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On the invariance properties and the hamiltonian of the unified affine electromagnetism and gravitation theories

by

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ABSTRACT. — The invariance properties of affine unified theories of gravitation are studied. The hamiltonian for the unified theory of electromagnetism and gravitation proposed by Ferraris and Kijowski is derived. The energy formulas for different formulations of the theory of interacting electromagnetic and gravitational fields are compared.

RÉSUMÉ. — Une étude des propriétés d'invariance des théories affines unitaires de la gravitation a été faite. Les résultats de cette étude permettent de dériver l'hamiltonien de la théorie unitaire d'électromagnétisme et de gravitation proposée par Ferraris et Kijowski. Les formules d'énergie pour différentes formulations de la théorie de gravitation et d'électromagnétisme sont brièvement comparées.

1. INTRODUCTION

A few years ago, Ferraris and Kijowski [*3*] proposed a unified theory of electromagnetism and gravitation, within the framework of

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purely affine theories of gravitation [*5*]. The unification is real, in the sense that:

a) the proposed theory is locally equivalent to the standard one,

b) it does not require supplementary space-time dimensions, as in the Kaluza-Klein unification [*4*] [*7*], and

c) it does not require any ad hoc assumptions about the algebraic structure of the Ricci tensor, as in the « already unified » approach [*8*] [*10*].

Point *a*) may be considered as a shortcoming of the proposed theory, because it suggests that the theory does not predict any new effects—the field equations are exactly the same as in the standard formulation. However, the transformation properties of the electromagnetic potential of the unified theory are different from the standard ones. Therefore, new predictions of the unified theory may be found by inspecting these physical quantities, which depend upon the transformation properties of the electromagnetic potential.

A symplectic analysis of field theories developed by Kijowski and Tulczyjew [*6*] leads naturally to a geometrical definition of the hamiltonian density for every Lagrangian theory. Kijowski and Tulczyjew proposed to interpret this hamiltonian density as the energy density of the theory, obtaining in this way a generally covariant expression for the energy density in general relativity [*5*] [*6*] (Kijowski and Tulczyjew use, by fiat, the name « energy density » to what is usually called « hamiltonian density », reserving the name « hamiltonian density » to a different generating function. To cut short any semantic confusion, the terms « hamiltonian density » and « energy density » are used here in their usual—and vague, as far as « energy density » is concerned—physicists meaning). The transformation properties of the fields enter explicitly in the formulas for the hamiltonian density derived in [*6*] therefore, if one adopts the point of view that the energy density is equal to the hamiltonian density, the formula for the energy of the unified theory differs from the standard one. This formula is derived in Section 4 of this paper. Certain identities, deriving from the invariance properties of the theory, are needed in order to show that the total energy of the theory is a boundary integral. These identities are derived in Sections 2 and 3, for a large class of purely affine theories, and the boundary hamiltonians for these theories are derived. It is interesting, that the final expressions for the hamiltonians do not make explicit reference to the Lagrangians. In Section 5, the formulas for the energy for different formulations of the theory of interacting gravitational and electromagnetic fields are compared.

2. THE FIELD EQUATIONS AND THE HAMILTONIAN

The most general coordinate invariant first order Lagrangian density for a GL(4, R) connection field theory must be of the form (1):

$$l(\Gamma^\lambda_{\mu\nu}, \Gamma^\lambda_{\mu\nu,\rho}) = l(Q^\lambda_{\mu\nu}, R^\lambda_{\mu\nu\rho}, Q^\lambda_{\mu\nu;\rho}). \tag{2.1}$$

We will restrict ourselves to theories, with a Lagrangian density depending only upon the curvature of $\Gamma^\alpha_{\beta\gamma}$

$$l = l(R^\lambda{}_{\mu\nu\rho}). \tag{2.2}$$

The momentum canonically conjugate to $\Gamma^\alpha_{\beta\gamma}$

$$\Pi_\lambda{}^{\mu\nu\rho} = \partial l/\partial\Gamma^\lambda_{\rho\mu,\nu} = 2\partial l/\partial R^\lambda{}_{\mu\nu\rho}, \tag{2.3}$$

is a tensor density antisymmetric in the last two indices. Using the identities

$$dl = \partial l/\partial R^\lambda{}_{\mu\nu\rho} \times dR^\lambda{}_{\mu\nu\rho} = \Pi_\lambda{}^{\nu\rho\mu} d\Gamma^\lambda_{\mu\nu,\rho} + (\Pi_\lambda{}^{\sigma\mu\rho}\Gamma^\nu_{\rho\sigma} + \Pi_\rho{}^{\nu\sigma\mu}\Gamma^\rho_{\sigma\lambda}) \times d\Gamma^\lambda_{\mu\nu} = \partial l/\partial\Gamma^\lambda_{\mu\nu,\rho} \times d\Gamma^\lambda_{\mu\nu,\rho} + \partial l/\partial\Gamma^\lambda_{\mu\nu} \times d\Gamma^\lambda_{\mu\nu},$$

the field equations

$$\partial_\rho(\partial l/\partial\Gamma^\lambda_{\mu\nu,\rho}) = \partial l/\partial\Gamma^\lambda_{\mu\nu} \tag{2.4}$$

can be written in the following form:

$$\Pi_\lambda{}^{\mu\nu\alpha}{}_{;\alpha} = Q^\alpha_{\alpha\beta}\Pi_\lambda{}^{\mu\nu\beta} + Q^\nu_{\alpha\beta}\Pi_\lambda{}^{\mu\beta\alpha}/2. \tag{2.5}$$

The hamiltonian vector density can be obtained from the following formula [6]:

$$E^\alpha = \Pi_\lambda{}^{\nu\alpha\mu} L_X \Gamma^\lambda_{\mu\nu} - X^\alpha l \tag{2.6}$$

From (A8) and (2.5) one obtains

$$E^\alpha = t^\alpha{}_\beta X^\beta + \partial_\beta E^{\beta\alpha}, \tag{2.7}$$

where

$$t^\alpha{}_\beta = \Pi_\rho{}^{\mu\nu\alpha} R^\rho{}_{\mu\nu\beta} - \delta^\alpha_\beta l, \tag{2.9}$$

$$E^{\beta\alpha} = \Pi_\rho{}^{\mu\alpha\beta}(X^\rho{}_{;\mu} + Q^\rho_{\sigma\mu}X^\sigma), \tag{2.9}$$

and $E^{\alpha\beta}$ is an antisymmetric tensor density. Let us show, that $t^\alpha{}_\beta$ vanishes when the field equations are satisfied. The simplest proof of this fact relies upon the following proposition:

(1) Since the standard reference books differ by conventions on signs, factors of 1/2 and positions of indices, we have found it convenient to present in the Appendix the identities satisfied by the curvature tensor, in our conventions. The conventions used in this paper are also listed.

PROPOSITION 1. — If the field equations are satisfied then, for all X^α:

$$E^\alpha{}_{,\alpha} = E^\alpha{}_{;\alpha} - Q^\alpha_{\alpha\beta}E^\beta = 0\,. \tag{2.10}$$

Proof. — Choose coordinates in which $X = \partial/\partial x^0$. Then $L_X\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu,0}$, and from (2.3) combined with (2.6) one has

$$E^\alpha{}_{,\alpha} = (\partial l/\partial\Gamma^\lambda_{\mu\nu,\alpha})_{,\alpha}\Gamma^\lambda_{\mu\nu,0} + \partial l/\partial\Gamma^\lambda_{\mu\nu,\alpha}\Gamma^\lambda_{\mu\nu,\alpha 0} - l_{,0} = 0\,.$$

in virtue of the field equations (2.4), and of the invariance of the Lagrangian (the invariance guarantees, that the only dependence of l upon x^0 is through the fields $\Gamma^\lambda_{\mu\nu}$ and their first derivatives). Since equation (2.10) has an invariant character, it holds in every coordinate system, for every vector X^α (this straightforward proof is due to J. Kijowski).

Proposition 1 implies

PROPOSITION 2.

$$l\delta^\mu_\nu = \Pi_\alpha{}^{\beta\gamma\mu}R^\alpha{}_{\beta\gamma\nu}\,. \tag{2.11}$$

Proof. — It follows from (2.7) and (2.10) that

$$0 = t^\lambda{}_{\sigma,\lambda}X^\sigma + t^\lambda{}_\sigma X^\sigma{}_{,\lambda}\,. \tag{2.12}$$

Since equation (2.12) is true for any vector field X^α, it follows that $t^\lambda{}_\sigma = 0$, which proves equation (2.11).

Equations (2.11) and (2.7) show that the hamiltonian of the theory is a boundary integral:

$$H = \int_\Sigma E^\lambda\eta_\lambda = \int_\Sigma \partial_\nu E^{\nu\lambda}\eta_\lambda = \frac{1}{2}\int_{\partial\Sigma} E^{\nu\lambda}\eta_{\nu\lambda} \tag{2.13}$$

3. THE INVARIANCE PROPERTIES OF THE THEORY

Formula (2.11), derived using the « adapted coordinates trick », holds for any invariant Lagrangian. In its derivation, the invariance has been used in a rather involved manner, it is therefore interesting to relate (2.11) to the invariance properties of the theory by a more direct method. The Lagrangian density is said to be coordinate invariant if, under a (passive) change of coordinates

$$x^\mu \rightarrow x^{\mu'} = f^{\mu'}(x^\mu)$$

the following identity holds

$$A^f \equiv l(R^{\lambda'}{}_{\mu'\nu'\alpha'}(f^{\mu'}(x^\alpha)))dx^{0'} \wedge \ldots \wedge dx^{3'} = l(R^\lambda{}_{\mu\nu\alpha})dx^0 \wedge \ldots \wedge dx^3, \tag{3.1}$$

Let $f^\mu{}_\tau(x^\alpha)$ be a local one-parameter family of diffeomorphisms, generated by a vector field X. Formula (3.1) implies

$$dA^{f_\tau}/d\tau = 0\,, \tag{3.2}$$

because the right hand side of (3.1) does not depend upon τ. Eq. (3.2) implies, at $\tau = 0$,

$$X^{\mu}{}_{,\mu} l = \frac{1}{2} \Pi_{\lambda}{}^{\mu\nu\sigma} \{ R^{\lambda}{}_{\rho\nu\sigma} X^{\rho}{}_{,\mu} + R^{\lambda}{}_{\mu\rho\sigma} X^{\rho}{}_{,\nu} + R^{\lambda}{}_{\mu\nu\rho} X^{\rho}{}_{,\sigma} - R^{\rho}{}_{\mu\nu\sigma} X^{\lambda}{}_{,\rho} \} . \quad (3.3)$$

If X is taken to be of the following form:

$$X^{\mu} = \delta^{\mu}_{\beta} x^{\alpha} ,$$

from (3.3) one deduces

$$2l\delta^{\alpha}_{\beta} = 2\Pi_{\lambda}{}^{\mu\nu\alpha} R^{\lambda}{}_{\mu\nu\beta} + \Pi_{\lambda}{}^{\alpha\mu\nu} R^{\lambda}{}_{\beta\mu\nu} - \Pi_{\beta}{}^{\mu\nu\lambda} R^{\alpha}{}_{\mu\nu\lambda} . \quad (3.4)$$

A contraction over α and β yields

$$l = \Pi_{\alpha}{}^{\beta\gamma\delta} R^{\alpha}{}_{\beta\gamma\delta}/4 = \frac{1}{2} \partial l / \partial R^{\alpha}{}_{\beta\gamma\delta} R^{\alpha}{}_{\beta\gamma\delta} , \quad (3.5)$$

which shows that an invariant Lagrangian is a homogeneous function, of degree 2, of the components of the curvature tensor.

The identities (3.4) can also be derived from the requirement, that l depends upon the coordinates through the fields only. Therefore, by dragging the Lagrangian density along a vector field X (active change of coordinates), we must have

$$L_X l = \Pi_{\alpha}{}^{\beta\gamma\delta} L_X R^{\alpha}{}_{\beta\gamma\delta} . \quad (3.6)$$

Writing out explicitly the Lie derivative of $R^{\alpha}{}_{\beta\gamma\delta}$ in (3.6) one recovers formula (3.3), and the identities (3.4) follow.

To prove (2.11), one has to use the field equations, the Ricci identity and the Jacobi (cyclic) identity for the curvature tensor. The contracted Ricci identity for the tensor density $\Pi_{\alpha}{}^{\mu\nu\rho}$ reads (see the Appendix)

$$2\Pi_{\alpha}{}^{\beta\gamma\delta}{}_{;\delta\gamma} = 2\Pi_{\alpha}{}^{\beta\gamma\delta}{}_{;[\delta\gamma]} = R^{\beta}{}_{\sigma\gamma\delta} \Pi_{\alpha}{}^{\sigma\gamma\delta} - R^{\sigma}{}_{\alpha\gamma\delta} \Pi_{\sigma}{}^{\beta\gamma\delta} + 3R^{\sigma}{}_{[\delta\sigma\gamma]} \Pi_{\alpha}{}^{\beta\delta\gamma} + Q^{\sigma}_{\gamma\delta} \Pi_{\alpha}{}^{\beta\delta\gamma}{}_{;\sigma} \quad (3.7)$$

On the other hand, the left hand side of (3.7) can be calculated from (2.5):

$$2\Pi_{\alpha}{}^{\beta\gamma\delta}{}_{;\delta\gamma} = \{ 3Q^{\sigma}_{[\sigma\delta;\gamma]} + Q^{\sigma}_{\sigma\rho} Q^{\rho}_{\gamma\delta} \} \Pi_{\alpha}{}^{\beta\gamma\delta} + Q^{\sigma}_{\delta\gamma} \Pi_{\alpha}{}^{\beta\gamma\delta}{}_{;\sigma} \quad (3.8)$$

Equations (3.7), (3.8), the Jacobi identity (A6) and the antisymmetry of $\Pi_{\alpha}{}^{\beta\gamma\delta}$ in the last two indices imply

$$R^{\alpha}{}_{\mu\nu\rho} \Pi_{\beta}{}^{\mu\nu\rho} = R^{\mu}{}_{\beta\nu\rho} \Pi_{\mu}{}^{\alpha\nu\rho} \quad (3.9)$$

Equations (3.4) and (3.9) give therefore an alternative proof of (2.11), which in turn implies (2.13). It must be emphasized, that the identities (3.4) and (3.5) have a completely different character than equation (2.11). Equations (3.4) are an algebraic consequence of the invariance of the theory, and equation (2.11) holds only for fields satisfying the field equations.

4. THE UNIFIED THEORY OF ELECTROMAGNETISM AND GRAVITATION

As was shown in ref. [3] or, by other methods, in ref. [1], any (possibly non-linear) theory of interacting gravitational and electromagnetic fields may be formulated as a unified GL(4, R) gauge theory, if one restricts the Lagrangian to depend only upon $K_{\mu\nu}$ and $F_{\mu\nu}$:

$$l = l(K_{\mu\nu}, F_{\mu\nu}) \tag{4.1}$$

where

$$K_{\mu\nu} = - R^{\alpha}{}_{(\nu\mu)\alpha}, \qquad F_{\mu\nu} = R^{\alpha}{}_{\alpha\mu\nu}. \tag{4.2}$$

It follows from (4.1), that the canonical momenta $\Pi_{\lambda}{}^{\mu\nu\alpha}$ take the following form:

$$\Pi_{\lambda}{}^{\mu\nu\alpha} = 2\Pi^{\mu[\alpha}\delta^{\nu]}_{\lambda} + \hat{F}^{\alpha\nu}\delta^{\mu}_{\lambda}, \tag{4.3}$$

where

$$\Pi^{\mu\nu} = \partial l/\partial K_{\mu\nu} \tag{4.4}$$

is interpreted as the contravariant metric density [3]

$$\Pi^{\mu\nu} = \sqrt{- \det g_{\alpha\beta}}\, g^{\mu\nu}/16\pi, \tag{4.5}$$

and $\hat{F}^{\mu\nu}$ is the usual electromagnetic induction density field:

$$\hat{F}^{\mu\nu} = - 2\partial l/\partial F_{\mu\nu} = \sqrt{- \det g_{\alpha\beta}}\, g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}/4\pi. \tag{4.6}$$

It can be shown [1] [3], that the field equations (2.5) (which are equivalent to the Einstein-Maxwell equations) imply that $\Gamma^{\lambda}_{\mu\nu}$ takes the following form (the convention on the position of indices of the connection coefficients differs from the convention used in [1] or [3], see the Appendix):

$$\Gamma^{\lambda}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} + Y_{\mu}\delta^{\lambda}_{\nu}/4,$$
$$\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}, \qquad Y_{\mu} = A_{\mu} - \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\lambda}, \qquad A_{\mu} = \Gamma^{\lambda}_{\mu\lambda}, \tag{4.7}$$

where $\left\{ {\lambda \atop \mu\nu} \right\}$ is the Christoffel symbol of the metric $g_{\mu\nu}$, and A_{μ} has the interpretation of the electromagnetic potential. The Lagrangian for the linear electrodynamics is [1] [3]

$$l = \sqrt{- \det K_{\alpha\beta}}\, K^{\alpha\mu}K^{\beta\nu}F_{\mu\beta}F_{\nu\alpha}/16\pi, \tag{4.8}$$

and $K^{\alpha\beta}$ is defined as the inverse tensor to $K_{\mu\nu}$. From (4.3), (4.7) and (2.9) one obtains

$$E^{\nu\lambda} = 2\Pi^{\mu[\nu}X^{\lambda]}{}_{|\mu} + \hat{F}^{\nu\lambda}(\partial_{\alpha}X^{\alpha} + A_{\alpha}X^{\alpha}), \tag{4.9}$$

where | denotes covariant differentiation with respect to the metric connection. From the identity $dl = (\Pi_\lambda{}^{\nu\alpha\mu} d\Gamma^\lambda_{\mu\nu})_{,\alpha}$, which holds when the field equations are satisfied, one easily obtains

$$-d(lX^\beta) = 2(\Pi_\lambda{}^{\nu\mu[\alpha}X^{\beta]}d\Gamma^\lambda_{\mu\nu})_{,\alpha} + \Pi_\lambda{}^{\nu\mu\beta}d\mathrm{L}_X\Gamma^\lambda_{\mu\nu} + \mathrm{L}_X\Pi_\lambda{}^{\nu\mu\beta}d\Gamma^\lambda_{\mu\nu}, \quad (4.10)$$

and formula (2.13) leads to

$$dH = d\int_\Sigma (\Pi_\lambda{}^{\nu\beta\mu}\mathrm{L}_X\Gamma^\lambda_{\mu\nu} - X^\beta l)\eta_\beta = \int_\Sigma \{ \mathrm{L}_X\Pi_\lambda{}^{\nu\mu\alpha}d\Gamma^\lambda_{\mu\nu} - \mathrm{L}_X\Gamma^\lambda_{\mu\nu}d\Pi_\lambda{}^{\nu\mu\alpha} \}\,\eta_\alpha$$
$$+ \int_{\partial\Sigma} \Pi_\lambda{}^{\nu\mu[\alpha}X^{\beta]}d\Gamma^\lambda_{\mu\nu}\eta_{\alpha\beta} \quad (4.11)$$

From (4.3), (4.7) and (4.11) it follows, that

$$dH = \int_\Sigma \{ \mathrm{L}_X A^\alpha_{\mu\nu}d\Pi^{\mu\nu} - \mathrm{L}_X\Pi^{\mu\nu}dA^\alpha_{\mu\nu} + \mathrm{L}_X A_\nu d\hat{F}^{\nu\alpha} - \mathrm{L}_X\hat{F}^{\nu\alpha}dA_\nu \}\,\eta_\alpha$$
$$- \int_{\partial\Sigma} (\Pi^{\mu\nu}dA^{[\alpha}_{\mu\nu} - \hat{F}^{\nu[\alpha}dA_\nu)X^{\beta]}\eta_{\alpha\beta}, \quad (4.12)$$

where

$$A^\alpha_{\mu\nu} = \mathring{\Gamma}^\alpha_{\mu\nu} - \delta^\alpha_{(\mu}\mathring{\Gamma}^\beta_{\nu)\beta} \quad (4.13)$$

5. COMPARISON WITH STANDARD FORMULATIONS

There exist up to now at least three different formulations of the theory of interacting electromagnetic and gravitational fields. Depending upon the approach, the electromagnetic field is described as:

(F) a covector field on space-time,
(C) a connection on a U(1) principal bundle [9], and
(UT) a connection form on a scalar density R^+ principal bundle [3].

The formulation (C) is actually widely accepted.

The field equations are, of course, the same for all three theories, but due to different transformation properties of the electromagnetic potentials, the expressions for the energy of the theory turn out to be different. This is due to the fact, that in the expression for the hamiltonian density for any lagrangian theory

$$E^\lambda(X) = \Pi_A{}^\lambda \mathrm{L}_X\varphi^A - X^\lambda l, \qquad \Pi_A{}^\lambda = \partial l/\partial\varphi^A{}_{,\lambda} \quad (5.1)$$

the Lie derivative of the fields φ^A appears. The Lie derivative is defined naturally in the (F) and (UT) descriptions of electromagnetism. In the (C) case, the electromagnetic field is described by a one-form on the bundle space, where no natural lift of the action of the diffeomorphisms group exists. Following [6], we will define the Lie derivative of a connection form ω as being the projection on space-time of the Lie derivative (in the bundle space) of ω with respect to the horizontal lift of the vector field X. This leads to the following Lie derivatives:

$$\text{(F)}\quad L_X A_\mu = A_{\mu,\nu}X^\nu + X^\nu{}_{,\mu}A_\nu .$$
$$\text{(C)}\quad L_X A_\mu = F_{\nu\mu}X^\nu , \tag{5.2}$$
$$\text{(UT)}\quad L_X A_\mu = A_{\mu,\nu}X^\nu + X^\nu{}_{,\mu}A_\nu + X^\nu{}_{,\nu\mu} \tag{5.3}$$

The appearence of second derivatives of X in (5.3) is related to the following transformation properties of the electromagnetic potential under coordinate transformations:

$$A_{\mu'} = A_\mu \partial x^\mu/\partial x^{\mu'} + \partial_{\mu'}\lambda \qquad \lambda = \ln|\det(\partial x^\alpha/\partial x^{\alpha'})| \tag{5.4}$$

(a change of coordinates induces a gauge transformation of the potential). To obtain the energy of the theory in cases (F) and (C), one can adopt to the conventions used in this paper the formulas derived by Kijowski [5], or simply use formulas (4.3) and (4.7) with $\hat{F}^{\alpha\nu}$ and Y_μ put to zero by hand (it can easily be checked, using the methods of [1], that such an approach is justified). One obtains the following expressions:

$$\text{(F)}\quad E^\alpha(X) = \left(\frac{1}{8\pi}\sqrt{-\det g_{\alpha\beta}}R^\alpha{}_\sigma + \hat{F}^{\alpha\mu}F_{\mu\sigma} - \delta^\alpha_\sigma l\right)X^\sigma - \hat{F}^{\mu\alpha}{}_{,\mu}X^\nu A_\nu + \partial_\mu E^{\mu\alpha}_F$$
$$\text{(C)}\quad E^\alpha(X) = \left(\frac{1}{8\pi}\sqrt{-\det g_{\alpha\beta}}R^\alpha{}_\sigma + \hat{F}^{\alpha\mu}F_{\mu\sigma} - \delta^\alpha_\sigma l\right)X^\sigma + \partial_\mu E^{\mu\alpha}_C \tag{5.5}$$

with

$$E^{\alpha\beta}_F = \sqrt{-\det g_{\alpha\beta}}\{-X^{[\alpha|\beta]} + 2F^{\alpha\beta}A_\gamma X^\gamma\}/8\pi ,$$
$$E^{\alpha\beta}_C = -\sqrt{-\det g_{\alpha\beta}}X^{[\alpha|\beta]}/8\pi . \tag{5.6}$$

Equations (5.6) have to be compared with

$$E^{\alpha\beta}_{UT} = \sqrt{-\det g_{\alpha\beta}}\{-X^{[\alpha|\beta]} + 2F^{\alpha\beta}(A_\gamma X^\gamma + \partial_\gamma X^\gamma)\}/8\pi . \tag{5.7}$$

It follows from the field equations, that the energy densities (5.5) are also of the form $\partial_\mu E^{\mu\alpha}$, therefore the energy is given by integrating (5.6) or (5.7) over the boundary of the domain under consideration.

With the definitions

$$H_i = \frac{1}{2}\int_{\partial\Sigma} E^{\alpha\beta}_i \eta_{\alpha\beta} \qquad i = \text{F, C, UT}$$

the generating equations read:

$$dH_i = \text{“} \int_\Sigma \Theta_\Sigma \text{”} + \frac{1}{2}\int_{\partial\Sigma} \Theta^i_{\partial\Sigma}, \quad i = \mathrm{F}, \mathrm{UT},$$

$$dH_C = \text{“} \int_\Sigma \Theta_\Sigma \text{”} + \int_\Sigma \hat{F}^{\alpha\beta}{}_{,\beta} X^\nu dA_\nu \eta_\alpha + \frac{1}{2}\int_{\partial\Sigma} \Theta^C_{\partial\Sigma} \tag{5.8}$$

$$\Theta_\Sigma = \{ L_X A^\alpha_{\mu\nu} d\Pi^{\mu\nu} - L_X \Pi^{\mu\nu} dA^\alpha_{\mu\nu} + L_X A_\nu d\hat{F}^{\nu\alpha} - L_X \hat{F}^{\nu\alpha} dA_\nu \} \eta_\alpha, \tag{5.9}$$

$$\Theta^F_{\partial\Sigma} = \left\{ -\frac{1}{8\pi}\sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu} dA^{[\alpha}_{\mu\nu} X^{\beta]} - 2X^{[\alpha}\hat{F}^{\beta]\gamma} dA_\gamma \right\} \eta_{\alpha\beta}, \tag{5.10}$$

$$\Theta^C_{\partial\Sigma} = \left\{ -\frac{1}{8\pi}\sqrt{-\det g_{\alpha\beta}}\, g^{\mu\nu} dA^{[\alpha}_{\mu\nu} X^{\beta]} - 3X^{[\alpha}\hat{F}^{\beta\gamma]} dA_\gamma \right\} \eta_{\alpha\beta}, \tag{5.11}$$

$$\Theta^{UT\text{“}}_{\partial\Sigma} = \text{”}\Theta^F_{\partial\Sigma} \tag{5.12}$$

(inverted commas in (5.8) and in (5.12) denote the fact, that Θ_Σ is only formally the same in all theories, and $\Theta^{UT}_{\partial\Sigma}$ is only formally the same as $\Theta^F_{\partial\Sigma}$. These special symplectic forms are defined on spaces of completely different objects). From all the expressions for the energy, only H_C is gauge-invariant, and can therefore be factorized to the space of gauge orbits. The gauge dependence of H_{UT} and H_F is not surprising, because, as opposed to H_C, H_{UT} and H_F generate gauge-dependent equations of motion ([2]) (the Lie derivatives (5.2) (F) and (5.3) are obviously gauge-dependent).

6. CONCLUSIONS

The results of section 5 show, that the interpretation of the hamiltonian, proposed in [*6*] and [*5*], as the energy of the fields, seems not to be physically justified. The same theory—electromagnetism and gravitation—has at least three different hamiltonians, but certainly cannot have three different expressions for the energy. It seems crucial, for a better understanding of the energy in general relativity, to find some energy exchange processes on which different expressions for the energy could be tested. It is possible that one of the expressions (5.6) and (5.7) provides the « true energy formula » for electromagnetism interacting with gravitation. If the unified theory energy expression turned out to be the fundamental one, we would probably have to revise our opinion about gauge-independence of physics.

Some of the results presented here have been derived independently by M. Ferraris [*2*].

([2]) This has been pointed out to the author by prof. I. Bialynicki-Birula.

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APPENDIX

In this paper, $G = c = \hbar = 1$. Moreover, the following conventions are used: the derivative with respect to a set of independent variables which are components of an *a priori* symmetric or antisymmetric tensor, for example $\{ g_{\mu\nu}, \mu \leqslant \nu \}$, is one half of the usual one. Therefore, $dA(g_{\alpha\beta}) = \partial A/\partial g_{\alpha\beta} dg_{\alpha\beta}$, and not $\frac{1}{2}\partial A/\partial g_{\alpha\beta} dg_{\alpha\beta}$. The signature of the metric is $- + + +$, greek indices range from 0 to 3, repeated indices imply a summation, round (square) brackets over indices denote a complete symmetrization (antisymmetrization) with the appropriate combinatorial factor, L_X denotes a Lie derivative with respect to the vector field X, a coma or a ∂ denotes a partial derivative, a semi-colon denotes covariant differentiation with respect to the non-metric connection, a bar ($|$) denotes covariant differentiation with respect to the metric connection, the connection symbols are defined as follows:

$$X^{\lambda}{}_{;\mu} = X^{\lambda}{}_{,\mu} + \Gamma^{\lambda}_{\mu\sigma} X^{\sigma}. \tag{A1}$$

The components of the curvature tensor are

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}_{\delta\beta} - \partial_{\delta}\Gamma^{\alpha}_{\gamma\beta} + \Gamma^{\alpha}_{\gamma\sigma}\Gamma^{\sigma}_{\delta\beta} - \Gamma^{\alpha}_{\delta\sigma}\Gamma^{\sigma}_{\gamma\beta}. \tag{A2}$$

The components of the torsion tensor are defined by

$$Q^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}. \tag{A3}$$

The following identity holds for the covariant divergence of an antisymmetric tensor density:

$$A^{\nu\lambda}{}_{;\nu} = A^{\nu\lambda}{}_{,\nu} + Q^{\lambda}_{\sigma\nu} A^{\sigma\nu}/2 + Q^{\alpha}_{\alpha\nu} A^{\nu\lambda}. \tag{A4}$$

The Ricci identity for tensors reads

$$2X_{\alpha\ldots}{}^{\beta\ldots}{}_{;[\gamma\delta]} = R^{\sigma}{}_{\alpha\gamma\delta} X_{\sigma\ldots}{}^{\beta\ldots} + \ldots + R^{\beta}{}_{\sigma\delta\gamma} X_{\alpha\ldots}{}^{\sigma\ldots} + \ldots + Q^{\sigma}_{\gamma\delta} X_{\alpha\ldots}{}^{\beta\ldots}{}_{;\sigma}. \tag{A5}$$

If $X_{\alpha..}{}^{\beta..}$ is a tensor density, a supplementary term $R^{\mu}{}_{\mu\gamma\delta} X_{\alpha..}{}^{\beta..}$ has to be added on the right hand side of (A5). The Jacobi and Bianchi identities are:

$$R^{\alpha}{}_{[\mu\nu\rho]} = Q^{\alpha}_{[\mu\nu;\rho]} + Q^{\alpha}_{\beta[\mu} Q^{\beta}_{\nu\rho]}, \tag{A6}$$

$$R^{\alpha}{}_{\beta[\mu\nu;\rho]} = - R^{\alpha}{}_{\beta\gamma[\mu} Q^{\gamma}_{\nu\rho]}. \tag{A7}$$

The Lie derivative of a connection $\Gamma^{\alpha}_{\beta\gamma}$ can be defined by the formula

$$L_X\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\sigma}X^{\sigma} + X^{\sigma}{}_{,\mu}\Gamma^{\lambda}_{\sigma\nu} + X^{\sigma}{}_{,\nu}\Gamma^{\lambda}_{\mu\sigma} - X^{\lambda}{}_{,\sigma}\Gamma^{\sigma}_{\mu\nu} + X^{\lambda}{}_{,\mu\nu} = R^{\lambda}{}_{\nu\sigma\mu}X^{\sigma} + \left(Q^{\lambda}_{\sigma\nu}X^{\sigma} + X^{\lambda}{}_{;\nu}\right)_{;\mu} \tag{A8}$$

The forms $\eta_{\alpha..\beta}$ are defined as follows:

$$\begin{aligned} \eta &= dx^0 \wedge \ldots \wedge dx^3, \\ \eta_{\mu} &= \partial_{\mu} \lrcorner\, \eta, \\ \eta_{\alpha\mu..\nu} &= \partial_{\alpha} \lrcorner\, \eta_{\mu..\nu}, \end{aligned} \tag{A9}$$

where $(X \lrcorner\, \alpha)(\ldots) = \alpha(X, \ldots)$. (A9) leads to the following identity

$$dx^{\mu} \wedge \eta_{\alpha_1\ldots\alpha_k} = k\delta^{\mu}_{[\alpha_1}\eta_{\alpha_2\ldots\alpha_k]},$$

from which one easily obtains

$$d\left(X^{\alpha_1\ldots\alpha_k}\eta_{\alpha_1\ldots\alpha_k}\right) = kX^{\alpha_1\ldots\alpha_k}{}_{,\alpha_1}\eta_{\alpha_2\ldots\alpha_k}.$$

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