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Borel summability beyond the factorial growth

by

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ABSTRACT. — A criterion for a generalized Borel summability is given. In this way we define a unique sum for series diverging faster than any factorial power. Possible applications in quantum mechanics and quantum field theories are discussed.

RÉSUMÉ. — On donne un critère de sommabilité de Borel généralisée. De cette façon, on peut définir une somme unique pour des séries divergeant plus vite que toute puissance de factorielle. On discute des applications possibles en mécanique quantique et en théorie quantique des champs.

I. INTRODUCTION

Non-polynomial anharmonic oscillators have been recently treated by Doglov and Popov [*3*] and one of us [*10*]. In particular for the operator $p^2 + x^2 + ze^x$, by using WKB methods, a divergency of the perturbation expansion coefficients of the order of $e^{k^2/4}$ is shown [*3*]. On the basis of this result Doglov and Popov draw the conclusion that the perturbation series is not Borel summable: in fact the extension of the Borel sum introduced by Le Roy (see e. g. Ref. 6) does not apply if the growth of the coefficients is faster than $(nk)!$ for all $n \in \mathbb{N}$.

As a matter of fact there exists a general class of summation methods, called by Hardy « moment constant methods » (Ref. 6, pp. 81-86), defined as follows. Let $\Sigma a_k z^k$ be a formal power series, whose « sum » we wish to define. Let $\{ \mu_k \}_{k=0}^{\infty}$ be a sequence of strictly positive numbers given as the moments of some function on $\mathbb{R}_+$: $\mu_k = \int_0^\infty t^k \rho(t) dt$. We say that the formal power series $\Sigma a_k z^k$ is « μ-ρ-Borel summable » if:

a) $B(t) = \sum_{k=0}^{\infty} a_k(\mu_k)^{-1} t^k$ converges in some disk $|t| < \varepsilon$;

b) $B(t)$ has an analytic continuation to a neighbourhood of the positive real axis;

c) $g(z) = z^{-1} \int_0^\infty B(t)\rho(tz^{-1})dt$ converges for some $z \neq 0$.

$B(t)$ is called the « μ-ρ-Borel transform » of the series $\Sigma a_k z^k$ and $g(z)$ is its « μ-ρ-Borel sum ». The ordinary Borel summation method takes $\mu_k = k!$, $\rho(t) = e^{-t}$. The Borel-Le Roy method of order (α, β) takes $\mu_k = \Gamma(\alpha k + \beta)$, $\rho(t) = \alpha^{-1} \exp(-t^{1/\alpha}) t^{(\beta/\alpha)-1}$.

In the present article we study the method (Ref. 3, pp. 84-86) defined by $\mu_k = (\pi/\alpha)^{1/2} e^{k^2/4\alpha}$, $\rho(t) = t^{-1} e^{-\alpha(\log t)^2}$, for fixed $\alpha > 0$. We prove a criterion for this summation method analogous to the Watson-Nevanlinna theorem (Ref. 6, 11, 15, 17) for the ordinary Borel (or Borel-Le Roy) summation method. More precisely, let $f(z)$ be a function analytic on the Riemann surface of the logarithm for $0 < |z| < R_0$, which has $\Sigma a_k z^k$ as its asymptotic series in a suitably strong sense (condition (2) below). Then the μ-ρ-Borel sum $g(z)$ (see the expression (14), or the equivalent one (5)) exists and equals the original function $f(z)$. Moreover, a remainder estimate of the type (2) is a necessary condition, too, so a large class of such logarithmic Borel sums is characterized.

The techniques of the proof are similar to the ones in Ref. 11, 15 after the choice of a suitable new variable and a redefinition of the function and of its transform. Besides, for what concerns the estimate on the remainders, in this case the problem is to combine the divergence in the index and the radial behaviour with a necessary angular divergence.

For what concerns the above quoted models, we do not give here the applicability result because the usual operator theory techniques appear inadequate to verify all the criterion hypotheses. On the other hand, by a heuristic analytis and by partial results it seems plausible to us that the proved criterion can be applied in these cases. We quote some summability examples from the context of simplified models recently introduced [*9*], and for the characteristic function of a probability distribution introduced by Kolmogorov [*4*], relating to the dimensions of particles yielded by a body pulverization.

In Section II the summability theorem is proved and an example and some remarks are given. In Section III possible applications to quantum mechanics are discussed.

II. LOGARITHMIC BOREL SUMMABILITY

THEOREM. — Let $f(z)$ be analytic on the Riemann surface of the logarithm for $-\infty < \mathrm{Re}(\log z) < c_0$ (for some $c_0 \in \mathbb{R}$) and let it satisfy the following properties:

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z) \tag{1}$$

$$| R_N(z) \leq A\delta^N e^{N^2/4\alpha} | z |^N e^{\alpha\theta^2 - 2\alpha\phi_0|\theta|} \tag{2}$$

where the positive constants A, δ, α, ϕ_0 ($\phi_0 < \pi$) are independent of $| z |$, of $\theta = \arg(z)$ and of $N \in \mathbb{N}$. Then the series

$$B(t) = \sum_{k=0}^{\infty} (\alpha/\pi)^{1/2} a_k e^{-k^2/4\alpha} t^k \tag{3}$$

is convergent for $| t | < \delta^{-1}$ and has an analytic continuation to a region $S_t(\delta, \phi_0) = \{ t/ | t | < \delta^{-1}$ or $| \arg(t) | < \phi_0 \}$. This region corresponds, in the variable $x = -2\log t$, to $S_x(\delta, \phi_0) = \{x/\mathrm{Re}\, x > 2\log\delta$ or $| \mathrm{Im}\, x | < 2\phi_0\}$. If $\beta(x)$ is the analytic continuation of $(1/2)e^{-\alpha x^2/4}B(e^{-x/2})$ then:

$$| \beta^{(m)}(x) | \leq A_1(A_2)^m m! \, | c |^m e^{\alpha c^2} e^{\alpha c\, (\mathrm{Re}\, x)} \tag{4 a}$$

uniformly for $m \in \mathbb{N}_0$, $| \mathrm{Im}\, x | \leq 2\phi_1$ and $c \in (-\infty, c_0)$ (for fixed $\phi_1 < \phi_0$). Moreover

$$| \beta(x) | \leq D_0 | e^{-\alpha x^2/4} | \tag{4 b}$$

uniformly for $\mathrm{Re}\, x \geq x_0$ (for fixed $x_0 > 2\log\delta$).

Besides, $f(z)$ can be represented as an absolutely convergent integral:

$$f(z) = e^{-\alpha(\log z)^2} \int_{-\infty}^{+\infty} e^{-\alpha(\log z)x} \beta(x) dx \tag{5}$$

for all z such that $-\infty < \mathrm{Re}(\log z) < c_0$.

Conversely, if the function $\beta(x)$ is such that $B(t) = 2e^{\alpha(\log t)^2}\beta(-2\log t)$ is analytic (for suitable α) in the region $S_t(\delta, \phi_0)$, and if it satisfies (4a)-(4b), then the function $f(z)$ defined by (5) is analytic in the region

$$-\infty < \mathrm{Re}(\log z) < c_0$$

and satisfies a bound of the type (2) with formal expansion coefficients

$$a_k = (\pi/\alpha)^{1/2} e^{k^2/4\alpha}(k!)^{-1} B^{(k)}(0), \; k = 0, 1, 2, \ldots$$

Proof. — Let us consider the integral obtained from (5) by the Riemann-

Fourier inversion formula of the two-sided Laplace transformation (see e. g. Ref. 2):

$$\beta(x) = (2\pi i)^{-1}\alpha \int_{-i\infty+c}^{+i\infty+c} e^{\alpha x u} e^{\alpha u^2} f(e^u) du \tag{6}$$

where the integration variable is $u = \log z$, and $c \in (-\infty, c_0)$. By hypotheses (1), (2) the integral in (6) is convergent for all x such that $|\operatorname{Im} x| < 2\phi_0$, and for such x is independent of $c \in (-\infty, c_0)$. In fact by using (1), (2) with $N = 1$, and setting $u = c + i\theta$, we have:

$$|\beta(x)| \leq (\alpha/2\pi) \int_{-\infty}^{+\infty} e^{\alpha(c \operatorname{Re} x - \theta \operatorname{Im} x)} e^{\alpha(c^2-\theta^2)} \{ |a_0| + A\delta e^{1/4\alpha} e^c e^{\alpha(\theta^2 - 2\phi_0|\theta|)} \} d\theta$$
$$\leq D(\phi_1) e^{\alpha c^2} e^{\alpha c (\operatorname{Re} x)} \tag{7}$$

if we consider all x such that $|\operatorname{Im} x| \leq 2\phi_1$, with $\phi_1 < \phi_0$, and $c \in (-\infty, c_0)$. Notice that for $\operatorname{Re} x > -2c_0$, we can choose $c = -\frac{1}{2}\operatorname{Re} x$ and thereby obtain $e^{\alpha c^2} e^{\alpha c (\operatorname{Re} x)} \leq e^{-\alpha (\operatorname{Re} x)^2/4}$. Inserting (1) into (6) (now with arbitrary N),

$$\beta(x) = (\alpha/2\pi i) \int_{-i\infty+c}^{+i\infty+c} e^{\alpha x u} e^{\alpha u^2} \sum_{k=0}^{N-1} a_k e^{ku} du$$
$$+ (\alpha/2\pi i) \int_{-i\infty+c}^{+i\infty+c} e^{\alpha x u} e^{\alpha u^2} R_N(e^u) du \tag{8}$$

The first term in (8), computed by writing $u = c + i\theta$ and performing the Gaussian integral, gives $(1/2)e^{-\alpha x^2/4} \sum_{k=0}^{N-1} (\alpha/\pi)^{1/2} a_k e^{-k^2/4\alpha} e^{-kx/2}$. Therefore by hypothesis (2) again setting $u = c + i\theta$, (8) implies:

$$\left| \beta(x) - (1/2)e^{-\alpha x^2/4} \sum_{k=0}^{N-1} (\alpha/\pi)^{1/2} a_k e^{-k^2/4\alpha} e^{-kx/2} \right|$$
$$\leq (A\alpha/2\pi) \int_{-\infty}^{+\infty} e^{\alpha(c \operatorname{Re} x - \theta \operatorname{Im} x)} e^{\alpha c^2} \delta^N e^{N^2/4\alpha} e^{Nc} e^{-2\alpha\phi_0|\theta|} d\theta \to 0 \tag{9}$$

as $N \to \infty$ whenever both $\operatorname{Re} x > 2 \log \delta$ and $|\operatorname{Im} x| < 2\phi_0$: it is sufficient to choose $c = -N/2\alpha$, so that

$$e^{\alpha c (\operatorname{Re} x)} e^{(\sqrt{\alpha c} + N/2\sqrt{\alpha})^2} \delta^N = e^{-N(\operatorname{Re} x)/2} e^{N(\log \delta)} \to 0 \quad \text{if} \quad \operatorname{Re} x > 2 \log \delta .$$

On the other hand this series is just convergent in the half-plane $\operatorname{Re} x > 2 \log \delta$, by the bound $|a_k| \leq A\delta^k e^{k^2/4\alpha}$ implicit in (2). So the function $\beta(x)$ defined by (6) is analytically continuable to a right half-plane, where it is uniquely determined by the sequence $\{ a_k \}_{k \in \mathbb{N}}$.

Let us consider the m-th derivative of $\beta(x)$:

$$\beta^{(m)}(x) = (\alpha/2\pi i) \int_{-i\infty+c}^{+i\infty+c} e^{\alpha xu} (\alpha u)^m e^{\alpha u^2} f(e^u) du . \tag{10}$$

The integral in (10) exists for any x such that $| \operatorname{Im} x | < 2\phi_0$ and, by hypotheses (1), (2) with $N = 1$, it satisfies the bound

$$| \beta^{(m)}(x) | \leq (\alpha/2\pi) \int_{-\infty}^{+\infty} e^{\alpha(c \operatorname{Re} x - \theta \operatorname{Im} x)} | \alpha(c + i\theta) |^m e^{\alpha(c^2 - \theta^2)} \{ | a_0 | + A\delta e^{1/4\alpha} e^c e^{\alpha(\theta^2 - 2\phi_0 |\theta|)} \} d\theta \leq A_1 (A_2)^m m! \, | c |^m e^{\alpha c^2} e^{\alpha c (\operatorname{Re} x)} \tag{11}$$

where A_1, A_2 are independent of m, c and x in the strip $| \operatorname{Im} x | < 2\phi_1$, for any fixed $\phi_1 < \phi_0$.

Thus the function $\beta(x)$ is differentiable to all orders in the strip $| \operatorname{Im} x | < 2\phi_0$ and the Taylor series expansion

$$\beta_{x_1}(x) = \sum_{m=0}^{\infty} (m!)^{-1} \beta^{(m)}(x_1)(x - x_1)^m \tag{12}$$

is convergent in some neighbourhood of x_1, for any x_1 in the strip. Moreover $\beta_{x_1}(x) = \beta_{x_2}(x)$ if x belongs to both the convergence disks. So the analytic continuation to the whole strip $| \operatorname{Im} x | < 2\phi_0$ is single valued and, by (9), $\beta(x)$ is uniquely determined by the coefficients of the formal expansion (1). Besides, $\beta(x)$ satisfies the bounds (4): indeed (4a) is proved by (11), while by (9) we have

$$| \beta(x) | \leq D'_0 \, | e^{-\alpha x^2/4} | (1 - \delta e^{-(\operatorname{Re} x)/2})^{-1} \leq D_0 \, | e^{-\alpha x^2/4} | \tag{13}$$

uniformly for $\operatorname{Re} x \geq x_0$ (with $x_0 > 2 \log \delta$).

As a consequence, inserting $\beta(x)$ in (5), the integral is absolutely convergent at $-\infty$ by (4 a) and absolutely convergent at $+\infty$ by (4 b). On the other hand the integral in (6) is absolutely convergent by (7): therefore, by the inversion formula of the Laplace transform [*2*], inserting (6) in (5) we have an identity and the criterion is proved.

Conversely, let $\beta(x)$ be given, such that $B(t) = 2e^{\alpha(\log t)^2} \beta(-2 \log t)$ is analytic in the region $S_t(\delta, \phi_0)$ (for suitable $\alpha > 0$), and let it satisfy (4 a)-(4 b). Then the function defined by (5) is clearly analytic for $-\infty < \operatorname{Re}(\log z) < c_0$. By the substitution $x = -2 \log t$, $f(z)$ can be represented in the form

$$f(z) = \int_0^{\infty} B(te^{i\phi}) e^{-\alpha(\log tz^{-1} + i\phi)^2} t^{-1} dt \tag{14}$$

for any ϕ such that $|\phi| < \phi_0$, since B(t) is analytic in the sector $|\arg(t)| < \phi_0$.

Setting $S_N(te^{i\phi}) = B(te^{i\phi}) - \sum_{k=0}^{N-1} (\alpha/\pi)^{1/2} a_k e^{-k^2/4\alpha}(te^{i\phi})^k$ we have:

$$
\begin{aligned}
| R_N(z) | &= \left| \int_0^\infty e^{-\alpha(\log t - \log z + i\phi)^2} S_N(te^{i\phi}) t^{-1} dt \right| \\
&\leq e^{\alpha(\theta-\phi)^2} \int_0^\infty e^{-\alpha(\log t - \log|z|)^2} t^N (N!)^{-1} | B^{(N)}(\varepsilon te^{i\phi}) | t^{-1} dt \quad (15)
\end{aligned}
$$

for some $\varepsilon = \varepsilon(t)$, $0 \leq \varepsilon \leq 1$. Now, since B($t$) is analytic for $|t| < \delta^{-1}$, we have, by Cauchy's integral formula $(N!)^{-1} | B^{(N)}(\varepsilon te^{i\phi}) | \leq D_1(2\delta)^N$ on the set $\Delta_1 = \{ t/0 < \varepsilon t < (3\delta)^{-1} \}$. Moreover, $B^{(N)}(t) = \frac{d^N}{dt^N}(e^{\alpha(\log t)^2}\beta(-2\log t))$ can be bounded by means of (4 a), taking into account that $x = -2\log t$. In general, setting $y = \log t$, we have

$$
\begin{aligned}
\left| \frac{d^r}{dt^r} f(\log t) \right| &= \left| \left(e^{-y}\frac{d}{dy} \right)^r f(y) \right| \\
&= \left| e^{-ry}\left(\frac{d}{dy} - r + 1 \right)\left(\frac{d}{dy} - r + 2 \right) \dots \frac{d}{dy} f(y) \right| \\
&\leq e^{-ry} \sum_{k=1}^{r} 2^{r-1} r^{r-k} | f^{(k)}(y) | \\
&\leq e^{-ry} \sum_{k=1}^{r} 2^{r-1} r^r k^{-k} | f^{(k)}(y) | .
\end{aligned}
$$

On the other hand (e. g. Ref. 5, p. 1033)

$$
\left| \frac{d^k}{dy^k} e^{y^2} \right| = | H_k(iy) e^{y^2} | \leq 2^{2k} | e^{y^2} | \sum_{j=0}^{[k/2]} j! \, | y |^{k-2j} .
$$

By combining everything one has the estimate:

$$
| B^{(N)}(t) | \leq e^{\alpha(\log t)^2} D^N t^{-N} \sum_{r=0}^{N} \binom{N}{r} (N - r)! \sum_{j=0}^{r} r!(j!)^{-1} | \log t |^j e^{-2\alpha c(\log t)} \quad (16)
$$

Since the right hand side of (16) is monotone increasing for large t and monotone non-increasing for $t < 1$, and since $0 \leq \varepsilon \leq 1$, setting

$$\Delta_2 = \{ t/(3\delta)^{-1} < \varepsilon t < 1 \},$$

by (15) and (16) we have:

$$\begin{aligned}
|R_N(z)| \leq & e^{\alpha(\theta-\phi)^2}\left\{ \int_{\Delta_1} e^{-\alpha(\log t|z|^{-1})^2} D_1(2\delta)^N t^{N-1} dt \right. \\
& + \int_{\Delta_2} e^{-\alpha(\log t|z|^{-1})^2} t^{N-1} \sum_{r=0}^{N} D^N \sum_{j=0}^{r} (j!)^{-1} e^{\alpha(\log 3\delta)^2} (3\delta)^N |\log 3\delta|^j dt \\
& \left. + \int_1^\infty e^{-\alpha(\log|z|)^2} e^{2\alpha(\log|z|)(\log t)} D_2 D^N \sum_{r=0}^{N} \sum_{j=0}^{r} (j!)^{-1} |\log t|^j e^{-2\alpha c(\log t)} t^{-1} dt \right\} \\
\leq & e^{\alpha(\theta-\phi)^2} \{ D_3(D_4)^N e^{N^2/4} |z|^N + (D_5)^N e^{-\alpha(\log|z|)^2} \} \\
\leq & e^{\alpha(\theta-\phi)^2} D_6(D_7)^N e^{N^2/4} |z|^N
\end{aligned} \tag{17}$$

where the constants are independent of z and N. Choosing $\phi = \phi_1$ $(0 < \phi_1 < \phi_0)$ when $\theta > 0$ and $\phi = -\phi_1$ when $\theta < 0$, (17) is an estimate of the required type and the theorem is proved.

REMARK 1. — As a consequence, one has the following annulment criterion: if $f(z)$ is analytic for $0 < |z| < R_0$ on a logarithmic Riemann surface and, for A, α, ϕ_0 independent of z,

$$|f(z)| \leq A\, |e^{-\alpha(\log z)^2}|\, e^{-2\alpha\phi_0|\theta|} \tag{18}$$

then $f(z) = 0$. In fact (18) implies that $a_k = (k!)^{-1} f^{(k)}(0+) = 0$ for all k, while (2) is satisfied since $e^{-\alpha(\log|z|)^2} \leq |z|^N e^{N^2/4\alpha}$, $\forall N \in \mathbb{N}$. Analytic non-zero functions on these domains, such as $e^{\pm(\log z)^n}$, $n \in \mathbb{N}$, fail to satisfy (18) uniformly with respect to $|z|$ and θ.

Example. — The function $f(z) = \int_0^\infty e^{-(\log t)^2}(1 + tz)^{-1} dt$, $|\arg(z)| < \pi$, with formal expansion, $\Sigma a_k z^k$, $a_k = (-1)^k e^{(k+1)^2/4} \pi^{1/2}$, satisfies the hypotheses of the above theorem and $\Sigma a_k z^k$ admits a unique sum given by (5), with $\alpha = 1$ and $\beta(x) = (1/2)e^{1/4} e^{-x^2/4}(1 + e^{(1-x)/2})^{-1}$. In the context of the theory of Stieltjes summability, we can consider the function

$$f_0(z) = \int_0^\infty e^{-(\log t)^2} \sin(2\pi \log t)(1 + tz)^{-1} dt,$$

which fails to satisfy hypothesis (2). The moments

$$\mu_k = \int_0^\infty t^{-\log t} \sin(2\pi \log t)(-t)^k dt = 0$$

(in Ref. 13 and 16) can be thought as the formal expansion coefficients of $f_0(z)$.

Both $f(z)$ and $f(z) + f_0(z)$ are Stieltjes functions, and they have the same asymptotic series expansion. Thus $\Sigma a_k z^k$ is a Stieltjes series, which turns out not to be Stieltjes summable [*13*]. However, as above remarked, it is summable by this generalized Borel method.

REMARK 2. — This summation method is regular, since it obviously sums a convergent series in the interior of the convergence circle. However in general it cannot be used to analytically continue the function beyond the convergence radius. Indeed Hardy shows (Ref. 6, p. 85) that it cannot sum Σz^k for $z = -1$. This is not surprising, since the Borel-Le Roy sum, of index $n > 2$, of this series: $(1 - z)^{-1} = \int_0^\infty e^{-t} \sum_{k=0}^\infty ((nk)!)^{-1}(t^n z)^k dt$ is valid in the interior of a compact set $(|z| \leq (\cos(\theta/n))^{-n}$ for $|\theta| \leq \pi$: see Ref. 6, p. 197), which tends to the disk of radius 1 as $n \to \infty$.

III. APPLICABILITY

(A) The above mentioned operator $H(z) = p^2 + x^2 + ze^x$ in $L^2(\mathbb{R})$ with its formal perturbation series, displays the difficulty of application of this criterion in the usual framework of operator theory, while this application appears reasonable by the partial results obtained and by the foregoing discussion (in (B)) of a simplified model.

Recall [*10*] that $H(z)$ is a compact resolvent operator for $|\arg(z)| < \pi$, and the eigenvalues are stable as $z \mapsto 0$ by norm resolvent convergence. By a translation $x \to (x - \log z)$, or $x \to (x - i\theta)$, the eigenvalues can be continued to a logarithmic Riemann surface in such a way, however, that they are not controlled as $|z|$ is kept constant and $\theta = \arg(z) \to \pm\infty$. A growth of the function, for $|z|$ fixed, of the order of $e^{\theta^2 - 2\phi_0|\theta|}$ is allowed by hypothesis (2) of the above theorem. On the other hand an usual estimate by projection operators (which are Cauchy integrals) would presuppose the perturbed eigenvalue to be near the unperturbed one uniformly as $\theta \to \pm\infty$.

However by an estimate in any fixed angular sector $|\theta| \leq \theta_0$, one checks just the bound on the remainder required by (2) for the behaviour with respect to $|z|$. Indeed, the actual eigenvalues of

$$\mathrm{H}(-i\theta, |z|) = p^2 + (x - i\theta)^2 + |z| e^x$$

are the analytic continuation of the eigenvalues of H(z) beyond the first sheet. For $|\theta| \leq \theta_0$, $\mathrm{R}_z = (\mathrm{H}(-i\theta, |z|) - \mathrm{E})^{-1}$ is uniformly norm convergent as $z \hookrightarrow 0$. Thus it is sufficient to evaluate (see e. g. Ref. 12, 7) the norm of the vectors:

$$(e^x \mathrm{R}_0)^{[(\mathrm{N}+1)/2]} \psi_k = (e^x(\mathrm{H}(-i\theta, 0) - \mathrm{E})^{-1})^{[(\mathrm{N}+1)/2]} \psi_k(x - i\theta),$$
$$(e^x \mathrm{R}_0^*)^{[\mathrm{N}/2]} e^x \mathrm{R}_z^* \overline{\psi}_k$$

in order to estimate the perturbation series remainders. Setting $w = e^{if(x-i\theta)}$, $f(x) = (i/2j)(x^2 - (k+1)\log(2 + x^2))$ we have:

$$(e^x \mathrm{R}_0)^j \psi_k = (e^x w)(w^{-1} \mathrm{R}_0 w)(e^x w)(w^{-2} \mathrm{R}_0 w^2) \ldots (e^x w)(w^{-j} \mathrm{R}_0 w^j) w^{-j} \psi_k$$

and analogously for $(e^x \mathrm{R}_0^*)^{j-1} e^x \mathrm{R}_z^* \overline{\psi}_k$. Now, one checks that $\| w^{-s} \mathrm{R}_z w^s \| \leq c$, for some $c > 0$ and for $1 \leq s \leq j$. Indeed we have

$$w^{-s} \mathrm{R}_z w^s = (w^{-s} \mathrm{H}(-i\theta, |z|) w^s - \mathrm{E})^{-1}$$

and

$$w^{-s} \mathrm{H}(-i\theta, |z|) w^s = \left(p + s\frac{df}{dx}\right)^2 + (x - i\theta)^2 + |z| e^x$$

has invariant domain and numerical ranges contained in the half-plane $\mathrm{Re}\,\mathrm{E} \geq -\theta^2 - c(k)$, $c(k) \geq 0$, for $0 \leq s \leq j$.

Moreover $|e^x w|^j \leq \mathrm{A} j^{k+1} e^{j^2/2} \leq \mathrm{A} \mathrm{B}^j e^{j^2/2}$.

It follows that the remainders are uniformly bounded by $c' c^{\mathrm{N}} e^{\mathrm{N}^2/4} |z|^{\mathrm{N}}$, when $|\theta| \leq \theta_0$, for any fixed θ_0, so that the required bound (2) holds at least in this region.

(B) Instead of single eigenvalues, let us consider another unitary invariant of H(z), that is $\mathrm{Tr}\,(e^{-t\mathrm{H}(z)})$. To calculate it, one can use the Trotter formula for the kernel of $e^{-it\mathrm{H}(z)}$:

$$\mathrm{K}(x, y; t) = \lim_{n \to \infty} \mathrm{K}^{(n)}(x, y; t)$$

where the $\mathrm{K}^{(n)}$'s are defined in Ref. 14 (p. 6). Following G.'t Hooft [9] we can consider a simplified model, in which K is replaced by $\mathrm{K}^{(1)}$. The trace corresponding to $\mathrm{K}^{(1)}(x, y; -it)$ is given by

$$\mathrm{T}_t(z) = \int_{-\infty}^{+\infty} \mathrm{K}^{(1)}(x, x; -it)dx = \int_{-\infty}^{+\infty} (2\pi t)^{-1/2} e^{-t(x^2 + ze^x)} dx.$$

For each $t > 0$, the formal expansion coefficients are

$$a_k = -(-t)^{k-1}(k!)^{-1} 2^{-1/2} e^{k^2/4t}:$$

$T_t(z)$ satisfies the hypotheses of the above theorem and admits the representation (5) (with $\alpha = t$, $\beta(x) = (2\pi t)^{-1/2}(1/2)e^{-tx^2/4}e^{-te^{-x/2}}$) uniquely determined by the a_k's. Notice that to our knowledge all the quantum mechanical models admitting usual Borel summability after such simplification, are actually proved to be Borel summable (e. g. $p^2 + x^2 + zx^{2n}$) [*12*]. As far as quantum field theories are concerned, the summability method could perhaps be applied to exponential interaction models introduced in Ref. 8, 1.

(C) Let us finally quote an example provided by the characteristic function of the logarithmic normal probability distribution defined as

$$\rho(t) = \begin{cases} 0 & (t < 0) \\ (2\pi)^{-1/2}(\sigma t)^{-1} \exp\left(-(2\sigma^2)^{-1}(a - \log t)^2\right) & (t > 0) \end{cases}$$

introduced by Kolmogorov (see Ref. 4, p. 32, and related references) for the dimensions of particles yielded by a body pulverization. If we consider the characteristic function of such probability, $f(z) = \int_0^\infty e^{itz}\rho(t)$, it satisfies the hypotheses of the theorem. As a consequence, the moments of such measure, simply related to the formal expansion coefficients of $f(z)$, uniquely determine the characteristic function via the logarithmic Borel sum.

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