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A note on a hermitian analog of Einstein spaces (*)

by

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ABSTRACT. — It is shown that a Hermitian Einstein space, i. e. a hermitian space such that the Ricci tensor of the hermitian connection is proportional to the hermitian metric, is Kählerian.

RÉSUMÉ. — On montre qu'un espace d'Einstein hermitien, c'est-à-dire un espace hermitien tel que le tenseur de Ricci de la connection hermitienne soit proportionnel à la métrique hermitienne, est kählerien.

INTRODUCTION

An Einstein space is a manifold of dimension n endowed with a riemannian metric g proportional to the Ricci tensor of the Levi-Civita connection of g. Besides their intrinsic interest from the mathematical point of view, Einstein spaces have been introduced also in physics, mainly in

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connection with the « space-time foam » approach to the quantization of the gravitational field (see e. g. 1) for a review.It can be noticed that most of the work on this last topic has been done by assuming (just for mathematical convenience and not for any physical reasons) that the manifolds considered are Kählerian. This requirement simplifies, in particular, the study of problems about existence and classification of Einstein's metrics, at least for 4-dimensional manifolds.

However we can observe, at first, that on a complex manifold there are two different connections which have geometrical meaning, namely the Levi-Civita connection and the hermitian connection. Therefore, the condition of being Einstein's could be defined for both the Ricci tensors at disposal, and the two definitions coincide *a priori* only in the Kählerian case. Moreover we recall that for the Levi-Civita connection the condition (Ricci $= \lambda$. metric) implies the constancy of λ (for manifold of dimension $\geqslant 3$). While from the expression (Ricci $= \lambda$. metric) for the Ricci tensor of the hermitian connection, it follows λ = constant only if the hermitian metric is also Kähler. In this paper we prove that an Einstein hermitian manifold (i. e. a hermitian manifold such that its hermitian metric is Einstein with $\lambda \neq 0$ constant) is necessarily Kähler and hence it is Einstein also in the usual sense.

Note that the converse is not true, i. e. there exist complex manifolds (e. g. the complex projective space $\mathbb{C}P^2$ with a point blown up) with a riemannian metric which is Einstein but not Einstein-Kähler.

We shall recall in the first paragraph some definitions and results on the theory of hermitian manifolds; the reader may refer to 2), 3), 4) for a more extensive and complete survey of the subject.

1. THE HERMITIAN CONNECTION

Let M be a complex manifold, $\dim_{\mathbb{C}} M = n = 2 \dim_{\mathbb{R}} M = 2m$ and $C^\infty(M)$ the space of all C^∞ complex-valued functions on M.

Let us denote by ${}_{\mathbb{C}}TM$ and ${}_{\mathbb{C}}TM^*$ the complex tangent and the complex cotangent bundle of M respectively. We can form the complex exterior product $\Lambda^r {}_{\mathbb{C}}TM^*$ which is called the bundle of complex r-forms on M. It is well known that ${}_{\mathbb{C}}TM$ splits as a sum of complex vector bundles:

$$ {}_{\mathbb{C}}TM = TM \oplus \overline{TM} \tag{1a} $$

where TM and $\overline{TM}$ are respectively the holomorphic and the antiholomorphic tangent bundles of M. Similarly we have:

$$ {}_{\mathbb{C}}TM^* = TM^* \oplus \overline{TM}^* \tag{1b} $$

where TM^* [resp. $\overline{TM}^*$] is the holomorphic [resp. antiholomorphic] cotangent bundle of M. The splitting (1*b*) induces the following decomposition of $\Lambda^r {}_{\mathbb{C}}TM^*$:

$$\Lambda^r {}_{\mathbb{C}}TM^* = \Lambda^r(TM^* \oplus \overline{TM}^*) = \bigoplus_{p+q=r} \Lambda^{p,q}(M) \tag{2}$$

where $\Lambda^{p,q}(M)$ denotes the image of $\Lambda^p TM^* \otimes \Lambda^q \overline{TM}^*$ in $\Lambda^r {}_{\mathbb{C}}TM^*$ and is called the bundle of complex (p, q)-forms on M.

Before dealing with connections, we spend a few words about C^∞ sections of the complex vector bundles which we introduced above. If E is anyone of these bundles, we shall denote its C^∞ sections by $C^\infty(E)$. (For example: $C^\infty({}_{\mathbb{C}}TM)$ are the complex vector fields on M, $C^\infty(\Lambda^{p,q}(M))$ are the fields of complex (p, q)-forms on M, and so on). We shall also need C^∞-sections of bundles built up by tensor product of two other bundles (for example: $C^\infty({}_{\mathbb{C}}TM^* \otimes {}_{\mathbb{C}}TM)$ are the fields of vector-valued 1-forms on M, $C^\infty(\Lambda^{p,q}(M) \otimes E) \equiv C^\infty(\Lambda^{p,q}(E))$ are the fields of E-valued (p, q)-forms on M, and so on).

A connection on ${}_{\mathbb{C}}TM$ is a complex linear map:

$$\nabla : C^\infty({}_{\mathbb{C}}TM) \to C^\infty({}_{\mathbb{C}}TM^* \otimes {}_{\mathbb{C}}TM)$$

which satisfies the condition:

$$\nabla(f\sigma) = df \otimes \sigma + f \cdot \nabla\sigma\,; \qquad \forall f \in C^\infty(M), \quad \forall \sigma \in C^\infty({}_{\mathbb{C}}TM)$$

The connection ∇ induces (by using the Leibnitz rule) connections on all the tensor bundles on M; we shall denote them by the same symbol ∇. In particular, for $\Lambda^r {}_{\mathbb{C}}TM^*$ we have:

$$\nabla : C^\infty(\Lambda^r {}_{\mathbb{C}}TM^*) \to C^\infty({}_{\mathbb{C}}TM^* \otimes \Lambda^r {}_{\mathbb{C}}TM^*)$$

Using the splittings (1*b*) and (2) one can write $\nabla = \nabla' + \nabla''$ where:

$$\nabla' : C^\infty(\Lambda^{p,q}(M)) \to C^\infty(TM^* \otimes \Lambda^{p,q}(M)) \tag{3a}$$

$$\nabla'' : C^\infty(\Lambda^{p,q}(M)) \to C^\infty(\overline{TM}^* \otimes \Lambda^{p,q}(M)) \tag{3b}$$

A connection ∇ is said of type (1, 0) (or compatible with the complex structure of M) if

$$\nabla s(\overline{W}) = 0$$

for any *holomorphic* section s of TM and for any $\overline{W} \in C^\infty(\overline{TM})$. A hermitian metric on M is a positive definite quadratic form $h \in C^\infty(TM^* \otimes \overline{TM}^*)$ such that

$$h(Z, \overline{W}) = \overline{h(W, \overline{Z})} \qquad \forall Z \in C^\infty(TM), \qquad \forall \overline{W} \in C^\infty(\overline{TM}) \tag{4}$$

It is a well known result [1] that there exists a *unique* connection D which is of type (1, 0) and such that

$$Dh = 0$$

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[1] See ref. 3, p. 73.

D is called the hermitian connection of M associated with the metric h.

The torsion form of a complex connection is an element of

$$C^\infty(\Lambda^2{}_{\mathbb{C}}TM^* \otimes {}_{\mathbb{C}}TM);$$

it can be shown [2] that the torsion form T of the hermitian connection D spits into: $T = S + \overline{S}$, where $\overline{S}$ is the conjugate of S and:

$$S \in C^\infty(\Lambda^{2,0}(TM)) \tag{5a}$$

$$\overline{S} \in C^\infty(\Lambda^{0,2}(TM)) \tag{5b}$$

Hence, taking into account the splittings (3*a*, *b*) we have:

$$DT(Z) \in C^\infty(\Lambda^{2,0}(TM) \oplus \Lambda^{0,2}(\overline{TM}))\,; \qquad Z \in C^\infty(TM) \tag{6a}$$

$$DT(\overline{W}) \in C^\infty(\Lambda^{2,0}(TM) \oplus \Lambda^{0,2}(\overline{TM}))\,; \qquad \overline{W} \in C^\infty(\overline{TM}) \tag{6b}$$

The curvature R of a complex connection is a 2-form with values in the space of linear complex maps ${}_{\mathbb{C}}TM \to {}_{\mathbb{C}}TM$, that is:

$$R \in C^\infty(\Lambda^2{}_{\mathbb{C}}TM^* \otimes {}_{\mathbb{C}}TM \otimes {}_{\mathbb{C}}TM^*)$$

For a hermitian connection it can be shown, for example by direct computation in local components and using splittings (1*a*, *b*), that

$$R \in C^\infty(\Lambda^2{}_{\mathbb{C}}TM^* \otimes TM \otimes TM^* \oplus \Lambda^2{}_{\mathbb{C}}TM^* \otimes \overline{TM} \otimes \overline{TM}^*)$$

However, since it is known [3] that, for the hermitian connection, R is a form of type (1, 1), we can write:

$$R \in C^\infty(\Lambda^{1,1}(M) \otimes TM \otimes TM^* \oplus \Lambda^{1,1}(M) \otimes \overline{TM} \otimes \overline{TM}^*) \tag{7}$$

We shall need in the following the Bianchi's identities expressed in terms of T, R and their covariant derivatives [4].

For any A, B, $C \in C^\infty({}_{\mathbb{C}}TM)$ the first and the second identities are respectively:

$$\{R(A, B)C\}_{(A,B,C)} = \{T(T(A, B), C) + (D_A T)(B, C)\}_{(A,B,C)} \tag{8a}$$

$$\{(D_A R)(B, C) + R(T(A, B), C)\}_{(A,B,C)} = 0 \tag{8b}$$

where $\{\quad\}_{(A,B,C)}$ denotes sum over the cyclic permutation of the arguments A, B, C.

2. HERMITIAN EINSTEIN SPACES

We recall [5] that the Ricci tensor is defined, for any A, B, $C \in C^\infty({}_{\mathbb{C}}TM)$, as:

$$\operatorname{Ric}(B, C) = \text{Trace of } [A \to R(A, B)C] \tag{9}$$

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(2) See ref. 2, p. 224.
(3) See ref. 4, vol. 2, p. 181.
(4) See ref. 4, vol. 1, p. 135.
(5) See ref. 4, vol. 1, p. 248.

By a *hermitian Einstein space* we mean a complex manifold with a hermitian metric h such that

$$\mathrm{Ric} = \lambda h ; \qquad \lambda \in \mathbb{C} - \{ 0 \} \tag{10}$$

We shall show that h is Kähler.

Proof. — If Z, W, $V \in C^\infty(TM)$ condition (10) implies in particular:

$$\mathrm{Ric}\,(Z, W) = 0 \tag{11a}$$

$$\mathrm{Ric}\,(\overline{Z}, \overline{W}) = 0 \tag{11b}$$

Furthermore, using the form of R given in (7) and condition (10), one can see easily that the only non-vanishing part of (9) is

$$\mathrm{Ric}\,(Z, \overline{W}) = \text{Trace of } [\overline{V} \to R(\overline{V}, Z)\overline{W}] \tag{12}$$

Let us use Bianchi's first identity (8*a*) to transform $R(\overline{V}, Z)\overline{W}$. Taking the cyclic permutation on $(\overline{V}, Z, \overline{W})$ we obtain explicitly:

$$\begin{aligned} R(\overline{V}, Z)\overline{W} + R(\overline{W}, \overline{V})Z + R(Z, \overline{W})\overline{V} = \\ = T(T(\overline{V}, Z)\overline{W}) + T(T(\overline{W}, \overline{V})Z) + T(T(Z, \overline{W})\overline{V}) + \\ + (D_{\overline{V}}T)(Z, \overline{W}) + (D_{\overline{W}}T)(\overline{V}, Z) + (D_Z T)(\overline{W}, \overline{V}) \end{aligned}$$

A lot of terms in the last expression vanish. In fact, using (7) we have that $R(\overline{W}, \overline{V})Z = 0$; from (5, *a*, *b*) we see that all the terms which are quadratic in T vanish too; finally, (6 *a*, *b*) imply that the only non-null derivative of T is obtained from the term $(D_Z T)(\overline{W}, \overline{V})$.

Summing up these results, we can rewrite (12) as:

$$\mathrm{Ric}\,(Z, \overline{W}) = \text{Trace of } [\overline{V} \to (-R(Z, \overline{W})\overline{V} + (D_Z T)(\overline{W}, \overline{V})] \tag{13}$$

We are going to show now that Trace of $[\overline{V} \to D_Z T(\overline{W}, \overline{V})]$ vanishes. It turns out that the computation is easier when local components are used; let $z^i = \{ z^\alpha, \overline{z^\beta} \}$ $(i=1, \ldots, n; \alpha, \overline{\beta}=1, \ldots, m)$ be a local chart of coordinates. Then the components of T and R, in terms of the local bases $\left\{ \frac{\partial}{\partial z^i} \right\}$ and $\{ dz^k \}$ of the tangent and cotangent spaces, are defined by:

$$R^i{}_{jkm} = \left\langle dz^i,\ R\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^m}\right)\frac{\partial}{\partial z^j} \right\rangle$$

$$T^i{}_{jk} = \left\langle dz^i,\ T\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) \right\rangle$$

where $\langle , \rangle$ is the usual pairing between 1-forms and vectors. If we write down the second Bianchi's identity (8*b*) in local components, contract it twice and use condition (10), we get (see Appendix A):

$$2\lambda T^i{}_{ki} - R^{jm}{}_{ik} T^i{}_{mj} = 0 \tag{14}$$

From (7) we have that $R^{\alpha\beta}{}_{ik} = 0$ and $R^{\bar{\alpha}\bar{\beta}}{}_{ik} = 0 \ \forall i, k$, from (5 a, b) we have $T^{i}{}_{\bar{\alpha}\beta} = 0$ and $T^{i}{}_{\alpha\bar{\beta}} = 0 \forall i$ and so the second term in (14) vanishes anyway. Then, as $\lambda \neq 0$, we have $T^{i}{}_{ki} = 0$ or, in the coordinate-independent form:

$$\text{Trace of } [A \to T(B, A)] = 0$$

$\forall A, B \in C^{\infty}({}_{\mathbb{C}}TM)$. That implies also:

$$\text{Trace of } [\overline{V} \to (D_Z T)(\overline{W}, \overline{V})] = 0 \tag{15}$$

$\forall \overline{V}, \overline{W} \in C^{\infty}(\overline{TM})$, $\forall Z \in C^{\infty}(TM)$.
Expression (13) with (15) reads:

$$\mathrm{Ric}\,(Z, \overline{W}) = \text{Trace of } [\overline{V} \to -R(Z, \overline{W})\overline{V}] = -\text{Trace of } [R(Z, \overline{W})]\,. \tag{16}$$

By using equations (11 a, b) and the condition (7) we can extend the (16) for $A, B \in C^{\infty}({}_{\mathbb{C}}TM)$:

$$\mathrm{Ric}\,(A, B) = -\text{ Trace of } [R(A, B)]\,.$$

The right hand side of (16), by a direct computation, can be written as $-$ [Trace of R](A, B). Moreover, as it is known (e. g. from Bianchi's identities) that the two form (Trace of R) is closed, we see that the fundamental form of h is also closed, namely h is Kähler.

APPENDIX A

In this appendix we give the details of the derivation of the equation (14). From eq. 8 *b* we have, in local components,

$$R^i_{jkm;l} + R^i_{jlk;m} + R^i_{jml;k} + R^i_{jam}T^a_{lk} + R^i_{jak}T^a_{ml} + R^i_{jal}T^a_{km} = 0 .$$

By contracting on *ik* we have:

$$R_{jm;l} - R_{jl;m} + R^i_{jml;i} + R^i_{jam}T^a_{li} - R_{ja}T^a_{ml} + R^i_{jal}T^a_{im} = 0 .$$

Another contraction on *jm* gives:

$$R_{;l} - R^m_{l;m} - R^i_{l;i} + R^i_a T^a_{li} - R^m_a T^a_{ml} + R^{im}{}_{al} T^a_{im} = 0$$

and hence using the condition (10) we finally obtain the equation (14).

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