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Quantum detailed balance

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ABSTRACT. — For infinite quantum systems a family of dissipative maps satisfying the condition of detailed balance is constructed. We conjecture that these maps provide a new way of characterizing the equilibrium states. We prove the conjecture for finite systems and for the infinite free Fermion gas.

RÉSUMÉ. — On construit une famille d'applications dissipatives satisfaisant la condition du bilan détaillé pour des systèmes quantiques infinis. On conjecture que ces applications fournissent une nouvelle caractérisation des états d'équilibre. On démontre la conjecture pour des systèmes finis et pour le gaz de Fermions libres infini.

I. INTRODUCTION

Classical lattice N-level systems do not have a natural time-evolution. However there is a definite notion of equilibrium state, as the generalization of the Gibbs state or Gibbs measure.

Attempts to characterize the Gibbs measures lead to the introduction of Markov processes of which the so-called Glauber dynamics [*1*] is a

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widely used example. With this type of time evolutions one studies fluctuations from equilibrium. One considers a system which is perturbed, either externally or by internal variations and observes the rate and manner of return to equilibrium. Within the possible models of fluctuation from equilibrium it is natural to choose those which satisfy the property of reversibility as it is called in the mathematical literature [*2*] [*3*].

Considerations of reversibility have been employed by physicists under the name of the principle of detailed balance. We continue to use the latter terminology. The Glauber dynamics satisfies the condition of detailed balance. Interesting properties were derived for this dynamics [*4*]. We recall the result that every space-homogeneous state converges weakly to an equilibrium state. Therefore, this kind of dynamics has given an intrinsic value to the condition of detailed balance, in such a way that it looked interesting to study evolutions satisfying this condition also for systems which dơ have a natural Hamiltonian evolution like classical continuous systems and quantum systems. Relevant problems are then: what is the explicit form of evolutions satisfying the detailed balance condition, what is their relation to the conservative dynamics, what about return to equilibrium, etc.?

In this paper we confine ourselves to quantum systems.

Up to now the most convenient mathematical description of irreversible dynamics for quantum systems is provided by the theory of dynamical semigroups [*5*] [*6*].

Although the semigroup property, or physically the absence of memory, is to be considered as an approximation holding in some limiting situations [*7*]-[*11*], it is used in a satisfactory way in many physical applications. In particular the dynamical semigroups have shown to be a powerful mathematical tool to derive results in equilibrium statistical mechanics [*12*] [*13*].

Valuable attempts gave rise to a rigorous mathematical definition of the property of detailed balance for quantum systems [*9*] [*10*] [*11*] [*14*]. Apart from the case of finite systems (i. e. systems with a discrete energy spectrum) one never exhibited an explicit form of such a dynamics. Therefore there was no hope of proving results of the type mentioned above, namely the characterization of a KMS-state of the system as a state linked to an irreversible evolution.

In this paper we define explicitly for the infinite quantum systems an infinite set of dissipative maps satisfying the detailed balance condition.

For finite systems it is proved (see [*11*]) that the detailed balance condition provides a characterization of the equilibrium states (KMS-states) of a heat bath.

Here we consider only closed systems and conjecture that every state which is left invariant under the set of dissipative maps satisfying the detailed balance condition is necessarily an equilibrium state. We prove this conjecture for finite systems and for the infinite free Fermion system. We are unable to prove it yet in its full generality.

II. DISSIPATIVE MAPS SATISFYING THE CONDITION OF DETAILED BALANCE

An infinite quantum dynamical system is given by a pair $(\mathscr{A}, (\tau_t)_{t\in\mathbb{R}})$ where $\mathscr{A}$ is a (non-abelian) C*-algebra with unit, and $t \in \mathbb{R} \rightarrow \tau_t$ a strongly continuous representation of $\mathbb{R}$ into the group of $*$-automorphisms of $\mathscr{A}$.

The set $\mathscr{A}_\tau$ of entire analytic elements for τ is then a norm-dense τ-invariant $*$-subalgebra of $\mathscr{A}$.

A state ω of $\mathscr{A}$ satisfies the (τ, β)—KMS—condition (for the evolution τ at inverse temperature $\beta \geqslant 0$) if the following holds:

$$\omega(x\tau_{i\beta}y) = \omega(yx), \qquad \text{for all } x, y \in \mathscr{A}_\tau .$$

A linear map L from a dense $*$-subalgebra into $\mathscr{A}$ is called dissipative if

$$L(1) = 0$$
$$L(x^*) = L(x)^*$$
$$L(x^*x) - L(x^*)x - x^*L(x) \geqslant 0$$

for all x in the domain of L.

Under supplementary conditions L is the generator of a Markov semigroup. In the following we work on the level of the generators and disregard the problem of exponentiation.

DÉFINITION II.1. — Let L be a dissipative linear map. L satisfies the condition of (τ, β)-detailed balance (at inverse temperature $\beta \geqslant 0$ and for the evolution τ), whenever the following holds

$$\omega(xL(y)) = \omega(L(x)y)$$

for all (τ, β)-KMS-states ω and all x, y in the domain of L.

Now our aim is, to construct for a given evolution τ and a fixed inverse temperature $\beta \geqslant 0$ a class $\mathscr{L}(\tau, \beta)$ of dissipative maps, satisfying the condition of (τ, β)-detailed balance.

We want the class $\mathscr{L}(\tau, \beta)$ to be large enough to characterize equilibrium, in the following way: Any state ω which is left invariant by all the dissipative maps of $\mathscr{L}(\tau, \beta)$ (i. e. $\omega \circ L = 0$ for all $L \in \mathscr{L}(\tau, \beta)$), is necessarily a (τ, β)-KMS-state. The class of dissipative maps which we are going to define appears to be the generalization of the well known solutions for systems with discrete energy spectrum [9] [10] [11] [14].

In the case of the infinite system we have to impose the condition of

L^1-asymptotic abelianness on the dynamical systems which we consider, specifically we assume from now on that there exists a norm dense $*$-subalgebra $\mathscr{A}_0$ such that

i) $$\tau\mathscr{A}_0 \subseteq \mathscr{A}_0 \subseteq \mathscr{A}_\tau,\ 1 \in \mathscr{A}_0$$

ii) for all $x, y \in \mathscr{A}_0$: $\displaystyle\int_{-\infty}^{\infty} ds \, \| [\tau_s x, y] \| < \infty$.

Define $\mathscr{L}(\tau, \beta)$ to be the convex hull of the set of linear maps L_x^f of $\mathscr{A}_0$ into $\mathscr{A}$ of the form

$$L_x^f(y) = \int dt ds f(t)(\tau_s x [y, \tau_{s+t} x] + [\tau_s x, y]\tau_{s+t} x) \quad . \tag{2}$$

where the integrals are in the norm sense and where $x = x^* \in \mathscr{A}_0$ and f is in the set $\mathscr{F}_\beta$ of functions which are analytic in the strip $D_\beta = \{ z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \beta \}$ continuous and bounded on the closure $\overline{D}_\beta$ and such that

a) $\forall t \in \mathbb{R} : f(-t + i\beta) = f(t)$
b) $t \in \mathbb{R} \ \to \ f(t)$ is of positive type
c) $t \in \mathbb{R} \ \to \ f(t)$ is an $L^1(\mathbb{R}, dt)$ function.

Finally we close $\mathscr{L}(\tau, \beta)$ for the strong topology (i. e. $L_\alpha \to L$ if $\| L_\alpha(x) - L(x) \| \to 0$ for all $x \in \mathscr{A}_0$). We continue to denote this closure by $\mathscr{L}(\tau, \beta)$.

Now we prove:

THÉORÈME II.2. — Let $(\mathscr{A}, \tau)$ be an L^1-asymptotic abelian dynamical system, then all elements of $\mathscr{L}(\tau, \beta)$ are dissipative maps satisfying the (τ, β)-detailed balance condition.

Proof. — First check that for all $x = x^* \in \mathscr{A}_0$ and $f \in \mathscr{F}_\beta$:

$$L_x^f(y^* y) - L_x^f(y^*)y - y^* L_x^f(y) = 2\int dt ds f(t-s)[y, \tau_s x]^* [y, \tau_t x] \geqslant 0$$

which follows from the fact that $t \to f(t)$ is of positive type.

Hence L_x^f is dissipative.

Before proceeding to the proof of the (τ, β)-detailed balance condition remark the following: let F be an analytic function in D_β, continuous and bounded on the strip $\overline{D}_\beta$ such that

$$t \to F(t) \text{ is } L^1$$

and

$$t \to F(t + i\beta) \text{ is } L^1$$

then

$$\int_{\mathbb{R}} F(t) dt = \int_{\mathbb{R}} F(t + i\beta) dt \, . \tag{3}$$

Indeed let $g_n(z) = e^{-z^2/n^2}$, $n \in \mathbb{N}$, then for each $n \in \mathbb{N}$

$$\int_{\mathbb{R}} F(t) g_n(t) dt = \int_{\mathbb{R}} F(t + i\beta) g_n(t + i\beta) dt$$

by Cauchy's theorem. Then (3) follows by Lebesgue's dominated convergence theorem.

Now consider

$$\int dt ds F(t, s)$$

where

$$F(t, s) = f(t)(\omega(\tau_{s-t} x [z, \tau_s x] y) - \omega(z [\tau_s x, y] \tau_{s+t} x)).$$

Since $x, y, z \in \mathscr{A}_0$, $f \in \mathscr{F}_\beta$ the function $t \in \mathbb{R} \to F(t, s)$ extends for all $s \in \mathbb{R}$ to a bounded and continuous function on $\overline{D}_\beta$, which is analytic in D_β. Furthermore as $f \in \mathscr{F}_\beta$

$$t \in \mathbb{R} \to F(t, s)$$

and

$$t \in \mathbb{R} \to F(t + i\beta, s)$$

are L^1-functions.

Hence by the remark (3)

$$\int_{\mathbb{R}} dt F(t, s) = \int_{\mathbb{R}} dt F(t + i\beta, s). \tag{4}$$

Since ω is (τ, β)-KMS and since $f \in \mathscr{F}_\beta$

$$F(t + i\beta, s) = f(-t)(\omega [z, \tau_s x] y \tau_{s-t} x) - \omega(\tau_{s+t} x z [\tau_s x, y])) \tag{5}$$

Now one checks that

$$\omega(L_x^f(z) y) - \omega(z L_x^f(y))$$
$$= \int dt ds (F(t, s) + f(t)(\omega([\tau_s x, z] \tau_{s+t} x y) - \omega(z \tau_s x [y, \tau_{s+t} x]))).$$

Using (4) and (5) one gets

$$\omega(L_x^f(z) y) - \omega(z L_x^f(y)) = 0.$$

The statement follows from a continuity argument. ■

Here we want to make some remarks about the connection of our presentation and result with the existing ones.

As mentioned above the notion of quantum detailed balance was only studied for finite systems [*9*] [*10*] [*11*] [*14*]. In that case the authors introduced it with respect to a single state.

In our approach the definition is given for a dynamical system $(\mathscr{A}, \tau)$

and a fixed inverse temperature $\beta \geqslant 0$, which is more in the spirit of the current definitions for the classical lattice systems.

For the finite quantum systems, there exists a unique KMS-state at a given temperature, and hence both approaches coincide.

III. DETAILED BALANCE AND EQUILIBRIUM

In this section we investigate in which way the set of dissipative maps $\mathscr{L}(\tau, \beta)$, constructed in section II, characterize the equilibrium states for τ at inverse $\beta \geqslant 0$.

To fix our ideas we treat first the finite quantum system i. e. take $\mathscr{A} = \mathrm{M}_n$, the algebra of $n \times n$ complex matrices, $\tau_t = \exp it \operatorname{ad} \mathrm{H}$ where $\mathrm{H} = \mathrm{H}^*$ is any diagonal $n \times n$ matrix.

Let $\{ \mathrm{E}_{kl} ; k, l = 1, \ldots, n \}$ be the set of matrix units of M_n, then define the time evolution by:

$$\tau_t(\mathrm{E}_{kl}) = \mathrm{E}_{kl} \exp i(\varepsilon_k - \varepsilon_l)t$$

where $\{ \varepsilon_k ; k = 1, \ldots, n \}$ are the eigenvalues of H. In this case the set $\mathscr{L}(\tau, \beta)$ of dissipative (τ, β)-detailed balance generators consists of all generators of the type: for $f \in \mathscr{F}_\beta$, $x = x^* \in \mathrm{M}_n$:

$$\mathrm{L}_x^f(y) = \int dt \mathscr{M}_s f(t)(\tau_s x [y, \tau_{s+t} x] + [\tau_s x, y]\tau_{s+t} x), \; y \in \mathrm{M}_n$$

where $\mathscr{M}_s$ stands for the mean over the variable s.

Now we want to show that if the state ω is invariant under semigroups having their generators in $\mathscr{L}(\tau, \beta)$ then ω is a (τ, β)-KMS or equilibrium Gibbs state.

Suppose $\omega(\mathrm{L}_x^f(y)) = 0$ for all $f \in \mathscr{F}_\beta$ and $x = x^* \in \mathrm{M}_n$.
First note that ω is time-invariant i. e. $\omega(\mathrm{E}_{kl}) = 0$ for $k \neq l$. Indeed take $x = \mathrm{E}_{kk}$, $y = \mathrm{E}_{kl}$ $(k \neq l)$ and $f \in \mathscr{F}_\beta$ such that $\hat{f}(0) \neq 0$ where

$$\hat{f}(p) = \left(\frac{1}{2\pi}\right)^{1/2} \int e^{-itp} f(t) dt ,$$

then the invariance follows from the fact that $\mathrm{L}_x^f(y) = -(2\pi)^{1/2} \hat{f}(0) \mathrm{E}_{kl}$.

Furthermore, take $x = \mathrm{E}_{kl} + \mathrm{E}_{lk}$, $y = \mathrm{E}_{kk}$ and $f \in \mathscr{F}_\beta$ such that $f(\varepsilon_k - \varepsilon_l) \neq 0$, then

$$\mathrm{L}_x^f(y) = (2\pi)^{1/2} 2 \{ -\hat{f}(\varepsilon_k - \varepsilon_l)\mathrm{E}_{kk} + \hat{f}(\varepsilon_l - \varepsilon_k)\mathrm{E}_{kl} \} .$$

As $f(p) = e^{\beta p} f(-p)$, $\omega(\mathrm{L}_x^f(y)) = 0$ yields

$$\frac{\omega(\mathrm{E}_{kk})}{\omega(\mathrm{E}_{ll})} = \exp - \beta(\varepsilon_k - \varepsilon_l)$$

proving that ω is necessarily the equilibrium state.

Although this result is for finite systems it offers a new way of looking to the problem of return to equilibrium under semigroups satisfying the detailed balance condition.

As by now a class of dissipative maps satisfying the condition of detailed balance is available (formula (2)) we are tempted to conjecture that the above property holds on for infinite systems as well: i. e. if ω is a state of $\mathscr{A}$ such that $\omega \circ L_x^f = 0$ for all $x = x^* \in \mathscr{A}$ and all $f \in \mathscr{F}_\beta$, then, may be modulo some technicalities of a minor nature, ω is a (τ, β)-KMS-state.

We are yet unable to prove this statement in its full generality. However we can obtain complete results for the free infinite system implying the conjecture for this case.

We consider explicitly the example of the free evolution of a Fermion system. The essential ingredients related to our conjecture could be repeated for a Boson system. We leave the Bose system as an exercise to the reader.

We start with recalling some standard results about the CAR-algebra $\mathscr{A}$ over a complex Hilbert space $H = L^2(\mathbb{R}^\nu)$. It is a simple unital C*-algebra generated by bounded operators $a(\phi)$ and $a^+(\phi) = a(\phi)^*$ which are conjugate linear respectively linear with respect to $\phi \in H$ and satisfy the anti-commutation relations

$$a(\phi)a^+(\psi) + a^+(\psi)a(\phi) = (\phi, \psi)1$$
$$a(\phi)a(\psi) + a(\psi)a(\phi) = 0$$

for all $\phi, \psi \in H$.

There exists a $*$-automorphism θ of $\mathscr{A}$ such that $\theta^2 = 1$ and $\theta(a(\phi)) = -a(\phi)$ for all $\phi \in H$. We denote

$$\mathscr{A}_{\text{even}} = \{ x \in \mathscr{A} \mid \theta(x) = x \}$$
$$\mathscr{A}_{\text{odd}} = \{ x \in \mathscr{A} \mid \theta(x) = -x \}.$$

It is clear that every element $x \in \mathscr{A}$ can be written in a unique way as $x = x_e + x_0$ where $x_e \in \mathscr{A}_{\text{even}}$ and $x_0 \in \mathscr{A}_{\text{odd}}$.

The time evolution is described by a one-parameter group $(\tau_t)_{t \in \mathbb{R}}$ of $*$-automorphisms of $\mathscr{A}$ defined by

$$\tau_t a(\phi) = a(\phi_t) \qquad t \in \mathbb{R}, \phi \in H$$

where

$$\hat{\phi}_t(k) = e^{it\varepsilon(k)}\hat{\phi}(k).$$

For simplicity we take $\varepsilon(k) = k^2$. More general quasi-free evolutions could be analyzed in an analogous way.

Because of the special algebraic structure of the CAR-algebra we reformulate the condition of L^1-asymptotic abelianness as follows: for all x, y in a dense $*$-subalgebra $\mathscr{A}_0$:

$$\int ds \, \| [\alpha_s(x), y]_\pm \| < \infty$$

where

$$[x, y]_\pm = [x_e, y_e] + [x_0, y_e] + [x_e, y_0] - \{ x_0, y_0 \}$$

and $x = x_e + x_0$, $y = y_e + y_0$, $[\cdot, \cdot]$ is the commutator, $\{ \cdot, \cdot \}$ is the anticommutator.

For the free evolution this condition is satisfied if $\mathscr{A}_0$ is the $*$-subalgebra generated by $\{ a^+(\phi) \mid \phi \in H_0 \}$ where

$$H_0 = \left\{ \phi \in L^2(\mathbb{R}^+) \mid \hat{\phi} \text{ of compact support and } \lim_{k \to 0} \frac{\hat{\phi}(k)}{|k|^{1/2}} < \infty \right\}.$$

Now $\mathscr{L}(\tau, \beta)$ is the closed convex hull generated by the dissipative maps of the form:

$$L_x^f(y) = \int dtds f(t)(\tau_s x [y, \tau_{s+t} x]_\pm + [\tau_s x, y]_\pm \tau_{s+t} x), \; y \in \mathscr{A}_0$$

with $x \in \mathscr{A}_{\text{even}} \cap \mathscr{A}_0$ or $x \in \mathscr{A}_{\text{odd}} \cap \mathscr{A}_0$ and $f \in \mathscr{F}_\beta$.

This slightly unusual form is due to the anticommutation relations (see also [*15*]).

As the evolution is the free evolution, it is natural to expect that it will be sufficient to consider only gauge-invariant quasi-free generators to establish already return to equilibrium.

Explicitly we concentrate on the maps

$$\begin{aligned} \Gamma_\phi^f(y) = \int dtds f(t)(a^+(\phi_s)[y, a(\phi_{s+t})]_\pm + [a^+(\phi_s), y]_\pm a(\phi_{s+t}) \\ + a(\phi_s)[y, a^+(\phi_{s+t})]_\pm + [a(\phi_s), y]_\pm a^+(\phi_{s+t})) \qquad y \in \mathscr{A}_0 \end{aligned} \tag{6}$$

Note that $\Gamma_\phi^f \in \mathscr{L}(\tau, \beta)$ since

$$\Gamma_\phi^f = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \alpha_\lambda L^f_{(a(\phi) + a^+(\phi))} \alpha_{-\lambda}$$

where

$$\alpha_\lambda a^\pm = e^{\pm i\lambda} a^\pm, \; \lambda \in [0, 2\pi)$$

is the gauge automorphism.

LEMMA III.1. — Let Γ_ϕ^f be the map of $\mathscr{A}_0$ into $\mathscr{A}$ defined in (6), then it extends to the generator of a one parameter semi-group $\Theta_t^{f,\phi}$ of strongly continuous, completely positive unity preserving mappings of $\mathscr{A}$.

Proof. — For $\psi_1, \psi_2 \in H_0$ we compute

$$\Gamma_\phi^f(a^+(\psi_1)a(\psi_2)) = a^+(\tilde{L}\psi_1)a(\psi_2) + a^+(\psi_1)a(\tilde{L}\psi_2) + (\psi_2, S\psi_1)$$

where

$$\tilde{L}\psi = -\int dtds(f(t) + f(-t))(\phi_{s+t}, \psi)\phi_s \tag{7}$$

$$S\psi = 2\int dtds f(-t)(\phi_{s+t}, \psi)\phi_s .$$

Consider the decomposition of the test function space $H = H_+ \oplus H_-$ where

$$H_\pm = \{ \psi \in H \mid \hat{\psi}(-p) = \pm \hat{\psi}(p) \}$$

and denote

$$\Phi(p)\begin{pmatrix} \hat{\psi}_+(p) \\ \hat{\psi}_-(p) \end{pmatrix} = \begin{pmatrix} |\hat{\phi}_+|^2(p) & \hat{\phi}_+\overline{\hat{\phi}}_-(p) \\ \overline{\hat{\phi}}_+\hat{\phi}_-(p) & |\hat{\phi}_-|^2(p) \end{pmatrix}\begin{pmatrix} \hat{\psi}_+(p) \\ \hat{\psi}_-(p) \end{pmatrix}$$

then

$$\widehat{\tilde{L}\psi}(p) = -(1 + e^{-\beta p^2})f(p^2)\frac{\hat{\Phi}(p)}{|p|}\begin{pmatrix} \hat{\psi}_+(p) \\ \hat{\psi}_-(p) \end{pmatrix}.$$

Since $\tilde{L}$ is a negative multiplication operator, it extends to the generator of a contraction semigroup $T_t = \exp t\tilde{L}$.

Furthermore

$$\widehat{S\psi}(\beta) = 2e^{-\beta p^2} f(p^2)\frac{\hat{\Phi}(p)}{|p|}\begin{pmatrix} \hat{\psi}_+(p) \\ \hat{\psi}_-(p) \end{pmatrix}$$

therefore

$$0 \leqslant S \leqslant -(\tilde{L}^* + \tilde{L}) .$$

Hence the equation

$$\frac{dQ_t}{dt} = T_t^* S T_t ; Q_0 = 0$$

has a unique solution

$$Q_t = \int_0^t ds T_s^* S T_s$$

satisfying

$$0 \leqslant Q_t \leqslant 1 - T_t^* T_t$$

$$Q_{t_1+t_2} = Q_{t_1} + T_{t_1}^* Q_{t_2} T_{t_1}$$

For later use remark that

$$Q_t = R_\beta(1 - T_t^* T_t) \tag{8}$$

where

$$\widehat{R_\beta\psi}(p) = \frac{1}{1 + e^{\beta p^2}}\hat{\psi}(p) .$$

Next we compute Γ_p^f on an arbitrary mononial in the creation and annihilation operators: let χ_i $(i = 1, \ldots, n)$, ψ_j $(j = 1, \ldots, m) \in H_0$ then

$$\Gamma^f_\phi a^+(\chi_1) \ldots a^+(\chi_n)a(\psi_1) \ldots a(\psi_m)$$
$$= \sum_{i=1}^{n} a^+(\chi_1) \ldots a^+(\tilde{L}\chi_i) \ldots a^+(\chi_n)a(\psi_1) \ldots a(\psi_m)$$
$$+ \sum_{j=1}^{m} a^+(\chi_1) \ldots a^+(\chi_n)a(\psi_1) \ldots a(\tilde{L}\psi_j) \ldots a(\psi_m)$$
$$+ (-1)^{n-1} \sum_{i=1}^{n} \sum_{i=1}^{m} (-1)^{i+j} a^+(\chi_1) \ldots a^+(\chi_{i-1})a^+(\chi_{i+1}) \ldots a^+(\chi_n)$$
$$a(\psi_1) \ldots a(\psi_{j-1})a(\psi_{j+1}) \ldots a(\psi_m)(\psi_j, S\chi_i).$$

Hence the map Γ^f_p is completely described by the operators $\tilde{L}$ and S on the testfunction space H.

Therefore consider the strongly continuous one-parameter semigroup of unity preserving, quasi-free, gauge invariant, completely positive maps of $\mathscr{A}$ [*16*] [*17*]

$$\Theta^{f,\phi}_t(a^\#(\psi_1) \ldots a^\#(\psi_n))$$
$$= \sum_{k=1}^{n} \sum_{\pi} \text{sign}\, \pi a^\#(T_t\psi_{\pi(1)}) \ldots a^\#(T_t\psi_{\pi(k)}\omega_{Q_t}(a^\#(\psi_{\pi(k+1)}) \ldots a^\#(\psi_{\pi(n)}))) \quad (9)$$

where $a^\#$ stands for a^+ or a, the sum $\sum_\pi$ runs over all permutations π: $(1, \ldots, n) \to (\pi(1), \ldots, \pi(n))$ such that $\pi(1) < \pi(2) < \ldots < \pi(k)$; $\pi(k+1) < \ldots \pi(n)$.

Moreover $(T_t)_{t\in\mathbb{R}^+}$ is the semigroup $T_t = \exp t\tilde{L}$ and ω_{Q_t} is the quasi-free functional of $\mathscr{A}$ determined by $\omega_{Q_t}(a^+(\psi_1)a(\psi_2)) = (\psi_2, Q_t\psi_1)$ where Q_t is found above.

Finally the Lemma follows by checking that

$$\frac{d}{dt}\Theta^{f,\phi}_t \Big|_{t=0} = \Gamma^f_\phi \quad \text{on } \mathscr{A}_0.$$

■

It is instructive to note that for a given $f \in \mathscr{F}_\beta$ and $\phi \in H_0$ the following result of return to equilibrium holds:

$$s - \lim_{t\to\infty} \Theta^{f,\phi}_t = 1 \hat{\otimes} \omega_{R_\beta} \quad (10)$$

where $1 \hat{\otimes} \omega_{R_\beta}$ is the conditional expectation on the C*-subalgebra $\mathscr{A}(H_{f,\phi})$ with respect to the (τ, β)-KMS-state ω_{R_β}; $\mathscr{A}(H_{f,\phi})$ is the C*-algebra generated by $\{ a^{\#}(\psi) ; \psi \in H_{f,\phi} \}$ and $H_{f,\phi} = \{ \psi \in H ; L^f_\phi \psi = 0,\ L^f_\phi$ is given by (7) $\}$.

Indeed T_t converges strongly to the orthogonal projection P onto $H_{f,\phi}$ when t tends to infinity, where as by (8): $s - \lim_{t \to \infty} Q_t = R_\beta(1 - P)$ and the result follows from (9).

Finally we prove the conjecture for the infinite free Fermion system.

THEOREM III.2. — Let ω be any state of $\mathscr{A}$ such that $\omega \circ \Gamma^f_\phi = 0$ for all $f \in \mathscr{F}_\beta$ and $\phi \in H_0$ than $\omega = \omega_{R_\beta}$.

Proof. — Let $x = a^{\#}(\psi_1) \ldots a^{\#}(\psi_n)$ where $\psi_i \in H_0$.

Take $f \in \mathscr{F}_\beta$ such that $f > 0$ on $\bigcup_{k=1}^{n} \operatorname{supp} \hat{\psi}_k$ and take $\phi \in H_0$ such that $\hat{\phi} = \sum_{k=1}^{n} \chi_k \hat{\psi}_k$ where χ_k are the characteristic functions on the sets Λ_k which are found as follows. Take the points p_k $(k = 1, \ldots, n)$ in the interior of $\operatorname{supp} \hat{\psi}_k$ $(k = 1, \ldots, n)$ such that $p_k \neq p_1$ for $k \neq 1$; then take Λ_k $(k = 1, \ldots, n)$ an open neighbourhood of p_k $(k = 1, \ldots, n)$ pairwise disjoint.

Then it is easy to check that $\tilde{L}\psi_k \neq 0$, in particular $\widehat{\tilde{L}\psi_k}(p_k) \neq 0$. Hence

$$\lim_{t \to \infty} \Theta_t^{f,\phi}(x) = \omega_{R_\beta}(x).$$

Therefore using the invariance of ω one gets $\omega(x) = \omega_{R_\beta}(x)$. The result follows from a continuity argument. ■

Remark that the technique used to prove the conjecture in the free case, namely the construction of the semigroup, leads to a much more powerful result than the one stated in the conjecture. In particular it enables us to prove explicitly the property of return to equilibrium (see formula (10)). For general systems a proof closer in spirit to the one given for the finite system seems us appropriate.

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