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by

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ABSTRACT. — Some statements about Jordan automorphisms including a structure theorem are proved, and a continuous one-parameter group of Jordan automorphisms is constructed which is not a group of *-automorphisms.

RÉSUMÉ. — On démontre quelques résultats sur les automorphismes de Jordan, en particulier un théorème de structure, et on construit un groupe continu à un paramètre d’automorphismes de Jordan qui n’est pas un groupe de *-automorphismes.

I

A Jordan automorphism \( \alpha \) of a C*-algebra \( \mathcal{A} \) is a bijective \*-*preserving linear map of \( \mathcal{A} \) onto itself which respects anticommutators:

\[
\alpha(xy + yx) = \alpha(x)\alpha(y) + \alpha(y)\alpha(x).
\]

Groups of Jordan automorphisms show up in physics when in an algebraic framework the time evolution of a quantum system is considered in the Schrödinger picture \([1, 2, 3]\). In these cases \( \mathcal{A} \) is a von Neumann algebra. For any Jordan automorphism \( \alpha \) of a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \), there is a projection \( E \) such that

\[
\alpha(AB)(1 - E) = \alpha(B)\alpha(A)(1 - E),
\]
i. e. \( \alpha \) looks like an antimorphism on \((1 - E)H\). The composition of two antimorphisms is a morphism; thus one might be inclined to conclude that any one-dimensional group \( \{ \alpha_t \}_{t \in \mathbb{R}} \) of Jordan automorphisms is actually a group of common \(*\)-automorphisms, because \( \alpha_t = \alpha_{t/2} \circ \alpha_{t/2} \). This is not true in general, although it holds in many cases of interest in physics. Actually, a group of Jordan automorphisms is exhibited in [4] although in a somewhat implicite form. Evidently, this is not well known among physicists.

The aim of this note is to prove some statements (including a structure theorem) about Jordan automorphisms and—based upon these results—give an explicite construction of a continuous one-parameter group \( \{ \alpha_t \}_{t \in \mathbb{R}} \) of Jordan automorphisms which are not \(*\)-automorphisms.

Section II contains some statements on Jordan automorphisms which are needed in the sequel and might have some interest in their own. Section III is concerned with the structure of Jordan automorphisms. In Section IV a genuine group of Jordan automorphisms is exhibited. It is conjectured that this construction can be generalized to yield groups of Jordan automorphisms of fairly arbitrary von Neumann algebras.

II

Let us recall some known results on Jordan automorphisms [1, 5].

By \( \text{J-Aut}(\mathcal{A}) \) we denote the set of all Jordan automorphisms of \( \mathcal{A} \). Since we are mainly interested in Jordan automorphisms of von Neumann algebras we shall not give the most general statements.

**Proposition 2.1.** Let \( \mathcal{M} \) be a von Neumann algebra acting on a Hilbert space \( H \), and \( \alpha \in \text{J-Aut}(\mathcal{M}) \).

1. \( \alpha \) is order preserving.
2. \( \alpha \) is an isometry.
3. There is a projection \( E \) in the centre \( \mathcal{Z}(\mathcal{M}) \) of \( \mathcal{M} \) such that for all \( A, B \in \mathcal{M} \)
   \[
   \alpha(AB) = \alpha(A)\alpha(B)E + \alpha(B)\alpha(A)(1 - E). \tag{1}
   \]

The proof can be found e. g. in [5, Th. 3.2.3]. A projection \( E \in \mathcal{Z}(\mathcal{M}) \) satisfying (1) will be called a decomposing projection of \( \alpha \). Note that \( \alpha \in \text{J-Aut}(\mathcal{M}) \) implies

\[
\alpha^{-1} \in \text{J-Aut}(\mathcal{M}). \tag{2}
\]

**Corollary 2.2.** The mapping \( A \in \mathcal{M} \mapsto \alpha(A) \) is continuous with respect to the ultraweak topology.
This is a consequence of Prop. 2.1 (i) and of (2): \( \alpha \) is a positive normal map, hence ultraweakly continuous [6, Th. I.4.2].

We shall now prove some minor results concerning decomposing projections of Jordan automorphisms, which will be used subsequently.

Let the von Neumann algebra \( \mathcal{M} \) act on the Hilbert space \( \mathcal{H} \). Then we define the subspace \( \mathcal{H}_0 \) and the projection \( E_c \) onto \( \mathcal{H}_0 \) by

\[
\mathcal{H}_0 = E_c \mathcal{H} := \{ \Psi \in \mathcal{H} ; [A, B] \Psi = 0 \quad \text{for all} \quad A, B \in \mathcal{M} \}.
\]

\([A, B] = AB - BA\) denotes the commutator, \( E_c \) is a central projection: clearly, \( \mathcal{M}' \mathcal{H}_0 \subset \mathcal{H}_0 \); furthermore, we have for all \( C \in \mathcal{M} \) and \( \Psi \in \mathcal{H}_0 \)

\[
[A, B] C \Psi = A [B, C] \Psi + [AC, B] \Psi = 0,
\]

hence \( \mathcal{M}' \mathcal{H}_0 \subset \mathcal{H}_0 \) and thus

\[
E_c \in \mathcal{D}(\mathcal{M}).
\]

**Lemma 2.3.** — Consider \( \alpha \in J\text{-Aut}(\mathcal{M}) \).

1. \( \mathcal{D}(\mathcal{M}) \) is invariant under \( \alpha \).
2. There exists a unique maximal decomposing projection \( E_m = E_m[\alpha] \); a decomposing projection \( E \) of \( \alpha \) is maximal if and only if \( E \subseteq E_c \).
3. \( \alpha(E_c) = E_c \).
4. If \( \alpha \) and \( \beta \) are Jordan automorphisms of \( \mathcal{M} \) with decomposing projections \( E_\alpha \) and \( E_\beta \), respectively, then \( \alpha \circ \beta \in J\text{-Aut}(\mathcal{M}) \) with decomposing projection

\[
E_{\alpha \circ \beta} = E_\alpha \alpha(E_\beta) + (1 - E_\alpha) \alpha(1 - E_\beta),
\]

and maximality of \( E_\alpha \) and \( E_\beta \) implies maximality of \( E_{\alpha \circ \beta} \).

As is easily seen, \( \alpha^{-1}(E_\alpha) \) is a decomposing projection for \( \alpha^{-1} \); and, as a consequence of (iv), we have

\[
E_m[\alpha^{-1}] = \alpha^{-1}(E_m[\alpha]).
\]

Note that in general \( \mathcal{D}(\mathcal{M}) \) is not pointwise invariant.

**Proof.** — Part i) is Lemma (14) of [2].

ii) Let \( E \) be any decomposing projection of \( \alpha \). On \( E_c \mathcal{H} \), a morphism cannot be distinguished from an antimorphism, hence

\[
E_m[\alpha] := E + E_c - EE_c \in \mathcal{D}(\mathcal{M})
\]

defines a decomposing projection for \( \alpha \). Now assume that we could find a decomposing projection \( E' \) which is not majorized by \( E_m[\alpha] \). Then there is a \( \Psi \in E'(1 - E_m) \mathcal{H}, \Psi \neq 0 \), and for all \( A, B \in \mathcal{M} \) we have

\[
\alpha(AB) \Psi = \alpha(AB)(1 - E_m) \Psi = \alpha(B) \alpha(A) \Psi
\]

and

\[
\alpha(AB) \Psi = \alpha(AB) E' \Psi = \alpha(A) \alpha(B) \Psi.
\]
Since \( \alpha \) maps \( M \) onto \( M \), it follows that \( \Psi \) is in \( \mathcal{H}_0 \), thus \( E'(1 - E_m) \leq E_c \) in contradiction to \( 1 - E_m \leq 1 - E_c \) which holds by definition (7). Hence \( E_m \) is maximal and unique. By the same reasoning one shows that any decomposing projection \( E' \) containing \( E_c \) has to be maximal, i.e. equal to \( E_m \); otherwise \( E_m(1 - E') \mathcal{H} \) would contain a \( \Psi \neq 0 \) giving rise to a contradiction.

(iii) According to (2), we have \( \alpha^{-1}([A, B])E_c = [\alpha^{-1}(A), \alpha^{-1}(B)]E_c = 0 \) for all \( A, B \in M \), hence \( 0 = \alpha(\alpha^{-1}([A, B]E_c)) = [A, B]\alpha(E_c) \) (note that \( \alpha(AC) = \alpha(A)\alpha(C) \) if \( C \in \mathcal{D} \)), and hence \( \alpha(E_c) \leq E_c \). Using \( \alpha^{-1} \) instead of \( \alpha \) we get \( \alpha^{-1}(E_c) \leq E_c \) and therefore \( \alpha(E_c) = E_c \) because \( \alpha \) is order preserving.

(iv) Obviously, \( \alpha \circ \beta \) is linear, \(*\)-preserving and bijective. \( E_{\alpha \beta} \) as defined above is a decomposing projection of \( \alpha \circ \beta \) as is seen by straightforward calculation. According to (ii), \( E_c \) is contained in \( E_m[\alpha] \) and in \( E_m[\beta] \); therefore, due to (iii), \( E_c = \alpha(E_c) \leq \alpha(E_m[\beta]) \) and \( E_c \leq E_m[\alpha] \alpha(E_m[\beta]) \). Since \( E_c(1 - E_m[\alpha]) = 0 \) it follows that \( E_c \leq \text{r.h.s. of (5)} \), and, again using (ii), we get (5). \( \square \)

Now let \( t \mapsto \alpha_t \) be a map of the additive group \( \mathbb{R} \) into \( \text{J-Aut}(M) \), and assume that the map \( t \mapsto \alpha_t(A) \) is weakly continuous. It is possible to choose decomposing projections \( E_t \) of \( \alpha_t \), \( t \in \mathbb{R} \), such that \( E_t \) is ultrastrongly continuous in \( t \). This holds especially for \( E_m[\alpha_t] \):

**Lemma 2.4.** — If \( t \to \alpha_t(A), \alpha_t \in \text{J-Aut}(M) \), is weakly continuous, then \( t \mapsto E_m[\alpha_t] \) is ultrastrongly continuous.

**Proof.** — See Lemma (22) of [2]; the group property is not needed. \( \square \)

In the sequel we shall always choose \( E_t = E_m[\alpha_t] \); \( \{ \alpha_t \}_{t \in \mathbb{R}} \) will always denote a one-parameter group of Jordan automorphisms. Applying Lemma 2.3 (iv) we then get

\[
E_{t_1 + t_2} = E_t \alpha_t(E_{t_2}) + (1 - E_t)\alpha_t(1 - E_{t_2}).
\]

The decomposing projections \( E_t \) cannot be independent of \( t \) lest the \( \alpha_t \) be common \(*\)-automorphisms:

**Lemma 2.5.** — Assume that the decomposing projections \( E_\alpha \) of a group \( G \) of Jordan automorphisms do not depend on \( \alpha \in G \), for all \( \alpha \neq \text{id} \). If \( G \) contains at least three elements then \( G \) is a group of \(*\)-automorphisms.

This is intuitively clear; but reasoning by intuition, as referred to in the introduction, should be confined to this case.

A formal proof can be easily given by application of Lemma 2.3 (iv).
the structure of Jordan automorphisms of a von Neumann algebra with
a cyclic and separating vector. A related result is the following structure
theorem, which applies to von Neumann algebras with an \( \alpha \)-invariant
vector state given by a cyclic vector which, however, need not be separating.
It also allows to deal with C*-algebras with an invariant state.

**THEOREM 3.1.** — Let \( \mathcal{M} \) be a von Neumann algebra acting on the
Hilbert space \( \mathcal{H} \), \( \alpha \in J\text{-Aut}(\mathcal{M}) \). Assume that \( \Omega \in \mathcal{H} \) is cyclic for \( \mathcal{M} \) and
defines an \( \alpha \)-invariant state:

\[
(\Omega, \alpha(A)\Omega) = (\Omega, A\Omega) \quad \text{for all} \quad A \in \mathcal{M}.
\]

Let \( E \) be a decomposing projection of \( \alpha \). Then \( \bar{E} := \alpha^{-1}(E) \) is a decomposing
projection for \( \alpha^{-1} \); and there exist partial isometries \( V \) and \( \bar{V} \), where \( V \)
is linear, \( \bar{V} \) is antilinear, such that

\[
\begin{align*}
\frac{1}{2} VV^* &= E, & \frac{2}{2} VV^* &= 1 - E, \\
\frac{1}{2} V^*V &= \bar{E}, & \frac{2}{2} V^*V &= 1 - \bar{E},
\end{align*}
\]

and for all \( A \in \mathcal{M} \)

\[
\begin{align*}
\alpha(A) &= \frac{1}{2} VAV^* + \frac{2}{2} VA^*V^*, \\
\alpha^{-1}(A) &= \frac{1}{2} V^*AV + \frac{2}{2} V^*A^*V.
\end{align*}
\]

Furthermore,

\[
\begin{align*}
\frac{1}{2} V\Omega &= E\Omega, & \frac{2}{2} V\Omega &= (1 - E)\Omega.
\end{align*}
\]

The case of a group of Jordan automorphisms is dealt with by the following

**PROPOSITION 3.2.** — Consider a one-parameter group \( \{ \alpha_t \}_{t \in \mathbb{R}} \subset J\text{-Aut}(\mathcal{M}) \)
for which the cyclic vector \( \Omega \) defines an invariant state of \( \mathcal{M} \), then the iso-
metries \( \bar{V}_t \) and \( \bar{V}_t \) which, according to Theorem 3.1, exist for all \( t \in \mathbb{R} \),
fulfill the relations

\[
\begin{align*}
\frac{1}{2} V_t^* &= V_{-t}, & \frac{2}{2} V_t^* &= V_{-t}, \\
\frac{1}{2} V_t V_{t'} + \frac{2}{2} V_{t'} V_{t'} &= V_{t+t'}, \\
\frac{1}{2} V_t V_{t'} + \frac{2}{2} V_{t'} V_{t'} &= V_{t+t'},
\end{align*}
\]

provided the \( E_t \) are chosen to be maximal.

The maximality of \( E_t \) implies \( E_0 = 1, \frac{1}{2} V_0 = 1 \) and \( \frac{2}{2} V_0 = 0 \). There is a
partial converse to Theorem 3.1:
THEOREM 3.3. — Given two partial isometries $V_1$ and $V_2$ on a Hilbert space $\mathcal{H}$, $V_1$ linear, $V_2$ antilinear, which fulfil
\[ V_1 V_1^* + V_2 V_2^* = 1, \quad V_1^* V_1 + V_2^* V_2 = 1. \] (18)

Let $\mathcal{B}$ (resp. $\tilde{\mathcal{B}}$) denote the von Neumann algebra of all operators on $\mathcal{H}$ commuting with the final (resp. initial) projection $E = V V^*$ (resp. $\tilde{E} = V^* V$) of $V : \mathcal{B} = \{E\}'$, $\tilde{\mathcal{B}} = \{\tilde{E}\}'$.

Then
\[ \gamma(A) := V_1^* A V_1 + V_2^* A V_2, \quad A \in \tilde{\mathcal{B}} \] (19)

and
\[ \tilde{\gamma}(A) := V_1 A V_1^* + V_2 A V_2^*, \quad A \in \mathcal{B} \] (19')
define Jordan isomorphisms from $\tilde{\mathcal{B}}$ (resp. $\mathcal{B}$) onto $\mathcal{B}$ (resp. $\tilde{\mathcal{B}}$) with decomposing projections $E \in \mathcal{L}(\mathcal{B})$ (resp. $\tilde{E} \in \mathcal{L}(\tilde{\mathcal{B}})$) and
\[ \gamma^{-1} = \tilde{\gamma}, \quad E = \gamma(E), \quad \tilde{E} = \gamma(E). \] (20) (21)

If $\mathcal{M} \subset \mathcal{B} \cap \tilde{\mathcal{B}}$ is a von Neumann algebra which is mapped onto itself by application of $\gamma$, then $\alpha := \gamma |_\mathcal{M}$ is a Jordan automorphism of $\mathcal{M}$. If furthermore eqs. (14) are fulfilled for a vector $\Omega \in \mathcal{H}$, then $\Omega$ defines an $\alpha$-invariant state.

With the above definitions of $E$ and $\tilde{E}$, condition (18) is evidently equivalent to eqs. (10) and (11).

There is also an analogue to Proposition 3.2:

PROPOSITION 3.4. — Given a family $\{V_t, \tilde{V}_t\}_{t \in \mathbb{R}}$ of isometries, $V_t$ linear, $\tilde{V}_t$ antilinear, such that
\[ V_1^1 V_t V_1^1 + V_2^2 V_t V_2^2 = 1, \quad V_t^1 V_1^1 + V_t^2 V_2^2 = 1, \] (22)

and such that eqs. (15) to (17) hold.
Assume that for arbitrary $t \in \mathbb{R}$
\[ \alpha_t(A) := V_t A V_t^* + \tilde{V}_t A \tilde{V}_t^* \] (23)
defines a map of a von Neumann algebra $\mathcal{M}$ onto itself. Then the projections $E_t := V_t V_t^*$ fulfill eq. (8), and $\{\alpha_t\}_{t \in \mathbb{R}}$ is a one-parameter group of Jordan automorphisms of $\mathcal{M}$.
**Proof of Theorem 3.1.** — We define \( \overline{1} V \) and \( \overline{2} V \) by

\[
\begin{align*}
\overline{1} VA\Omega &= \alpha(A)E\Omega, \quad A \in \mathcal{M}, \\
\overline{2} VA\Omega &= \alpha(A^*)(1 - E)\Omega, \quad A \in \mathcal{M}.
\end{align*}
\] (24)

\( \overline{1} V \) (resp. \( \overline{2} V \)) is a well-defined linear (resp. antilinear) operator because \( A\Omega = 0 \) implies \( \alpha(A)E\Omega = 0 \) and \( \alpha(A^*)(1 - E) = 0 \):

\[
\| \alpha(A)E\Omega \|^2 = (E\Omega, \alpha(A^*)E\Omega) = (\Omega, \alpha(A^*A)\alpha^{-1}(E)\Omega) = (\Omega, A^*A\alpha^{-1}(E)\Omega) = 0
\]

if \( A\Omega = 0 \). Here we need assumption (9). Analogously:

\[
\| \alpha(A^*)(1 - E)\Omega \|^2 = (\Omega, \alpha(A)\alpha(A^*)(1 - E)\Omega) = (\Omega, \alpha(A^*A)(1 - E)\Omega) = \ldots = 0.
\]

Eqs. (24) and (25) define \( \overline{1} V \) and \( \overline{2} V \) on \( \mathcal{D} = \mathcal{M}\Omega \). According to eq. (9), we have

\[
\| A\Omega \|^2 = (\Omega, \alpha(A^*A)\Omega) = (\Omega, \alpha(A)^{\ast}\alpha(A)E\Omega) + (\Omega, \alpha(A)\alpha(A^*)(1 - E)\Omega),
\]

hence

\[
\| A\Omega \|^2 = \| \overline{1} VA\Omega \|^2 + \| \overline{2} VA\Omega \|^2.
\] (26)

Thus \( \overline{1} V \) and \( \overline{2} V \) are bounded and can be extended to \( \overline{\mathcal{D}} = \mathcal{H} \) (\( \Omega \) is assumed to be cyclic !). Next we want to show

\[
\begin{align*}
\overline{1} VB\Omega &= \alpha^{-1}(B)\overline{1} E\Omega, \quad B \in \mathcal{M}, \\
\overline{2} VB\Omega &= \alpha^{-1}(B^*)(1 - \overline{1} E)\Omega, \quad B \in \mathcal{M}.
\end{align*}
\] (27)

Again we need the invariance assumption (9), the cyclicity of \( \Omega \), and \( \overline{1} E = \alpha^{-1}(E) \in \mathcal{L}(\mathcal{M}) \):

\[
(B\Omega, \overline{1} VA\Omega) = (\Omega, B^*\alpha(A)E\Omega)
= (\Omega, \alpha(\alpha^{-1}(B^*)A)E\Omega) = (\Omega, \alpha(\alpha^{-1}(B^*)A)\alpha^{-1}(E)\Omega)
= (\Omega, \alpha^{-1}(B^*)A\overline{1} E\Omega) = (\alpha^{-1}(B)\overline{1} E\Omega, A\Omega),
\]

hence (27). Analogously, eq. (28) is proved.

By combining eqs. (24), (25), (27) and (28) it is a straightforward task to show eqs. (10) and (11).

In the same way, we demonstrate (12) and (13): Applying (27) and (24) we get

\[
\overline{1} VAV^*B\Omega = VA\alpha^{-1}(B)\overline{1} E\Omega = \alpha(A^\ast\alpha^{-1}(B)\overline{1} E)E\Omega = \alpha(A)B\alpha(\overline{1} E)E\Omega = \alpha(A)EB\Omega,
\]

where we used \( E = \alpha(\overline{1} E) \in \mathcal{L}(\mathcal{M}) \); and therefore,

\[
\overline{1} VAV^* = \alpha(A)E. \quad (29)
\]

From (28) and (25) it follows that

\[
\overline{2} VAV^* = \alpha(A^*)(1 - E). \quad (30)
\]
Eqs. (29) and (30) imply (12). Similarly, we get
\[ \begin{align*}
\frac{1}{2} V^* A^* V &= \alpha^{-1}(A) \tilde{E}, \\
\frac{2}{2} V^* A V &= \alpha^{-1}(A^*)(1 - E)
\end{align*} \]
and hence (13).

Eqs. (14) are special cases of (24) resp. (25). \( \square \)

**Proof of Proposition 3.2.** — We use \( E_t = \alpha_t(E_{-t}) \), which holds for maximal \( E_t \), see eq. (8). Since \( V_t A\Omega = \alpha_t(A)E_t\Omega \), we have
\[ (B\Omega, V^*_t A\Omega) = (\Omega, \alpha_t(\alpha_{-t}(B^*)A)E_t\Omega) = (\Omega, \alpha_t(\alpha_{-t}(B^*)AE_{-t})\Omega). \]

Due to the invariance of \( (\Omega, \Omega) \), we get
\[ (B\Omega, V^*_t A\Omega) = (\alpha_{-t}(B)E_{-t}\Omega, A\Omega) = (V^*_t B\Omega, A\Omega) \]
and thus \( V^*_t = V_{-t} \). In the same manner, we find \( (B\Omega, V^*_t A\Omega) = (A\Omega, \alpha_{-t}(V^* B)E_{-t}\Omega) \)
and therefore \( V^*_t = V_{-t} \) (remember that \( V_t \) is antilinear).

Next we calculate
\[ \frac{1}{2} V^*_t V_t A\Omega = \frac{1}{2} V_t \alpha_t(A)E_t^* \Omega = \alpha_t(\alpha_t(A)E_t^* \Omega) = \alpha_{t+r}(A)\alpha_t(E_t^* \Omega), \quad (31) \]
and analogously
\[ \frac{2}{2} V^*_t V_t A\Omega = \alpha_{t+r}(A)\alpha_t(1 - E_t^*)(1 - E_t)\Omega. \quad (32) \]

Adding (31) and (32) and using eq. (8) we arrive at
\[ (V^*_t V_t + V^*_t V_t) A\Omega = \alpha_{t+r}(A)E_{t+r} \Omega = V_{t+r} A\Omega, \]
hence eq. (16). The proof of (17) is left to the reader. \( \square \)

We need two further relations for \( V \) and \( V^* \):

**Lemma 3.5.** — (i) Eqs. (10) and (11) imply
\[ \begin{align*}
\frac{1}{2} V A V^* &= 0 \quad \text{for all} \quad A \in \mathcal{B} = \{ \tilde{E} \}', \\
\frac{1}{2} V^* A V &= 0 \quad \text{for all} \quad A \in \mathcal{B} = \{ E \}'.
\end{align*} \]
In the setting of Theorem 3.1, both relations hold for all \( A \in \mathcal{M} \).

(ii) Eqs. (10), (11) and (14) imply
\[ \begin{align*}
\frac{1}{2} V^* \Omega &= \tilde{E}\Omega, \\
\frac{2}{2} V^* \Omega &= (1 - \tilde{E})\Omega.
\end{align*} \]

**Proof.** — Trivial, since eqs. (10) and (11) imply
\[ \begin{align*}
\frac{1}{2} V &= \frac{1}{2} V \tilde{E} \\
\frac{2}{2} V^* &= (1 - \tilde{E})\frac{2}{2} V^*.
\end{align*} \quad (33) \]
**Proof of Theorem 3.3.** — We start noting that eqs. (21) are easily derived from (10) and (11) resp. (33) with the help of Lemma 3.5 (i) with $A = 1$.

It is obvious that $\gamma$ and $\bar{\gamma}$ are linear *-preserving maps which also preserve anticommutators. $\gamma$ and $\bar{\gamma}$ are injective: again using Lemma (3.5) (i) one sees that $\bar{\gamma}(\gamma(A)) = \bar{E} A \bar{E} + (1 - \bar{E}) A (1 - \bar{E})$, hence $\bar{\gamma}(\gamma(A)) = A$ if $A \in \{ \bar{E} \}' = \mathfrak{B}$,

$$\gamma : \mathfrak{B} \to \gamma(\mathfrak{B}) \text{ is injective, and } \bar{\gamma} = \gamma^{-1} \text{ on } \gamma(\mathfrak{B}). \quad (34)$$

Similarly,

$$\tilde{\gamma} : \mathfrak{B} \to \tilde{\gamma}(\mathfrak{B}) \text{ is injective, } \tilde{\gamma}^{-1} = \gamma \text{ on } \tilde{\gamma}(\mathfrak{B}) \quad (35)$$

As is easily seen, $E$ (resp. $\bar{E}$) commutes with all elements of $\gamma(\mathfrak{B})$ (resp. $\tilde{\gamma}(\mathfrak{B})$), that is, we have

$$\gamma(\mathfrak{B}) \subseteq \{ E \}' = \mathfrak{B}, \quad \tilde{\gamma}(\mathfrak{B}) \subseteq \{ \bar{E} \}' = \mathfrak{B}. \quad (36)$$

Eqs. (34) to (36) imply that $\gamma(\mathfrak{B}) = \mathfrak{B}$, and that $\gamma$ is a bijective map from $\mathfrak{B}$ onto $\mathfrak{B}$, hence a Jordan isomorphism. Clearly, $E \in \{ E \}' = \mathfrak{B}$; and $E \in \gamma(\mathfrak{B})'$ = $\mathfrak{B}$ according to (36), hence $E \in \mathfrak{B}(\mathfrak{B})$. Similarly one shows that $\bar{E} \in \mathfrak{B}(\mathfrak{B})$. By direct computation it follows that $E$ (resp. $\bar{E}$) are decomposing projections of $\gamma$ (resp. $\bar{\gamma}$).

It is now clear that $\gamma|_{\mathcal{M}} \in J\text{-Aut}(\mathcal{M})$ if $\gamma$ maps $\mathcal{M} \subseteq \mathfrak{B} \cap \mathfrak{B}$ onto itself.

**Proof.** Consider the universal representation $\pi_u$ of $\mathfrak{B}$, which is faithful, hence

$$\pi_u(1) = 1, \pi_u(2) = 0, \text{ hence } \alpha_{t=0} = \text{id};$$

it remains to be shown that the $E_t$ fulfil (8) and that $\alpha_t \circ \alpha_{t'} = \alpha_{t+t'}$. This can be verified by direct calculation with the help of Lemma 3.5 (i). \[\square\]

As already mentioned, mainly the case of a von Neumann algebra is of interest in physical applications of Jordan automorphisms. Nevertheless let us briefly consider the case of a C*-algebra $\mathfrak{A}$ with a Jordan automorphism $\alpha$.

**PROPOSITION 3.6.** — Let $\mathfrak{A}$ be a C*-algebra, $\alpha \in J\text{-Aut}(\mathfrak{A})$. Assume that there is an $\alpha$-invariant state $\omega$ on $\mathfrak{A}$. Then one can define a Jordan automorphism $\alpha^\omega$ on the von Neumann algebra $\mathfrak{B}_\omega = \pi_\omega(\mathfrak{A})'$ generated by the representation $\pi_\omega$ associated with $\omega$ such that

$$\alpha^\omega(\pi_\omega(x)) = \pi_\omega(\alpha(x)) \text{ for all } x \in \mathfrak{A}. \quad (37)$$

Subsequently, the above structure theorem can be applied.

**Proof.** — Consider the universal representation $\pi_u$ of $\mathfrak{A}$. $\pi_u$ is faithful, hence

$$\alpha_u : \pi_u(x) \to \pi_u(\alpha(x)) \quad (38)$$

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is well-defined, and $\alpha_u \in \text{J-Aut}(\pi_u(A))$. It can be extended to a Jordan automorphism on $\mathcal{B} = \pi_u(A)'$ which will also be denoted by $\alpha_u$ (see e.g. Lemma (2.4) of [1] and the remarks proceeding it). The state $\omega$ gives rise to a corresponding vector state $\varphi_\omega$ on $\mathcal{B}$, $\omega(A) = \varphi_\omega(\pi_u(A))$, and clearly, $\varphi_\omega$ is $\alpha_u$-invariant. Define $E_\omega$ to be the projection onto $\mathcal{B}_\omega$, and let $F_\omega$ be its central support. Due to the invariance of $\varphi_\omega$, $F_\omega$ is $\alpha_u$-invariant (Lemma (15) of [2]). Hence we have $\alpha_u(AF_\omega) = \alpha_u(A)F_\omega$, $A \in \mathcal{B}$, thence $\alpha_u \upharpoonright_{\mathcal{BH}} \in \text{J-Aut}(\mathcal{B}F_\omega)$; by the isomorphism $\mathcal{B}F_\omega \cong \mathcal{B}_\omega$, there is a corresponding Jordan automorphism $\alpha_\omega$ of $\mathcal{B}_\omega$. Eq. (37) is then a consequence of (38). \hfill \Box

IV

The aim of this section is to give an explicite construction of a group of genuine Jordan automorphisms.

Theorem 3.3 suggests a way of generating a Jordan isomorphism: choose a projection $E$, a unitary operator $U$ and a conjugation $J$ (i.e. $J = J^*$, $J^2 = 1$) such that $E$ and $J$ commute. Then $V := EU$ and $\overline{V} := (1 - E)U$ fulfill the assumption (18) of Theorem 3.3. If moreover $E$ is in the centre of a von Neumann algebra $\mathcal{M}$, and $U^*U = \mathcal{M}$, $JMJ \subset \mathcal{M}$, then $\alpha(A) = VAV^* + VA^*V^*$ defines a Jordan automorphism of $\mathcal{M}$ which is not a $*$-automorphism if $E_0 \leq E \neq 1$, where $E_0$ is defined by eq. (3).

In order to find a group of Jordan automorphisms one has to choose central projections $E_t$ for all $t$ in such a way that in addition eq. (8) is fulfilled:

$$E_{t+t'} = E_t\alpha_t(E_{t'}) + (1 - E_t)\alpha_t(1 - E_{t'}).$$

This is of course the essential difficulty. We shall proceed as follows:

1. Choose an abelian von Neumann algebra $\mathcal{D}$, a family $\{E_t\}_{t \in \mathbb{R}}$ of projections in $\mathcal{D}$, and a group $\{\alpha_t^0\}_{t \in \mathbb{R}} \subset \text{Aut}(\mathcal{D})$ such that (8) is fulfilled with $\alpha_t^0$ in place of $\alpha_t$.

2. Find a von Neumann algebra $\mathcal{M}$ such that $\mathcal{D} = \mathcal{D}(\mathcal{M})$ and such that $\alpha^0_t$ extends to $\alpha^0_t \in \text{Aut}(\mathcal{M})$, where $\alpha^0_t$ is unitarily implemented by $U_t$; $U_{t=0} = 1$, and $U_{t+t'} = U_tU_{t'}$.

3. Find a conjugation $J$ fulfilling

$$JE_t = E_tJ, \quad JU_t = U_tJ, \quad JMJ \subset \mathcal{M}. \quad (39 \ a, b, c)$$

4. Define

$$V_t := E_tU_t, \quad \overline{V}_t := (1 - E_t)JU_t$$

and

$$\alpha_t(A) = V_tAV_t^* + \overline{V}_tA^*\overline{V}_t^*, \quad A \in \mathcal{M}; \quad (41)$$

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finally, make sure, that
\[ E_t \leq E_t + 1 \] (42)
for at least one value of \( t \).

We claim that \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is a non-trivial group of Jordan automorphisms of \( \mathcal{M} \). Assume we have accomplished steps 1. to 4. In order to show that \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is a group of Jordan automorphisms we shall use Proposition 3.4. It is then clear that not all \( \alpha_t \) can be automorphisms due to (42) and Lemma 2.3 (ii). Assumption (22) of Prop. 3.4 is readily checked for \( V_t \) and \( V_t \) defined above. Because of (39 a) we can write eq. (8) as
\[ E_{t+t'} = E_t U_t E_{t'} U_t^* + (1 - E_t) U_t (1 - E_{t'}) U_t^*. \] (43)

Since \( U_0 = 1 \), it follows that \( E_0 = 1 \), and thus \( V_0 = E_0 U_0 = 1 \), \( V_0 = 0 \). With \( t' = -t \), eq. (43) reads
\[ 1 = E_t U_t E_{-t} U_t^* + (1 - E_t) U_t (1 - E_{-t}) U_t^*. \]
Multiplication from the left (resp. from the right) by \( U_t^* E_t \) (resp. \( U_t E_{-t} \)) yields \( U_t^* E_t = U_t^* (E_t U_t E_{-t}) U_t^* \) and \( U_t E_{-t} = E_t U_t E_{-t} \). Inserting the second equation into the first one and using \( U_t^* = U_{-t} \) we arrive at \( V_t^* = U_t^* E_t = E_{-t} U_t^* = V_{-t} \). Similarly, it follows that \( V_t^* = V_{-t} \); thus eqs. (15) hold. It is straightforward to check the validity of (16) and (17) using (39 a, b) and (43). Thus Proposition 3.4 is applicable. It remains to make sure that \( \alpha_t \) maps \( \mathcal{M} \) onto itself. But this is clear: since \( U_t A_U^* = \alpha_t^U(A), \alpha_U^U \in \text{Aut}(\mathcal{M}), E_t \in \mathcal{L}(\mathcal{M}) \) and \( J \mathcal{M} J = \mathcal{M}, \alpha_t \) maps \( \mathcal{M} \) into itself, and due to the group property which follows from (16) and (17), we have \( \alpha_t^{-1} = \alpha_{-t} \), thus \( \alpha_t \) is bijective.

Now let us present a concrete example. It will be evident from the definitions that \( E_t \) and \( U_t \) and hence \( \alpha_t \) depend continuously on \( t \).

1. Let \( \mathcal{L} = \{ T_f ; f \in L^\infty(\mathbb{R}, dx) \} \) denote the set of multiplication operators on the complex Hilbert space \( \mathcal{H}^0 = L^2(\mathbb{R}, dx) : (T_f \varphi)(x) = f(x) \varphi(x). \mathcal{L} \) is a maximal abelian algebra, i. e. \( \mathcal{L}' = \mathcal{L} \). The translation operators \( U_t^0 : \varphi(x) \in \mathcal{H}^0 \mapsto \varphi(x-t) \) are unitary, they implement an automorphism group \( \{ \alpha_t^0 \} \) of \( \mathcal{L} \). The projections \( E_t \) are defined as follows:
\[ 1 - E_t = T_{\chi_{(0,t)}}, \]
where \( \chi_t \) denotes the characteristic function of the intervall I, and
\[ I(t_1, t_2) = \{ x \in \mathbb{R} ; \min (t_1, t_2) < x < \max (t_1, t_2) \} \]
It is easy although somewhat lengthy to calculate
\[ E_t \alpha_t^0 (1 - E_t) = T_{\chi(t,t')} \text{ and } (1 - E_t) \alpha_t^0 (1 - E_t) = T_{\chi(t,t')}, \]
and check that

\[ \chi_{(0,t)} + \chi_{(t,t')} - \chi_{(t,t')} = \chi_{(0,t+t')} \]

holds, hence

\[ 1 - E_{t+t'} = (1 - E_t) + E_t \alpha_t^0(1 - E_{t'}) - (1 - E_t) \alpha_t^0(1 - E_{t'}) \]

which is equivalent to eq. (8).

2. Now we consider \( \mathcal{H} = \mathcal{H}^0 \otimes \mathcal{H}^1 \), \( \mathcal{H}^1 = \mathcal{L}^2(\mathbb{R}, dy) \) and define \( \mathcal{M} := \mathcal{L} \otimes \mathcal{B}(\mathcal{H}^1) \). Consequently, \( \mathcal{M}' = \mathcal{L}' \otimes 1 = \mathcal{L} \otimes 1 \), and \( \mathcal{L}(\mathcal{M}) \simeq \mathcal{L} \).

We define \( U_t := U_t^0 \otimes 1 \), \( \alpha_t^U(A) := U_t^0 A U_t^0 \), \( A \in \mathcal{M} \), thus \( \alpha_t^U \) is \( \mathcal{M}' \) up to an isomorphism — an extension of \( \alpha_t^0 \), and \( \alpha_t^U \in \text{Aut}(\mathcal{M}) \).

3. The conjugation \( J \) is taken to be \( J : \psi(x, y) \rightarrow (J\psi)(x, y) = \overline{\psi}(x, y) \), where the bar denotes complex conjugation. It is evident that the conditions (39, a, b, c) are satisfied.

4. \( \{ \alpha_t \} \), as defined by (40) and (41) is the desired group of Jordan automorphisms. We claim that

\[ E_c = 0; \quad (44) \]

hence the \( E_t \) are maximal; and since \( E_t \neq 1 \) if \( t \neq 0 \), \( \alpha_t \) is not an automorphism. Finally, let us demonstrate (44): Any vector \( \Psi \in \mathcal{H} \) can be written as \( \Psi = \sum_{i,j} c_{ij} \varphi_i^0 \otimes \varphi_j^1 \), where \( \{ \varphi_i^0 \}_{i \in \mathcal{N}} \) (resp. \( \{ \varphi_j^1 \}_{j \in \mathcal{N}} \)) is an orthonormal basis of \( \mathcal{H}^0 \) (resp. \( \mathcal{H}^1 \)). If \( \Psi \neq 0 \) there is a \( c_{kl} = (\varphi_k^0 \otimes \varphi_l^1, \Psi) \neq 0 \). Let \( P_l \in \mathcal{B}(\mathcal{H}^1) \) be the projection onto \( \varphi_l^1 \). Since \( \dim \mathcal{H}^1 = \infty \), there is an isometry \( V \) with initial projection 1 and final projection \( 1 - P_l \), hence \( [V^*, V] = V^*V - VV^* = P_l \) and thus \( 1 \otimes P_l = [1 \otimes V^*, 1 \otimes V] \) is an element of the commutator set \( \mathcal{C}(\mathcal{M}) = \{ [A, B] ; A, B \in \mathcal{M} \} \). Due to our choice of \( k \) and \( l \), \( (\varphi_k^0 \otimes \varphi_l^1, 1 \otimes P_l \Psi) \neq 0 \), therefore \( \Psi \notin E_c \mathcal{H} \) (remember that \( \Psi \in E_c \mathcal{H} \) if and only if \( C\Psi = 0 \) for all \( C \in \mathcal{C}(\mathcal{M}) \)). Since \( \Psi \) was arbitrarily chosen, \( E_c \) must vanish.

It is rather plausible that the above construction can be generalized in several ways. Having in mind the central decomposition of a von Neumann algebra on a separable Hilbert space we are led to the conjecture that any such von Neumann algebra which is neither abelian nor a factor and which has a sufficiently large (non-discrete) centre admits a group of genuine Jordan automorphisms.

After having finished this paper I learned that a rather similar construction was given by G. A. Raggio [9]. (In his paper the maximality of the decomposing projections is not proven). To my knowledge, it has not been published.
REFERENCES


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