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Inner automorphisms of hyperfinite factors and Bogoliubov transformations

by

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ABSTRACT. — Let E be a real infinite dimensional separable Hilbert space and $\mathscr{C}(E)$ the Clifford algebra over E. We consider the quasifree state ω_C and representation π_C of $\mathscr{C}(E)$, defined by a skew adjoint operator C on E with $\| C \| \leqslant 1$ and ker C not odd dimensional. Then there is a complex structure J on E which commutes with C. If R is an orthogonal operator on E then it determines a Bogoliubov automorphism of $\mathscr{C}(E)$. Under the assumption that JC does not have one in its spectrum we show that there is a unitary $\Gamma(R) \in \pi_C(\mathscr{C}(E))''$ implementing the Bogoliubov automorphism determined by R if and only if either $R + I$ is Hilbert-Schmidt with dim ker $(R - I)$ even, or $R - I$ is Hilbert-Schmidt with dim ker $(R - I^v$ odd. This generalises a well known theorem of Blattner [2] for the case $C = 0$.

RÉSUMÉ. — Soit E un espace de Hilbert réel séparable de dimension infinie et C(E) l'algèbre de Clifford sur E. On considère l'état quasi libre ω_C et la représentation π_C de C(E) définis par un opérateur antiautoadjoint C sur E avec $\| C \| < 1$ et ker C de dimension non impaire. Alors il existe une structure complexe J sur E qui commute avec C. Si R est un opérateur orthogonal sur E, il détermine un automorphisme de Bogoliubov de C(E). Sous l'hypothèse que JC ne contient pas 1 dans son spectre, on montre qu'il existe un unitaire $\Gamma(R) \in \pi_C(C(E)''$ réalisant l'automorphisme de Bogoliubov déterminé par R si et seulement si ou bien $R - I$ est de Hil-

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bert-Schmidt avec dim ker (R − I) paire, ou bien R − I est de Hilbert Schmidt avec dim ker (R − I) impaire. Ce résultat généralise un théorème bien connu de Blattner [2] dans le cas C = 0.

1. INTRODUCTION

This paper concerns certain automorphisms of hyperfinite factors which are constructed via quasifree representations of the Clifford algebra over an infinite dimensional real Hilbert space. Some notation is required before the results can be described.

Let E denote an infinite dimensional real separable Hilbert space and $\mathscr{C}(\mathrm{E})$ the Clifford algebra over E which we take to be the unital C* algebra generated by $\{ c(u) \mid u \in \mathrm{E} \}$ where

$$c(u)^* = c(u), \qquad c(u)^2 = \| u \|^2 \cdot 1 .$$

Each orthogonal operator R on E defines an automorphism α_{R} of $\mathscr{C}(\mathrm{E})$ via its action on the generating elements

$$\alpha_{\mathrm{R}}(c(u)) = c(\mathrm{R}u) . \tag{1.1}$$

These automorphisms are usually referred to as Bogoliubov automorphisms. A quasifree state on $\mathscr{C}(\mathrm{E})$ is defined initially in the dense subalgebra of $\mathscr{C}(\mathrm{E})$ consisting of polynomials in the generating elements $c(u)$, $u \in \mathrm{E}$, by setting

$$\omega(c(u_1) \dots c(u_r)) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \mathrm{P}f\,[\omega(c(u_r)c(u_j))] & \text{if } r \text{ is even} \end{cases} \tag{1.2}$$

where $\mathrm{P}f\,[a_{ij}]$ denotes the Pfaffian of the array a_{ij} and ω is determined on products $c(u)c(v)$, $u, v \in \mathrm{E}$ by a skew adjoint operator C (the covariance) on E, with $\| \mathrm{C} \| \leqslant 1$, via

$$\omega(c(u)c(v)) = (u, v) + i(\mathrm{C}u, v) . \tag{1.3}$$

Let π_{C} denote the representation of $\mathscr{C}(\mathrm{E})$ determined by C, then $\pi_{\mathrm{C}}(\mathscr{C}(\mathrm{E}))''$ is a factor provided ker C is not odd dimensional. (Details of the above may be found in [6] and [9].)

A Bogoliubov automorphism α_{R} is said to be implemented in π_{C} if there is a unitary operator $\Gamma_{\mathrm{C}}(\mathrm{R})$ acting on the Hilbert space of π_{C} such that

$$\Gamma_{\mathrm{C}}(\mathrm{R})\pi_{\mathrm{C}}(c(u))\Gamma_{\mathrm{C}}(\mathrm{R})^{-1} = \pi_{\mathrm{C}}(c(\mathrm{R}u)) . \tag{1.4}$$

Let O(E) denote the orthogonal group on E and $\mathrm{SO(E)}_2$ the subgroup of O(E) consisting of operators R with R − I Hilbert-Schmidt and dim ker (R + I) even or infinite. Let G_2 denote the group

$$\mathrm{SO(E)}_2 \cup \{ \mathrm{R} \in \mathrm{O(E)} : \mathrm{R} + \mathrm{I} \ \text{ is Hilbert-Schmidt}, \ \ \dim\ker(\mathrm{R}-\mathrm{I}) \text{ odd} \} .$$

(This group was introduced by Blattner [2].) The assumption that $\pi_C(\mathscr{C}(E))''$ is a factor allows us to define a complex structure J on E by taking any complex structure on ker C and extending it to E by taking the isometric part in the polar decomposition of C. Our main result is

THEOREM 1.1. — *If* $R \in G_2$ *then* α_R *is implemented in* π_C. *When* $\pi_C(C(E))''$ *is a factor and* 1 *is not in the spectrum of* JC *a Bogoliubov automorphism* α_R *which is implemented in* π_C *is inner if and only if* $R \in G_2$.

This result has a number of corollaries and we discuss one here. The special case where $C = J(1 - 2\lambda)$ and $0 < \lambda < 1/2$ is of interest since then $\pi_C(\mathscr{C}(E))''$ is the hyperfinite $III_{\lambda/1-\lambda}$ factor while if $\lambda = 1/2$ then it is the hyperfinite II_1 factor.

COROLLARY 1.2. — *If* $C = J(1 - 2\lambda)$ *with* $0 < \lambda \leqslant 1/2$ *and* J *a complex structure on* E *then* α_R *is implemented in* π_C *and is inner if and only if* $R \in G_2$.

The case $\lambda = 1/2$ is due to Blattner [2] (see also de la Harpe and Plymen [4]).

The paper is organised as follows. Section 2 contains the main part of the proof of theorem 1.1. It turns out to be convenient to reformulate the problem in terms of Araki's self dual CAR algebra [1]. In this context we state a mild generalisation of theorem 1.1 (theorem 2.10).

In section 3 we translate back into the Clifford algebra notation and discuss some corollaries of the argument.

Questions not unrelated to those discussed here and other background material may be ground in [5], [7].

2. SELF-DUAL CAR ALGEBRAS

Introduce the following structure.

i) A complex Hilbert space $H = E \oplus E$ with complex structure $J \oplus -J$.

ii) $\Gamma : H \to H$, an antiunitary involution, defined by $\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

iii) An isomorphism of O(E) with the group unitary operators on H which commute with Γ by $R \in O(E) \to T(R) = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix}$ where

$$T_1 = 1/2(R - JRJ), \qquad T_2 = 1/2(R + JRJ).$$

Now introduce the self dual CAR algebra over H, denoted $\mathscr{A}_s(H)$, which is the unital C* algebra generated by $\{ B(f) \mid f \in H \}$ with

iv) $$B(f)^* = B(\Gamma f)$$

v) $$f \to B(f)^* \quad \text{complex linear from H into } \mathscr{A}_s(H).$$

vi) $$B(\Gamma f)B(g) + B(g)B(\Gamma f) = \langle g, f \rangle_H ; \qquad g, f \in H.$$

The map

$$\mathrm{B}(u\oplus 0)+\mathrm{B}(0\oplus u) \ \rightarrow \ c(u), \quad -i\mathrm{B}(u\oplus 0)+i\mathrm{B}(0\oplus u) \ \rightarrow \ c(\mathrm{J}u), \quad u\in\mathrm{E} \tag{2.1}$$

extends to an isomorphism of $\mathscr{A}_s(\mathrm{H})$ with $\mathscr{C}(\mathrm{E})$.

Quasifree states on $\mathscr{A}_s(\mathrm{H})$ are defined as in definition 3.1 of [*1*]. For the purpose of this paper it is sufficient to note the analogue of (1.3) namely that a quasifree state ω_A on $\mathscr{A}_s(\mathrm{H})$ is completely determined by a self-adjoint operator A on H with $0 \leqslant \mathrm{A} \leqslant \mathrm{I}$ and $\Gamma\mathrm{A}\Gamma = 1 - \mathrm{A}$, via

$$\omega_\mathrm{A}(\mathrm{B}(f)^*\mathrm{B}(g)) = \langle g, \mathrm{A}f \rangle . \tag{2.2}$$

The map (2.1) shows that the skew adjoint operator C on E with $\| \mathrm{C} \| \leqslant 1$ defines a quasifree state on $\mathscr{A}_s(\mathrm{H})$ via

$$\mathrm{A} = \begin{pmatrix} 1/2(1 - \mathrm{JC}) & 0 \\ 0 & 1/2(1 + \mathrm{JC}) \end{pmatrix}. \tag{2.3}$$

Now transport the notation of the introduction over to this context. Let π_A be the representation of $\mathscr{A}_s(\mathrm{H})$ determined by ω_A, $\alpha_{\mathrm{T(R)}}$ be the automorphism of $\mathscr{A}_s(\mathrm{H})$ defined by

$$\alpha_{\mathrm{T(R)}}(\mathrm{B}(f)) = \mathrm{B}(\mathrm{T(R)}f), \qquad \mathrm{R}\in 0(\mathrm{E})\,; \qquad f\in\mathrm{H}\,.$$

and $\Gamma_\mathrm{A}(\mathrm{T(R)})$ for a unitary implementing $\alpha_{\mathrm{T(R)}}$.

The representation π_A of $\mathscr{A}_s(\mathrm{H})$ can be realised as follows. Define a new Hilbert space $\mathrm{K} = \mathrm{H} \oplus \mathrm{H}$ and a projection P_A on K by

$$\mathrm{P}_\mathrm{A} = \begin{pmatrix} \mathrm{A} & \mathrm{A}^{1/2}(1 - \mathrm{A})^{1/2} \\ \mathrm{A}^{1/2}(1 - \mathrm{A})^{1/2} & 1 - \mathrm{A} \end{pmatrix}.$$

Let $\hat{\Gamma} = \Gamma \oplus (-\Gamma)$ and let $\mathscr{A}_s(\mathrm{K})$ denote the self dual CAR algebra over K. Now the corresponding representation $\pi_{\mathrm{P}_\mathrm{A}}$ of $\mathscr{A}_s(\mathrm{K})$ is irreducible. Under the identification of $\mathscr{A}_s(\mathrm{H})$ with $\mathscr{A}_s(\mathrm{H} \oplus 0) \subseteq \mathscr{A}_s(\mathrm{K})$, we find that $\pi_{\mathrm{P}_\mathrm{A}}$ restricted to $\mathscr{A}_s(\mathrm{H} \oplus 0)$ is equivalent to π_A. We record three important observations.

OBSERVATION 2.1. — *The assumption that* $\pi_\mathrm{C}(\mathscr{C}(\mathrm{E}))''$ *is a factor follows from the assumption that the operator* A *in* (2.3) *has* 1/2 *as an eigenvalue of even or infinite multiplicity. Moreover the assumption that* 1 *is not in the spectrum of* JC *is equivalent to asserting that* 0 *is not in the spectrum of* A. *The latter assumption on* A *will hold throughout the subsequent discussion.*

OBSERVATION 2.2. — *From Araki* ([*1*], 4.10) *we know that the G. N. S. cyclic vector* $\Omega_{\mathrm{P}_\mathrm{A}}$ *for* $\pi_{\mathrm{P}_\mathrm{A}}$ *is cyclic and separating for* $\pi_{\mathrm{P}_\mathrm{A}}(\mathscr{A}_s(\mathrm{H} \oplus 0))''$ *whenever* 0 (*and hence* 1) *is not an eigenvalue of* A. *Henceforth whenever the symbol* π_A *appears it means the representation* $\pi_{\mathrm{P}_\mathrm{A}}$ *restricted to* $\mathscr{A}_s(\mathrm{H} \oplus 0)$ *and correspondingly* Ω_A *means the vector* $\Omega_{\mathrm{P}_\mathrm{A}}$.

OBSERVATION 2.3. — *If* α_{-1} *denotes the automorphism*

$$\alpha_{-1}(B(k)) = -B(k), \qquad k \in K$$

of $\mathcal{A}_s(K)$ *then* α_{-1} *is implemented in* π_{P_A}. *Write* $\Gamma(-1)$ *for the implementing operator where the choice* $\Gamma(-1)\Omega_A = \Omega_A$ *fixes the phase. Notice that* $\Gamma(-1)B(0 \oplus h)$ *commutes with all the elements of* $\pi_{P_A}(\mathcal{A}_s(H \oplus 0))$ *for all* $h \in H$.

LEMMA 2.4. — α_{-1} *is not inner in any quasifree representation* π_A.

Proof. — Assume α_{-1} is inner. Then by observation 2.3

$$\Gamma(-1)\Gamma(-1)B(0 \oplus h)\Gamma(-1) = \Gamma(-1)B(0 \oplus h)$$

for all $h \in H$. But this implies

$$-\Gamma(-1)B(0 \oplus h) = \Gamma(-1)B(0 \oplus h)$$

a contradiction. □

DEFINITION 2.5. — *We say that an element of* $\pi_{P_A}(\mathcal{A}_s(H \oplus 0)''$ *is even or odd according to whether it commutes or anticommutes with* $\Gamma(-1)$.

We note that this definition arises from the observation that if $B \in \pi_{P_A}(\mathcal{A}_s(H \oplus 0)''$ then B commutes or anticommutes with $\Gamma(-1)$ exactly when $B\Omega_A$ is in the $+1$ or -1 eigenspace of $\Gamma(-1)$.

LEMMA 2.6. — *If* A *satisfies the conditions above and* $R \in O(E)$ *is such that* $\Gamma_A(T(R)) \in \pi_{P_A}(\mathcal{A}_s(H \oplus 0))''$ *then if* $\Gamma_A(T(R))$ *is even,* $R - I$ *is Hilbert-Schmidt while if* $\Gamma_A(T(R))$ *is odd,* $R + I$ *is Hilbert-Schmidt.*

Proof. — If $\Gamma_A(T(R))$ satisfies the condition of the lemma then $\Gamma_A(T(R))$ must commute with all the elements $\Gamma(-1)\pi_{P_A}(B(0 \oplus f))$, $f \in H$ by observation 2.3. Thus

$$\Gamma_A(T(R))\pi_{P_A}(B(0 \oplus f))\Gamma_A(T(R))^{-1} = \alpha\pi_{P_A}(B(0 \oplus f))$$

for all $f \in H$ with $\alpha = 1$ or -1 depending on whether $\Gamma_A(T(R))$ is even or odd respectively. Thus conjugation by $\Gamma_A(T(R))$ implements the Bogoliubov automorphism of $\mathcal{A}_s(H \oplus H)$ defined by the unitary operator $V(R) = \begin{pmatrix} T(R) & 0 \\ 0 & \alpha I \end{pmatrix}$ on $K = H \oplus H$. Now by Araki ([*1*], theorem 6) this latter automorphism is implemented if and only if $V(R)P_A - P_AV(R)$ is Hilbert-Schmidt. This last holds if and only if the three operators

$$T(R)A - AT(R), \qquad (T(R) - \alpha I)A^{1/2}(1-A)^{1/2}, \qquad A^{1/2}(1-A)^{1/2}(T(R) - \alpha I)$$

are all Hilbert-Schmidt. Now $A^{1/2}(1-A)^{1/2}$ is invertible since 0 (and hence 1) is not in the spectrum of A and so $T(R) - \alpha I$ is Hilbert-Schmidt, proving the result. □

The proof of lemma 2.6 also demonstrates the following

COROLLARY 2.7. — *If* $\mathrm{R} \pm \mathrm{I}$ *is Hilbert-Schmidt then* $\alpha_{\mathrm{T(R)}}$ *is implemented in* π_{A}.

REMARK 2.8. — $\alpha_{\mathrm{T(R)}}$ *extends to an automorphism of* $\mathscr{A}_s(\mathrm{K})$ *in many ways. Henceforth by* $\alpha_{\mathrm{T(R)}}$ *we will mean the automorphism of* $\mathscr{A}_s(\mathrm{K})$ *defined by* $\mathrm{V}_\pm(\mathrm{R}) = \begin{pmatrix} \mathrm{T(R)} & 0 \\ 0 & \pm \mathrm{I} \end{pmatrix}$ *depending on whether* $\mathrm{R} \mp \mathrm{I}$ *is Hilbert-Schmidt. We let* $\Gamma_{\mathrm{A}}(\mathrm{T(R)})$ *denote a unitary implementing this automorphism of* $\mathscr{A}_s(\mathrm{K})$.

LEMMA 2.9. — *If* $\mathrm{R} \in \mathrm{SO(E)}_2$ *then* $\alpha_{\mathrm{T(R)}}$ *is inner.*

Proof. — Araki shows ([*1*], p. 434) that we may choose an $\mathrm{R}' \in \mathrm{SO(E)}_2$, which commutes with R, and such that

a) $\ker(\mathrm{RR}' - \mathrm{I})$ is infinite dimensional

b) $\mathrm{R}' - \mathrm{I}$ is trace class.

It follows therefore that $\alpha_{\mathrm{T(R')}}$ is inner because $\Gamma_{\mathrm{A}}(\mathrm{R}') \in \pi_{\mathrm{A}}(\mathscr{A}(\mathrm{H}))$ ([*9*] or [*1*], theorem 5). Thus in order to show that for $\mathrm{R} \in \mathrm{SO(E)}_2$, $\alpha_{\mathrm{T(R)}}$ is inner it is sufficient to consider the case where $\ker(\mathrm{R} - \mathrm{I})$ is infinite dimensional.

By the preceding results we have to consider the operator

$$\mathrm{V(R)} = \begin{pmatrix} \mathrm{T(R)} & 0 \\ 0 & \mathrm{I} \end{pmatrix} \quad \text{on} \quad \mathrm{K}.$$

As $\mathrm{V(R)} - \mathrm{I}$ is Hilbert-Schmidt the spectral theorem gives us a sequence $\{ \mathrm{E}_n \}_{n=0}^{\infty}$ of spectral projections of V(R) each of which is $\hat{\Gamma}$ invariant and with even dimensional range for $n > 1$. E_0 we take as the projection onto the subspace corresponding to eigenvalue 1. Let $\mathrm{F}_n = \mathrm{E}_0 + \Sigma_{i \leqslant n} \mathrm{E}_i$. Now V(R) may be written as $\exp \mathrm{X}$ for X skew adjoint Hilbert-Schmidt with $\hat{\Gamma}\mathrm{X} = \mathrm{X}\hat{\Gamma}$. Then there is a one parameter group $t \to \mathrm{R}_t$ in $\mathrm{SO(E)}_2$ corresponding to the one parameter group $t \to \exp t\mathrm{X}$, i. e.

$$\mathrm{V}(\mathrm{R}_t) = \begin{pmatrix} \mathrm{T}(\mathrm{R}_t) & 0 \\ 0 & 1 \end{pmatrix} = \exp t\mathrm{X}.$$

Notice that $\exp t\mathrm{F}_n\mathrm{X} - \exp t\mathrm{X}$ converges to zero in Hilbert-Schmidt norm as $n \to \infty$. Let R_t^n be the element of $\mathrm{SO(E)}_2$ corresponding to $\exp t\mathrm{F}_n\mathrm{X}$. Then $\Gamma_{\mathrm{A}}(\mathrm{T}(\mathrm{R}_t^n))$ is in $\pi_{\mathrm{P_A}}(\mathscr{A}_s(\mathrm{H} \oplus 0))$ as $\exp(t\mathrm{F}_n\mathrm{X}) - 1$ is finite rank [*1*].

The method of proof is to show that the phase of $\Gamma_{\mathrm{A}}(\mathrm{T}(\mathrm{R}_t^n))$ for each n and of $\Gamma_{\mathrm{A}}(\mathrm{T}(\mathrm{R}_t))$ may be chosen so that as $n \to \infty$ the sequence $\Gamma_{\mathrm{A}}(\mathrm{T}(\mathrm{R}_t^n))$ converges strongly to $\Gamma_{\mathrm{A}}(\mathrm{T}(\mathrm{R}_t))$. To this end we exploit some results of Ruijsenaars [*8*]. In [*8*] a self-dual CAR algebra is introduced which may be identified with ours via the correspondences $\mathrm{K} \Leftrightarrow \mathscr{H}$, $\mathrm{P_A K} \Leftrightarrow \mathscr{H}_+$, $\hat{\Gamma} \Leftrightarrow \mathrm{C}$ where the latter symbols in each case are those of [*8*]. Then equations (4.1) and (5.3) of [8] show that whenever $\ker P_A V(R) P_A = (0)$ for $R \in SO(E)_2$, the phase of $\Gamma_A(T(R))$ may be fixed by requiring

$$\langle \Omega_A, \Gamma_A(T(R))\Omega_A \rangle > 0 . \tag{2.3}$$

Now for t sufficiently small (2.3) may be used to fix the phase of $\Gamma_A(T(R_t))$ and $\Gamma_A(T(R_t^n))$ independently of n since as $t \to 0$ $V(R_t)$ and $V(R_t^n)$ converge uniformly to I.

Consider (with this phase choice)

$$\begin{aligned}\langle \Omega_A, [\Gamma_A(T(R_t^n))^* - \Gamma_A(T(R_t))^*]\,[\Gamma_A(T(R_t^n)) - \Gamma_A(T(R))]\Omega_A \rangle \\ = 2 - 2\,\mathrm{Re}\,\langle \Omega_A, \Gamma_A(T(R_t^n))^*\Gamma_A(T(R_t))\Omega_A \rangle .\end{aligned}$$

As π_{P_A} is an irreducible representations of $\mathscr{A}_s(K)$ we have

$$\Gamma_A(T(R_t^n))^*\Gamma_A(T(R_t)) = \gamma_n \Gamma_A(T(R_t^{n*}R_t))$$

for some $\gamma_n \in \mathbb{C}$ with $|\gamma|_n = 1$ where the phase of $\Gamma_A(T(R_t^{n*}R_t))$ is again fixed by (2.3). (Note that

$$V(R_t^{n*}R_t) = \exp(t(1 - F_n)X)$$

and for sufficiently small t, $\ker P_A \exp(t(1-F_n)X)P_A = (0)$). Then I claim that the sequence $\{\gamma_n \Gamma_A(T(R_t^n))\}$ in $\pi_{P_A}(\mathscr{A}_s(H \oplus 0))''$ converges strongly to $\Gamma_A(T(R_t))$ as $n \to \infty$. To see this note that it is sufficient to show that

$$\| \gamma_n \Gamma_A(T(R_t^n))\Omega_A - \Gamma_A(T(R_t))\Omega_A \| \to 0$$

as $n \to \infty$ since this will then imply strong convergence on the dense subspace generated from Ω_A by polynomials in $\pi_A(B(k))$ for $k \in K$. But the preceding calculation gives

$$\| \gamma_n \Gamma_A(T(R_t^n))\Omega_A - \Gamma_A(T(R_t))\Omega_A \|^2 = 2 - 2\,\mathrm{Re}\,\langle \Omega_A, \Gamma_A(T(R_t^{n*}R_t))\Omega_A \rangle .$$

Ruijsenaars [8] has computed (equation (4.45))

$$\langle \Omega_A \Gamma_A(T(R_t^{n*}R_t))\Omega_A \rangle = \det(1 - P_A V_n^*(1 - P_A)V_n P_A)^{-1/4}$$

where $V_n = \exp(t(1-F_n)X)$. The right hand side of this $\langle \Omega_A, \Gamma_A \ldots \rangle$ expression depends continuously on the Hilbert-Schmidt norm of $V_n - I$ and since $V_n - I$ converges to zero as $n \to \infty$ we have the required result. □

The preceding argument when combined with the discussion on p.423-424 of [1] can be used to establish the following analogue of Theorem 1.1 for self dual CAR algebras.

Notice that G_2 may be identified with the group of unitary operators U on H which commute with Γ and such that $U - I$ is Hilbert-Schmidt and $\ker(U + I)$ is even dimensional *or* $U + I$ is Hilbert-Schmidt and $\ker(U - I)$ is odd dimensional.

THEOREM 2.10. — *If* $R \in G_2$, $\alpha_{T(R)}$ *is implemented in* π_A. *If* 0 *is not in the spectrum of* A *then a Bogoliubov automorphism* $\alpha_{T(R)}$ *which is implemented in* π_A *is inner if and only if* $R \in G_2$.

3. PROOF OF THEOREM 1.1

The results of the previous section show that if $R \in SO(E)_2$ then α_R is inner. We revert to the notation of the introduction and for the rest of the proof follow Blattner [2]. If $R + I$ is Hilbert-Schmidt and $\dim \ker (R - I)$ is odd then let $\{ e_i \}_{i=1}^n$ be a basis for $\ker (R - I)$. Let $R_{e_1 \dots e_n}$ denote the operator on E defined by

$$c(u) \rightarrow c(e_1) \dots c(e_n) c(u) c(e_n) \dots c(e_1), \qquad u \in E .$$

Then if $T = R_{e_1 \dots e_n} R$, $T \in SO(E)_2$ and $\dim \ker (T + I) = 0$. Thus by the preceding lemma, α_T is inner. But then α_R is inner. Thus all the elements of G_2 define inner automorphisms.

Conversely, if α_R is inner then $R - I$ (resp. $R + I$) is Hilbert-Schmidt whenever $\Gamma_c(R)$ is even (resp. odd) by lemma 2.6 .But if $\Gamma_c(R)$ is even (resp. odd) and $\dim \ker (R + I)$ (resp. $\dim \ker (R - I)$) is odd (resp. even) then $- R \in G_2$ and hence α_{-R} is also inner by the argument of the preceding paragraph. But then $- I = R(- R)$ so that α_{-I} is inner contradicting lemma 2.4. So $R \in G_2$ proving the theorem.

We note some corollaries. Firstly, an argument of de la Harpe and Plymen [4] generalises to our context. If G is a separable locally compact group and ρ is the regular representation of G acting on $L^2(G)$, then we can let ρ_∞ be an infinite direct sum of copies of ρ acting on a Hilbert space which we will call E. The natural complex structure on $L^2(G)$ defines a complex structure on E, say J, so we may form $\mathscr{C}(E)$ and the quasifree representation π_λ of $\mathscr{C}(E)$ given by the skew-adjoint operator $(2\lambda - 1)J$. The operators $\rho(g)$, $g \in G$ are unitary on E and so are implementable in π_λ. Moreover neither $\rho(g) + I$ or $\rho(g) - I$ can be Hilbert-Schmidt unless g is the identity element of G. Thus we have

PROPOSITION 3.1. — *If* G *is a separable locally compact group then* G *has a representation by automorphisms of* $\mathscr{C}(E)$, *implementable in* π_λ, *and hence a representation by automorphisms of the hyperfinite* $III_{\lambda/1-\lambda}$ *factor* $(0 < \lambda < 1/2)$, *such that only the identity element of* G *gives an inner automorphism.*

Remark 3.2. — *Our assumptions on* C *mean that we have not considered the case of hyperfinite* II_∞, III_0 *or* III_1 *factors. It would appear in particular that when a quasifree state on the self dual* CAR *algebra is determined by some self adjoint operator with zero in its spectrum, then the structure of the group of implementable automorphisms is considerably more complicated. An analysis for irreducible representations appears in* [3].

We note on final result. With $C = J(1 - 2\lambda)$, $0 < \lambda < 1/2$ denote by $\mathcal{O}$ the group of all orthogonal operators on E, implementable in π_λ. Then by [*1*] lemma 5.3 $R \in \mathcal{O}$ if and only if RJ − JR is Hilbert-Schmidt. From [*9*] we know that $\mathcal{O}$ is isomorphic to the group of orthogonal operators implementable in the Fock representation of $\mathscr{C}(E)$ determined by J. The content of theorem 1.5 of [*3*] is that this latter group is equipped with a natural topology which we transfer to $\mathcal{O}$, the map $j : \mathcal{O} \to \mathbb{Z}_2$ defined by

$$j(R) = \dim_{\mathbb{C}} \ker (JRJ - R) \pmod 2$$

is then a continuous homomorphism and moreover $\ker j$ is the connected component of the identity in $\mathcal{O}$.

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