APPROXIMATION AND AN APPLICATION

In this section we want to elaborate further on the formalism of section 2 and in particular on the possibility of developing approximation schemes and of testing their convergence. We shall consider here only approximations for ε^*. A natural procedure is to substitute in $(2.27)_{1,2}$ for $Q_{\varepsilon^0}AQ_{\varepsilon^0}$ (resp. $Q'_{\varepsilon^0}AQ'_{\varepsilon^0}$) another positive operator Z with which the expression can be calculated or approximated, and then to estimate the error made.

In view of the resolvent structure of $(2.27)_{1,2}$ we call this procedure a resolvent expansion. One has

$$(I - Q_{\varepsilon_0}AQ_{\varepsilon_0})^{-1} = (1-Z)^{-1} + (1-Z)^{-1}(Q_{\varepsilon_0}AQ_{\varepsilon_0} - Z).(1 - Q_{\varepsilon_0}AQ_{\varepsilon_0})^{-1} \qquad (4.1)$$

$$(1 + Q'_{\varepsilon_0}AQ'_{\varepsilon_0})^{-1} = (1+Z)^{-1} - (1+Z)^{-1}(Q'_{\varepsilon_0}AQ'_{\varepsilon_0} - Z)(1 + Q'_{\varepsilon_0}AQ'_{\varepsilon_0})^{-1} \qquad (4.2)$$

Our task is to control the second term on the r. h. s. of (4.1), (4.2).

A possible choice for Z is

$$Z = Q_{\varepsilon_0}A_0Q_{\varepsilon_0} \qquad (4.3)$$

where

$$A_0 > 0, \qquad \| A_0 \| \leq 1 \qquad (4.4)$$

A_0 *may* be the Hodge projection corresponding to different boundary conditions on ∂D. The following easy lemma indicates the results to be expected.

LEMMA 4.1. — Let Z be defined as in (4.3) and assume

$$A - A_0 \doteq \delta A \leq A_0 \qquad (4.5)$$

Then the iteration of (4.2) gives a uniformly convergent series. ■

Proof. — From (4.4) and the assumptions of the lemma, it follows that

$$\| (I + Q'_{\varepsilon_0}AQ'_{\varepsilon_0})^{-1/2} Q'_{\varepsilon_0}\delta A Q'_{\varepsilon_0}(I + Q'_{\varepsilon_0}AQ'_{\varepsilon_0})^{-1/2} \| < 1 \qquad (4.6)$$

The formal series obtained by iterating (4.2) is

$$(I + Q'_{\varepsilon_0}AQ'_{\varepsilon_0})^{-1} = (I + Q'_{\varepsilon_0}A_0Q'_{\varepsilon_0})^{-1} + \\ + \sum_{n\geq 1} (I + Q'_{\varepsilon_0}A_0Q'_{\varepsilon_0})^{1/2}K^n(1 + Q'_{\varepsilon_0}A_0Q'_{\varepsilon_0})^{1/2} \qquad (4.7)$$

where $K \doteq (1 + Q'_{\varepsilon_0}A_0Q'_{\varepsilon_0})^{-1/2}Q'_{\varepsilon_0}\delta AQ'_{\varepsilon_0}(1 + Q'_{\varepsilon_0}A_0Q'_{\varepsilon_0})^{1/2}$.

The conclusion of the lemma is then a consequence of (4.6). ■

Remark. — From the proof given, it is clear that a result corresponding to lemma 4.1 holds for an expansion of

$$\langle \underline{B}, Q'_{\varepsilon_0}(I + Q'_{\varepsilon_0}AQ'_{\varepsilon_0})^{-1}Q'_{\varepsilon_0}\underline{B} \rangle$$

if the weaker assumption holds

$$|\langle \psi, \delta A\psi \rangle| \leq \langle \psi, A\psi \rangle$$

for all ψ in the smallest subspace of $(L^2(D))^3$ which contains $\underline{B}$ and is left invariant by $Q'_{\varepsilon_0}AQ'_{\varepsilon_0}$ and by $Q'_{\varepsilon}A_0Q'_{\varepsilon}$. We shall use this fact in the example illustrated below. A second case of interest is the following. Let D_i, $i > 2$, have diameter $< l_0$ and satisfy $d(D_i, D_j) > L_0$ for $i \neq j$ and also $d(D_i, \partial D) > L_0$, where $L_0 \gg l_0$. Let $\varepsilon_i = c_i I$ $i = 1 \dots N$, and $\varepsilon_1 < \varepsilon_k$, $k \geq 2$. We do not assume that $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$.

This description corresponds to a homogeneous host medium in which N homogeneous inclusions are sparsely placed.

Let $\{ \underset{\sim}{D}_i, i > 2 \}$ be a collection of domains in D such that $\partial\underset{\sim}{D}_i$ are smooth and $D_i \subset \underset{\sim}{D}_j$ iff $i = j$, $d(D_i, \partial\underset{\sim}{D}_j) > L_0/2$ $\forall i, j$, $d(D_i, \partial D) > L_0/2$, and let $\underset{\sim}{D}_1 = D \backslash \bigcup_{j\geq 2} \underset{\sim}{D}_j$.

In words, $\underset{\sim}{D}_i$, $i \geq 2$, are widely separated and for each i, D_i is interior to $\underset{\sim}{D}_i$ and widely separated from its boundary.

Let A_i, $i = 2 \dots N$, be the Hodge operator in $\underset{\sim}{D}_i$ with Dirichlet boundary conditions on $\partial\underset{\sim}{D}_i$, and A_1 be the Hodge operator with Dirichlet boundary condition on $\partial\underset{\sim}{D}_1 \backslash \partial D$ and with the boundary conditions on ∂D which entered the definition of A. Consider the natural decomposition

$$(L_2(D))^3 = \bigoplus_1^N (L_2(\underset{\sim}{D}_i))^3$$

and let

$$A_0 \doteq \bigoplus_i^N A_i$$

Take $\varepsilon_0 = \varepsilon_1$, so that

$$Q = \sum_{i=2}^N \left(\frac{c_i}{c_1} - 1\right)^{1/2} \chi_i$$

and notice that

$$QA_0Q = \bigoplus_i Q_iA_iQ_i \tag{4.8}$$

Since

$$\langle QB, (I + QA_0Q)^{-1}QB \rangle = \sum_{i=1}^{N} \langle Q_iB, (I + Q_iA_iQ_i)^{-1}Q_iB \rangle$$

one has

$$\varepsilon^* = \varepsilon_1 + \sum_{2}^{N} (\varepsilon_1 - \varepsilon_i)^2 \int \frac{\lambda \mu_i(d\lambda)}{\varepsilon_1(1-\lambda)+\lambda\varepsilon_i} + \\ + \langle QB, [(I + QAQ)^{-1} - (I + QA_0Q)^{-1}]QB \rangle \tag{4.9}$$

where μ_i is the spectral measure of $\chi_iA_i\chi_i$ in the state χ_iB. With these notations, one has

LEMMA 4.2. — For L_0 sufficiently large, the series

$$\sum_{p \geq i} (QB, (I + QA_0Q)^{-1}(Q\delta AQ(1 + QA_0Q)^{-1})^pQB), \qquad \delta A \doteq A - A_0$$

is uniformly convergent and converges to zero when $L_0 \to \infty$. ■

Proof. — We regard an operator T on $(L^2(D))^3$ as a matrix of operators from $(L^2(D_i))^3$ to $(L^2(D_j))^3$. Since $(I + QA_0Q)^{-1} < I$, it suffices to prove that one can find $\Lambda > 0$ such that, for $L_0 > \Lambda$, the operator whose matrix is $((Q_i\delta AQ_j))$ has norm smaller than one.

Remark that $\chi_iA_0\chi_j = 0$, $i \neq j$.

It suffices therefore to prove that for any integer N one can find $\Lambda > 0$ such that, for $L_0 > \Lambda$

$$\max \left\{ \sup_{i \neq j} \left(\| \chi_iA\chi_j \| \left(\frac{c_i}{c_1} - 1\right)^{1/2} \left(\frac{c_j}{c_1} - 1\right)^{1/2} \right), \right. \\ \left. \sup_i \left(\| \chi_i(A - A_i)\chi_i \| \left(\frac{c_i}{c_1} - 1 \right) \right) \right\} < \frac{1}{N}$$

By a limiting argument, it is easily established that

$$\| \chi_iA\chi_j \| \leq \sup_{x \in \underset{\sim}{D}_i, y \in \underset{\sim}{D}_j} \ \sup_{\hat{a}, \hat{y} \in S^2} | (\hat{a}.\nabla_x)(b.\nabla_y)G(x, y) |, \qquad i \neq j \tag{4.10}$$

$$\| \chi_i(A - A_i)\chi_i \| \leq \sup_{x, y \in \underset{\sim}{D}_i} \ \sup_{\hat{a}, \hat{b} \in S^2} | (\hat{a}.\nabla_x)(b.\nabla_y)(G(x, y) - G_i(x. y)) | \tag{4.11}$$

where G is the Green function associated to A and G_i the Green function in $\underset{\sim}{D}_i$ with Dirichlet boundary conditions. The conclusions of lemma 4.2 follow then from standard results on the decay properties of Green functions for regular domains. ■

Remark. — We remark that the first term in the resolvent expansion, i. e. the second term in (4.9), is usually called the Clausius-Mossotti approximation [*3*a].

In the remaining part of this section, we work out the details of the approximation procedure for the following case [*11*]a, *b*, *c*. A cubic periodic array of balls of dielectric tensor $\varepsilon_2 . I$ and radius ρ is placed in a homogeneous host medium with dielectric tensor $\varepsilon_1 . I$. Periodic boundary conditions are assumed. We seek an expansion of ε^* in powers of ρ. One has, according to § 1, and assuming $\varepsilon_1 > \varepsilon_2$

$$\varepsilon^* = \varepsilon_1 p_1 + \varepsilon_2 p_2 - (\varepsilon_1 - \varepsilon_2)^2 \langle \chi\hat{k}, \chi A\chi(\varepsilon_1 - (\varepsilon_1 - \varepsilon_2)\chi A\chi)^{-1}\chi\hat{k} \rangle \quad (4.11)$$

where $p_2 = 4/3\pi\rho^3$, χ is the characteristic function of the ball at the origin, and we have chosen the vector $\underline{B}$ to be parallel to one of the principal axis of the lattice, which we denote by $\hat{k}$. We denote by W the unit cube centered at the origin, and by b the ball of radius ρ at the origin. We compute the action of $A\chi$ on special vectors.

Let $f \in (L^2(W))^3$, $\partial_i f_i = 0$ (weakly). By a limit process one verifies that $\zeta(x) \doteq (A\chi f)(x)$ has the following properties, which characterize it uniquely

a) ζ is curl-and divergence free except at most on ∂b;

b) on ∂b the normal component of ζ has a discontinuity equal to $-f.n$, where $\underline{n}(x)$ is the outward normal;

c) ζ has mean value zero in W and is periodic on ∂W.

Formally, $\zeta(x)$ is the electric field due to charges $-\frac{1}{4\pi} f.\underline{n}$ distributed on the surface of the balls at each lattice site.

Let A_0 be the restriction to $(L^2(W))^3$ of the Hodge projection on $(L^2(R_3))^3$. $(A_0\chi f)(x)$ is then $1/4\pi$ times the electric field due to a charge $-f.\underline{n}$ on ∂b. Define $\delta A'\chi$ through

$$\delta A'\chi f \doteq A\chi f - A'_0\chi f$$
$$A'_0\chi f(x) \doteq A_0\chi f(x) - \int_W (A_0\chi f)(y) d^3y \quad (4.12)$$

Recall that any vector-valued function g which has support in b, is of class C^1, is divergence free and is continuous up to ∂b, can be uniquely decomposed as

$$g = g^d + g'$$

where g^d is the « dipole » part of g, relative to the given axis $\hat{k}$.

If f is divergence-free, we defind $\delta A . \chi f$ by

$$\delta A . \chi f = \delta A'(\chi f)d + \hat{k}(\chi f)' \quad (4.13)$$

where $\hat{k}.(\chi f)'$ is the field in W due to the array of « image » spheres (with the exclusion of the one at the origin) each charged with a surface charge $-f.n$.

This field is given by a uniformly convergent series. In this notation, $\delta A.\chi f$ is by definition the field in W due to all spheres, each charged with surface charge $-f.n$. Moreover, since f is divergence-free

$$\delta A'\chi f(x) = \delta A\chi f(x) - \int_W \delta A\chi f(y) d^3 y$$

The action of $A_0\chi$ on the (divergence-free) vectors $\nabla Y_{lm} r^l$ can be computed to be

$$A_0 \chi \nabla Y_{lm} r^l = \frac{1}{2l+1} \nabla (Y_{lm} r^l) \tag{4.14}$$

In particular

$$\chi A'_0 \chi k = \frac{1}{3}(1-\phi)\chi\hat{k}, \qquad \phi = \frac{4}{3}\pi\rho^3 \tag{4.15}$$

Using invariance under permutation, one proves also

$$\delta A.\chi\hat{k}(0) = 0, \qquad \int_W \delta A.\chi\hat{k}(x) d^3 x = 0$$

Taking into account invariance under rotation of $\pi/2$ around $\hat{k}$, one verifies that

$$\delta A'\chi\hat{k}(x) = \delta A\chi\hat{k}(x) =$$
$$= -\nabla \left| \sum_{p \geq 1} \sum_{|m| < 2p+1,\ \frac{m}{4} \in Z} \left(a_{2p+1,m} \frac{\rho^3}{2p+3} Y_{2p+1,m} r^{2p+1} \right) \right. \tag{4.16}$$

The coefficients $a_{l,m}$ are obtained comparing (4.16) with the gradient of the potential given by the image charges

$$U(x) = \frac{\rho^3}{3} \sum_{\underline{n} \neq \underline{0}} \left(\frac{\partial}{\partial x_1} \frac{1}{|\underline{x} - \underline{n}|} \right)$$

One obtains

$$a_{l,m} = \frac{4}{3} \frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)} \sum_{\underline{n} \neq \underline{0}} \frac{Y_{l+1,m}(\theta_{\underline{n}}, \phi_{\underline{n}})}{|n|^{1+2l}} \tag{4.17}$$

where $\phi_{\underline{n}}$, $\theta_{\underline{n}}$ are the angular coordinates of the point $\underline{n} \in Z_3$.

The action of $\delta A\chi$ on more general divergence-free vectors requires only a slight generalization. An expression similar to (4.16) must be compared with the potential of an image array of multipoles at the lattice points.

For the present purposes it will be sufficient to know the first term in an expansion in spherical harmonics of $\delta A\chi\nabla(Y_{\zeta,0}r^3)$. One computes

$$\delta A\chi\nabla(Y_{\zeta,0}r^3)(x) = \rho^7 a'_{\zeta,0}\nabla(Y_{\zeta,0}r^3) + o(\rho^7) \tag{4.18}$$

where

$$a'_{\zeta,0} = -\frac{60}{7}\sum_{n\neq 0}\frac{P_6(\theta_{\underline{n}})}{|\underline{n}|^7} \tag{4.17$'$}$$

and P_6 is the Legendre polynomial of order 6. Remembering that $\nabla(Y_{\zeta,0}r^3)$ has mean zero in W, (4.18) provides also the value of $\delta A'\chi\nabla(Y_{\zeta,0}r^3)(x)$.

We are able now to compute explicitely the coefficients of ε^* in an expansion in powers of ρ, up to order 20. From (4.11), (4.7) one has

$$\begin{aligned}\varepsilon^* &= \underline{\varepsilon}.\underline{p} - (\varepsilon_1 - \varepsilon_2)(\chi\hat{k}, A'_0(I - \gamma\chi A'_0\chi)^{-1}\chi\hat{k}) + \\ &+ ((I - \gamma\chi A'_0\chi)^{-1}\chi\hat{k}, \gamma\chi\delta A'\chi(I - \gamma\chi A'_0\chi)^{-1}\chi\hat{k}) + \\ &+ ((I - \gamma\chi A'_0\chi)^{-1}\gamma\chi\delta A'\chi(I - \gamma\chi A'_0\chi)^{-1}\gamma\chi\delta A'\chi(I - \gamma\chi A'_0\chi)^{-1}\chi\hat{k}, \\ &\quad \gamma\chi\delta A'\chi(I - \gamma\chi A'\chi)^{-1}\chi\hat{k}) + \\ &+ ((I - \gamma\chi A'\chi)^{-1}\chi\delta A'\chi(I - \gamma\chi A'\chi)^{-1}\chi\hat{k}, \gamma\chi\delta A'\chi(I - \gamma\chi A'\chi)^{-1}\chi\hat{k}) + \\ &+ \text{terms with more than three } \delta A'.\end{aligned} \tag{4.19}$$

We have defined

$$\gamma = 1 - \frac{\varepsilon_2}{\varepsilon_1}, \qquad p_2 = \frac{4}{3}\pi\rho^3, \qquad p_1 = 1 - p_2$$

The first term in parenthesis gives the Clausius-Mossotti formula

$$\varepsilon^* = \underline{\varepsilon}.\underline{p} - (\varepsilon_1 - \varepsilon_2)\phi\frac{\gamma(1-\phi)}{3 - 9\gamma^{-1}(1-\phi)} = \varepsilon_1\left(1 - \frac{3\phi}{3\gamma^{-1} - 1 + \phi}\right) \tag{4.20}$$

where $\phi = \frac{4}{3}\pi\rho^3$.

The second term is zero due to orthogonality (inside b) of $\chi\hat{k}$ and $\delta A'\chi\hat{k}$.

The third term can be computed to any order in ϕ from the knowledge of $\delta A'\chi\hat{k}$. One obtains

$$\left(1 - \frac{\gamma}{3}(1-\phi)\right)^{-2}\gamma^2 \sum_{l\,\text{odd},\, l\geq 3} \frac{l a_{l,0}^2 + 2l\sum_{p<l/4} a_{l,4p}a_{l,-4p}}{1 - \frac{1}{2l+1}\gamma^l}\,\rho^{2l+7}. \tag{4.21}$$

The fourth term can be computed using (4.18); one obtains

$$3\gamma^2\left(1 - \frac{\gamma}{3}(1-\phi)\right)^2\left(1 - \frac{3}{7}\gamma\right)^2 a'_{3,0}(a_{3,0})^2\rho^{20} + o(\rho^{21}) \tag{4.22}$$

Therefore, up to order 21 included, ε^* is given by the sum of the expressions (4.20), (4.21), (4.22).

We want to point out that the (infinite) sums which occur in (4.17), (4.17)' are so rapidly converging that, with the exception of the $a_{3,0}$ term, they are very well approximated if one extends the sum only over the six nearest neighbours.

The figures we find are

$$\begin{aligned} a_{3,0} &= 5.55246 \\ a_{5,0} &= 1.22558 \\ a_{5,4} &= 1.70899 \\ a_{7,0} &= 7.955231 \\ a_{7,4} &= 0.74789 \\ a'_{3,0} &= 4.91425 \end{aligned}$$

These figures are in accordance with those given in [*11*]*a* with the exception of the ρ^{17} term, which is incorrect in [*11*]*a*, as pointed out in [*11*]*b*.

APPENDIX

RELATION WITH THE BOUNDARY LAYER METHODS

We give here a short sketch of the relation between the formalism of section 2 and the approach through boundary layer potentials. Again we take electrostatics as an example. Let D_i, $i = 1 \dots N$ be open disjoint sub-domains of D, with smooth boundary. Let $D_{N+1} \doteq D \setminus \bigcap_{i=1\dots N} D_i$. Let $\varepsilon(x)$ have the form

$$\varepsilon(x) = \sum_{1}^{N+1} \chi_i(x)\varepsilon_i I \tag{A.1}$$

where χ_i is the characteristic function of D_i. Assume that $\partial D_i \cap \partial D = \phi$ $i = 1 \dots N$, and let $n_i(x)$, $x \in \partial D_i$, $i = 1 \dots N$ be the outward normal to ∂D_i at x. Let g be a function of class C^1, supp $g \cap \partial D = \phi$, and let $b(x)$ be a continuous vector field on D.

By a limiting procedure it is easy to verify that, for $i, j = 1 \dots N$

$$\int_D d^3x \nabla g(x).(\chi_i A \chi_j b)(x) = \int_{\partial D_j} dy g(y) n_i(x) \nabla_y G(y, x) n_j(x) d^3x \tag{A.2}$$

(A.2) holds in a weak sense if g is only assumed to be continuous.

Under the assumption made on ε and ∂D_i, the vector-valued function $A\chi_j b$ has a continuous trace on ∂D_i, $i = 1 \dots N$. Therefore (3.21) can be iterated, and for each integer n one can write

$$(QB, (QAQ)^n QB), \qquad Q = \sum_{i=1}^{N+1} \left(1 - \frac{\varepsilon_i}{\varepsilon_0}\right)^{1/2} \chi_i$$

in terms of the operators $K_{i,j}$ from $L_2(\partial D_j)$ to $L_2(\partial D_i)$ $(i, j = 1 \dots N)$ defined by

$$(K_{ij} f)(y) \doteq n_i \int_{\partial D_j} \nabla G(x, y) f(x) dx \qquad y \in \partial D_i \tag{A.3}$$

From (2.14) (resp. (2.15)), expanding in power series in QAQ, one obtains an expression of E (resp. ε^*) in terms of the (boundary layer) operators K_{ij}.

We remark that it is somewhat difficult to establish directly the convergence of the series so obtained. This is of course due to the fact that the bound $\| A \| = 1$ is not immediately apparent in the boundary-layer formulation.

Only in the case $N = 1$ can one write ε^* in closed form in terms of $K_{1,1}$.

Since $K_{1,1}$ is compact (in fact, of Hilbert-Schmidt class), one concludes that, if $N = 1$ and $\partial D_1 \cap \partial D = \phi$, ∂D_1 of class C^1, the measure μ in $(2.23)_{1,2}$ is totally atomic [*4*]*a*; one also verifies that 1/2 is the only limit point of the atoms of μ.

From our general representation, it follows that in this case the operator $\chi_2 A \chi_2$ has non-empty point spectrum (the same is true of $\chi_1 A \chi_1$). It is not known if the continuous part of the spectrum of $\chi_2 A \chi_2$ is empty, nor what is the spectral type of QAQ if $N > 2$.

We describe briefly the case of a spatially periodic array of identical spherical inclusions of radius ρ with dielectric tensor $\varepsilon_1 . I$, imbedded in a host homogeneous medium of dielectric tensor $\varepsilon_1 . I$ [*12*].

If a constant homogeneous macroscopic electrical field B is imposed upon the material we have that

$$E(x) = \nabla T(x)$$
$$T(x) = u(x) + B.x \qquad (A.4)$$

with $u(x)$ a periodic function.

We suppose that $|B| = 1$.

Let the array be cubic, and let W be the unit cell.

Let B_ρ be the ball of radius ρ centered at the origin, S_ρ its boundary.

Maxwell's equations are then

$$\nabla(\varepsilon(x)(\nabla u + B)) = 0 \qquad x \in \overset{\circ}{W} \qquad (A.5)_1$$

$$\varepsilon_1\left[\frac{\partial u}{\partial n}(x) + B.n(x)\right] = \varepsilon_2\left[-\frac{\partial u}{\partial n}(x) + B.n(x)\right] \qquad x \in S_\rho \qquad (A.5)_2$$

where u is continuous and periodic.

Here $n(x)$ is the outward normal to S_ρ.

From potential theory we can determine $u(x)$, solution of $(A.5)_{1,2}$ and then find the electric field $E(x)$ in terms of the charge density on S_ρ. For $u(x)$ we make the following ansatz:

$$u(x) = \int_{S_\rho} G(x, y)z(y)dS(y) \qquad (A.6)$$

with

$$\int_{S_\rho} z(y)dS(y) = 0 \qquad (A.7)$$

In (A.6) $G(x, y)$ is the periodic Green's function, solution of the following problem

$$-\Delta G(x, y) = \delta(x - y) - 1 \qquad (A.8)$$

It can be represented as

$$G(x, y) = g(x, y) + \frac{|x - y|^6}{6} + S(x, y) \qquad (A.9)$$

where $g(x, y) = 1/4\pi|x - y|$ is the free Green's function and $S(x, y)$ is a harmonic function in W.

From the fact that the single layer potential $u(x)$ presents a discontinuity in the normal derivative on the surface of the inclusion we have that the unknown charge density $z(y)$ verifies

$$z(x) = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}\left\{\int_{S_\rho} K(x, y)z(y)dS(y) + 2B.n(x)\right\}, \qquad x \in S_\rho \qquad (A.10)$$

Here

$$K(x, y) = 2\frac{\partial G}{\partial n}(x^t\, y) \doteq K_0(x, y) + K_1(x, y) \qquad (A.11)$$

$$K_0 \doteq \frac{(x - y).x}{2\pi|x - y|^3} \qquad (A.12)$$

is, up to a sign, the normal derivative of the free Green's function.

We remark that since we are dealing with spheres the following equality holds:

$$K_0(x, y) = G_0(x, y) \qquad (A.13)$$

for $x, y \in S_\rho$.

Let

$$\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \doteq \alpha \qquad (A.14)$$

If α^{-1} does not belong to the spectrum of the integral operator K we have

$$z(x) = \sum_{i \geq 1} \alpha^i (K^{i-1}(B.n))(x) \quad (A.15)$$

and from (A.6)

$$u(x) = \sum_{i \geq 1} \alpha^i \int_{S_\rho} G(x, y)(K^{i-1}(B.n))(y) dS(y) \quad (A.16)$$

From (A.1) and (A.2) one obtains then the following representation for E

$$E = B + \sum_{i \geq 1} \alpha^i \int_{S_\rho} \nabla_x G(x, y)(K^{i-1}(B.n))(y) dS(y) \quad (A.17)$$

Using (A.1) and the distributional identity

$$\nabla.(\chi\nabla) = \nabla\chi.\nabla G + \chi(1 - \pi_0) \quad (A.18)$$

where $(\pi_0 f)(x) = \int (f.n)(y) dS(y)$ and performing a resummation, one easily verifies that (A.17) coincides with (2.14) for $\varepsilon_0 = \varepsilon_2$.

For the effective dielectric constant ε^* defined by

$$\varepsilon^* \doteq \int_W (E.D) dx \quad (A.19)$$

one has

$$\varepsilon^* = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2)\phi + \int_{S_\rho} u(x)\left[\varepsilon_2 \frac{\partial u}{\partial n} - \varepsilon_1 \frac{\partial u}{\partial n_+}\right] dS + 2(\varepsilon_2 - \varepsilon_1) \int_{S_\rho} (B.\hat{x}) u(x) dS \quad (A.20)$$

where ϕ denotes, as in section 4, the fraction of volume occuped by the sphere.

Using (A.6), (A.15), (A.18) and performing a partial resummation, one verifies that (A.20) coincides with (2.24) when $B = \hat{K}$, $\varepsilon_0 = \varepsilon_2$.

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