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Spinors on CP^2 in $d \geqslant 9$ Supergravity

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ABSTRACT. — When the internal space of a $d \geqslant 9$ supergravity theory is $CP^2 \times M$, for some M, it is shown how to put spinors on CP^2 using spin^c structures, without the introduction of any extra fields. When M contains S^1, the spinors couple to a U(1) field with the number of right-handed spinors (ν_+) greater than the number of left-handed spinors (ν_-). When M contains S^2, the spinors are SU(2) doublets with $\nu_+ - \nu_- = -1$.

RÉSUMÉ. — Lorsque l'espace interne d'une théorie de supergravité à $d \geqslant 9$ est $CP^2 \times M$ pour un certain M, on montre comment on peut définir des spineurs sur CP^2 en utilisant des spin^c-structures sans introduire de champs supplémentaires. Lorsque M contient S^1, les spineurs sont couplés à un champ U(1), le nombre de spineurs droits (ν_+) étant plus grand que le nombre de spineurs gauches (ν_-). Lorsque M contient S^2, les spineurs sont des doublets de SU(2) avec $\nu_+ - \nu_- = -1$.

There are now many solutions known of the bosonic field equations of $d = 11$ and $d = 10$ supergravity theories. The basic idea is to use the extra fields, whose presence is dictated by supergravity arguments, to split the d dimensional metric up into the direct sum of a 4-dimensional metric on a non-compact spacetime and a $(d-4)$ dimensional metric on a compact, internal, manifold. The isometry group of the internal manifold is identified with the gauge group of nature.

In particular, some of these solutions involve an internal manifold whose topology is that of $CP^2 \times M$, where M is a compact space [*1*]-[*5*]. One

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of the reasons that CP^2 attracts attention is that it admits an Einstein metric whose group of isometrics has the algebra of SU(3), so that the total gauge algebra of such solutions is that of $SU(3) \times G_M$ where G_M is the group of isometries of the metric on M. However, these solutions suffer from the difficulty that CP^2 does not admit a spinor structure [*5*]-[*8*] and so such a solution will not admit spinors. One way round this problem is to put a spin^c structure on CP^2 [*6*] [*7*] [*8*] i. e. put a U(1) field on CP^2 and then add spinor fields which couple to the U(1) field in such a way that they are defined consistently. This is the approach of Ref. [*1*], where the U(1) fields is simply added to the theory, but this is somewhat contrary to the spirit of supergravity theories.

In this paper, a method is presented of consistently putting spinors on CP^2 without the introduction of any extra bosonic fields, other than those already present in the supergravity theory. The spinors can couple to an SU(2) field as doublets on CP^2 or a U(1) field as singlets on CP^2, depending on the topology of M, though the total particle spectrum will depend on the spinor content of the other dimensions. This indicates a possible connection between the particles spectrum and the topology, as first suggested by Hawking and Pope [*6*].

In many solutions the equations of motion for the metric on the internal space, obtained by dimensional reduction from $d = 10$ or 11, are simply Einstein's equations on the internal manifold ([1]).

$$\mathbb{R}_{AB} \divideontimes e^{ABC} = \lambda \divideontimes e^{C} \tag{1}$$

where e^A, $A = 1, \ldots, d-4$ are a basis of orthonormal one-forms. $e^{AB\ldots C} = e^A \wedge e^B \wedge \ldots \wedge e^C$, $\divideontimes$ is the Hodge dual in $d-4$ dimensions, $\mathbb{R}_{AB}$ are the curvature 2-forms and λ is a constant coming from the curvature scalar of 4-dimensional space-time and any other fields present e. g. the 4-form field of $N=1, d=11$ supergravity. More generally, if the internal space is an outer product of two or more Einstein spaces, λ could be different for each space, e. g. if the internal space contains an S^1, as in the dimensional reduction of $N = 1$, $d = 11$ down to $N = 2$, $d = 10$ [*3*].

In what follows we shall examine the case where the internal space is a product space $CP^2 \times M$, where M is a compact manifold with a Riemannian metric. The Fubini-Study metric on CP^2 is given by [*7*]

$$g_{CP^2} = \sum_{a=1}^{4} e^a \otimes e^a$$

([1]) For the particular case of the $N = 2$, $d = 10$ chiral supergravity theory, no such solution has been constructed. However, it seems reasonable to suppose that some solutions of this theory may involve Einstein metrics, of the form (1).

where e^a are given by

$$e^1 = \frac{r\sigma^1}{(1+r^2)^{\frac{1}{2}}} \qquad e^2 = \frac{r\sigma^2}{(1+r^2)^{\frac{1}{2}}}$$
$$e^3 = \frac{r\sigma^3}{(1+r^2)} \qquad e^2 = \frac{dr}{(1+r^2)} \tag{2}$$

where $\sigma^i (i=1,2,3)$ are left invariant one forms on S^3 and obey $d\sigma^i = \varepsilon^{ijk}\sigma^j \wedge \sigma^k$. Explicitly

$$\sigma^1 = \frac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi)$$
$$\sigma^2 = -\frac{1}{2}(\cos\psi d\theta + \sin\theta\sin\psi d\phi) \tag{3}$$
$$\sigma^3 = \frac{1}{2}(\cos\theta d\phi + d\psi)$$

where $0 \leqslant \theta < \pi$, $0 \leqslant \phi < 2\pi$, $0 \leqslant \psi < 4\pi$, $0 \leqslant r < \infty$. Of course, CP^2 is not covered by this single co-ordinate patch, one must choose different co-ordinates to cover the points $r = \infty$.

The orientation is given by $\overset{*}{_4}1 = e^{1243}$ ($\overset{*}{_4}$ is the Hodge operator on CP^2).

The unique, torsion free, metric compatible connection obtained from (2) is

$$\omega_{14} = \frac{1}{r}e^1 \qquad \omega_{23} = \frac{1}{r}e^1$$
$$\omega_{24} = \frac{1}{r}e^2 \qquad \omega_{31} = \frac{1}{r}e^2 \tag{4}$$
$$\omega_{34} = \frac{1-r^2}{r}e^3 \qquad \omega_{12} = \frac{1+2r^2}{r}e^3$$

The curvature 2-forms, $R_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b$, are

$$R_{14} = e^{14} + e^{23} \qquad R_{23} = e^{14} + e^{23}$$
$$R_{24} = e^{24} + e^{31} \qquad R_{31} = e^{24} + e^{31}$$
$$R_{34} = 4e^{34} - 2e^{12} \qquad R_{12} = -2e^{34} + 4e^{12}$$

and satisfy

$$R_{ab} \wedge \overset{*}{_4}e^{abc} = 12\overset{*}{_4}e^c$$
$$\Rightarrow R_{ab} \wedge \overset{*}{_4}e^{ab} = 24\overset{*}{_4}1$$

One can show that chiral spin $\frac{1}{2}$ fields cannot be put on CP^2 by using the Atiyah-Singer index theorem [7] to calculate the difference between the number of right handed (ν_+) and the number of left handed (ν_-) massless spinors with the above curvature,

$$\nu_+ - \nu_- = -\frac{1}{24}\frac{1}{8\pi^2}\int_{CP^2} R_{ab} \wedge R^{ab}$$
$$= -\frac{1}{8}$$

Since this is not an integer, one cannot put chiral spinors on CP^2. However, if the spinors interact with a gauge field by minimal coupling then the Dirac operator is modified and the index theorem, for complex bundles, yields:

$$\nu_+ - \nu_- = -\frac{P}{8} + \frac{1}{8\pi^2}\int_{CP^2} \mathrm{Tr}(G \wedge G) \tag{5}$$

where G is the field strength 2-form for the gauge field, which may be non-abelian, hence the trace. P is the dimension of the representation space on which the group acts. Thus, by suitable choice of gauge-group and G one can hope to adjust the R.H.S. of equation (5) so as to make it an integer, and indeed this is what a spin^c structure does.

Suitable gauge fields, G, can be constructed from the Levi-Civita connection (ref. [9]). The connection is an SO(4) Lie Algebra valued one form and so can be decomposed into two SU(2) Lie Algebra valued one forms

$$\begin{aligned} B^1_\pm &= \frac{1}{2}(\omega^{41} \pm \omega^{23}) \\ B^2_\pm &= \frac{1}{2}(\omega^{42} \pm \omega^{31}) \\ B^3_\pm &= \frac{1}{2}(\omega^{43} \pm \omega^{12}) \end{aligned} \tag{6}$$

The resulting field strengths $G^i_\pm = dB^i_\pm + \varepsilon^{ijk}B^j_\pm \wedge B^k_\pm$ are (anti) self-dual. When this procedure is applied to the standard metric and connection on S^4, one obtains the BPST instanton [10].

Take the positive sign in equations (6). Then using (4)

$$B^1 = B^2 = 0, \qquad B^3 = \frac{3}{2}re^3 \equiv B \tag{7}$$

and the field is that of U(1) rather than SU(2). The field strength is

$$G = 3(e^{43} + e^{12}) = {}^*_4G \tag{8}$$

It is self-dual and is, in fact the Kahler 2-form on CP^2. To see how this field can be introduced into equations (1), we consider the case where the manifold, M, is the direct product of S^1 with an Einstein space. Then equations (1) split into three cases ($a, b, c = 1, \ldots, 4$ label CP^2; e^5 is the volume element of S^1; $\mu = 1, \ldots, d-9$ label the rest of M)

$$\mathbb{R}_{AB} \wedge \divideontimes e^{ABc} = \lambda_1 \divideontimes e^c \tag{9}$$

$$\mathbb{R}_{AB} \wedge \divideontimes e^{AB5} = \lambda_2 \divideontimes e^5 \tag{10}$$

$$\mathbb{R}_{AB} \wedge \divideontimes e^{AB\mu} = \lambda_3 \divideontimes e^\mu \tag{11}$$

Now consider S^1 over CP^2 in the light of a standard Kaluza-Klein theory and use the Killing symmetry of the metric on S^1 to construct a U(1) field over CP^2.

Let $e^5 = dx^5$, $0 \leqslant x^5 < 2\pi$, and consider a new, five dimensional metric

$$g_{(5)} = \sum_{a=1}^{4} e^a \otimes e^a + (e^5 + B) \otimes (e^5 + B)$$

where B is any one form on CP^2 (not necessarily that of equation (7)), with no x^5 dependence. This is the standard Kaluza-Klein ansatz, except that it is being applied purely to the internal space.

Of course the metric $g_{(5)}$ will not be a metric for $CP^2 \times S^1$, in general, but for some other space with a different global topology.

The reduction of equations (9) and (10), with e^5 replaced by $(e^5 + B)$ yields

$$d\overset{*}{_4}G = 0 \tag{12}$$

$$G \wedge \overset{*}{_4}G = \frac{2}{3}(2\tilde{\lambda}_1 - \tilde{\lambda}_2)\overset{*}{_4}1 \tag{13}$$

$$R_{ab} \wedge \overset{*}{_4}e^{abc} = \tilde{\lambda}_1 \overset{*}{_4}e^c + \frac{1}{2}[G \wedge i^c\overset{*}{_4}G - i^cG \wedge \overset{*}{_4}G] \tag{14}$$

where $\tilde{\lambda}_i = \lambda_i - \mathscr{R}$ with $\mathscr{R}$ the curvature scalar for the remaining dimensions, $G = dB$ and i^c is the interior derivative, sending p-forms to $(p-1)$ forms and satisfying $i^c(e^a) = \delta^{ca}$, $i^c(e^{ab}) = \delta^{ca}e^b - \delta^{cb}e^a$, $i^c\overset{*}{_4}(e^a) = \overset{*}{_4}e^{ac}$.

If we now use the one-form derived from the connection on CP^2, equation (7), then (12) is automatically satisfied since G is self-dual. (13) is satisfied if $\tilde{\lambda}_2 = 2\tilde{\lambda}_1 - 27$ and (14) reduces to (9) with $\tilde{\lambda}_1 = 12$. (11) will be satisfied by suitably adjusting the curvature scalar of the remaining Einstein space, $\mathscr{R}$.

Substituting equation (8) into (5) with P = 1, the dimension of S^1 (the volume of CP^2 is $\pi^2/2$)

$$\nu_+ - \nu_- = -\frac{1}{8} + \frac{9}{8} = 1$$

Thus this construction enables us to put spinors on CP^2 in a consistent fashion, with one more right-handed than left-handed spinor. In general the R.H.S. could be made into any positive integer, simply by rescaling B. This result is no more than a combination of the standard, 5 dimensional Kaluza-Klein ansatz, applied to the internal manifold, and the spin^c structures of ref. [*6*]. Its attractive feature is that it avoids the ad hoc introduction of an extra U(1) field.

Let us take the negative sign in equations (6). Then

$$B^1 = -\frac{1}{r}e^1, \qquad B^2 = -\frac{1}{r}e^2, \qquad B^3 = -\frac{(2+r^2)}{2r}e^3 \tag{15}$$

giving

$$G^1 = e^{41} - e^{23}, \qquad G^2 = e^{42} - e^{31}, \qquad G^3 = e^{43} - e^{12} \tag{16}$$

G is anti-self dual.

This field can be introduced in a consistent way if M is the direct product of S^2 with an Einstein space. Again, equations (1) split into three cases ($a, b, c = 1, \ldots, 4$ label CP^2, $m, n = 5, 6$ label S^2, μ labels the rest of M)

$$\mathbb{R}_{AB} \wedge ※ e^{ABc} = \lambda_1 ※ e^c \tag{17}$$

$$\mathbb{R}_{AB} \wedge ※ e^{ABm} = \lambda_2 ※ e^m \tag{18}$$

$$\mathbb{R}_{AB} \wedge ※ e^{AB\mu} = \lambda_3 ※ e^\mu \tag{19}$$

Now use the Killing symmetries of the standard metric on S^2 to construct an SU(2) gauge field over CP^2 in the manner of a non-abelian Kaluza-Klein theory [*11*]-[*14*].

Let $e^5 = d\alpha$, $e^6 = \sin\alpha\, d\beta$ $0 \leqslant \alpha < \pi$, $0 \leqslant \beta < 2\pi$ be a basis of orthonormal one forms on S^2, and consider a new, six dimensional metric

$$g_{(6)} = \sum_{a=1}^{4} e^a \otimes e^a + \sum_{m=5}^{6}\left[\left[e^m - 2\sum_{j=1}^{3} e^m(\tilde{K}_j)B^j\right] \otimes \left[e^m - 2\sum_{k=1}^{3} e^m(\tilde{K}_k)B^k\right]\right]$$

Here $\tilde{K}_j$ are the three Killing vectors of the metric on S^2 and the numbers $e^m(\tilde{K}_j)$ are the components of their metric dual one-forms in the e^m basis. B^j are SU(2) Lie Algebra valued one forms on CP^2, not necessarily those of equation (15), which are independent of α and β.

The Killing one forms K_j, metric dual to the Killing vectors, can be chosen as

$$\begin{aligned} K_1 &= \sin\beta e^5 + \cos\beta \cos\alpha e^6 \\ K_2 &= \cos\beta e^5 - \sin\beta \cos\alpha e^6 \\ K_3 &= \sin\alpha e^6 \end{aligned}$$

and satisfy

$$dK_i = -\varepsilon_{ijk} K_j \wedge K_k$$

It is convenient also to define the quantities

$$h_{ij} = \sum_{m=5}^{6} e^m(\tilde{K}_i) e^m(\tilde{K}_j)$$

which will be used for the contraction of group indices.

Note that

$$\delta^{jk}h_{jk} = 2 \qquad \text{and} \qquad \delta^{jk}K_j \wedge i^m \overset{*}{_2}K_k = - \overset{*}{_2}e^m \tag{20}$$

where $\overset{*}{_2}$ is the Hodge dual on $S^2(\overset{*}{_2}1 = e^{56})$.

The reduction of equations (17) and (18), with e^m replaced by

$$e^m - 2\sum_{j=1}^{3} e^m(\tilde{K}_j)B^j$$

yields

$$R_{ab} \wedge \overset{*}{_4}e^{abc} = (\lambda_1 - R_2)\overset{*}{_4}e^c + 2[G^j \wedge i^c \overset{*}{_4}G^k - i^c G^j \wedge \overset{*}{_4}G^k]h_{jk} \tag{21}$$

$$(G^j \wedge \overset{*}{_4}G^k)[3h_{jk}\overset{*}{_2}e^m + K_j \wedge i^m(\overset{*}{_2}K_k) + K_k \wedge i^m(\overset{*}{_2}K_j)] = \frac{1}{2}(R_4 - \lambda_2)\overset{*}{_4}1 \wedge \overset{*}{_2}e^m \tag{22}$$

$$D(\overset{*}{_4}G^j) = 0 \tag{23}$$

$R_2 = 2$ is the curvature scalar for S^2, $R_4 = 24$ is the curvature scalar for CP^2, and $D(\overset{*}{_4}G^j) = d(\overset{*}{_4}G^j) + 2\varepsilon^{jkl}B^k \wedge \overset{*}{_4}G^l$. If we now substitute (15) and (16), then (23) is automatically satisfied since G^i is anti-self dual. Since $G^j \wedge \overset{*}{_4}G^k = 2\overset{*}{_4}1\delta^{jk}$ then (22) is satisfied (using 20) with $\lambda_2 = 8$ and (21) reduces to (17) which is satisfied provided $\lambda_1 = 14$. Finally (19) can be satisfied by adjusting λ_3 (for $d = 11$, m has only one value as the remaining manifold must be S^1). This requires that $\lambda_1\lambda_2$ and λ_3 all be positive.

Now, putting (16) into (5) with $G = \tau_j G^j$ where τ_j are the generators of SU(2), normalised so that $\mathrm{Tr}(\tau_j\tau_k) = 2\delta_{jk}$, and $P = 2$, the dimension of S^2, one obtains

$$\nu_+ - \nu_- = -\frac{1}{4} - \frac{3}{4} = -1$$

Thus we now have one more left-handed spinor than right-handed ones. These spinors are SU(2) doublets, since $P = 2$.

The term $-\frac{3}{4}$ deserves some comment. It is, in fact, the topological charge of the SU(2) « instanton » field over CP^2 and being non-integral indicates a discontinuity in the fields. The construction (6) only gives us an ansatz for the Algebra. If we had used SO(3) as the group rather than SU(2), then the topological charge would have been -3 instead of $-\frac{3}{4}$. One factor of two arises because SU(2) is the double covering of SO(3) and another because the relevant quantity is the first Pontrjagin number for real bundles, as opposed to the second Chern number for complex bundles and, when a complex bundle is regarded as real by forgetting the complex structure, these differ by a factor of two (2).

Fractionally charged instanton configurations have been studied extensively in the literature, for a review see ref. [*15*].

CONCLUSION

It has been shown that it is possible to consistently define spinors on CP^2 within the context of supergravity theories, without the introduction of any extra fields. This is achieved by minimally coupling the spinors to gauge fields which arise as extra components of the metric. If the internal manifold has the topology of $CP^2 \times M$ before the introduction of the extra components, then the spectrum of the spinors depends on the topology of M, being SU(2) doublets if M contains an S^2 and U(1) singlets if M contain an S^1. If M is $S^2 \times S^1$, then a combination of both singlets and doublets could be achieved, there being more left-handed than right-handed doublets, and more right-handed than left-handed singlets. The overall particle spectrum of d-dimensional spinors will, however, depend on the spinor fields on the remaining $d-4$ dimensions apart from CP^2, in particular the chirality content of the total particle spectrum could well be different from that of the spinors on CP^2 only.

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