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Section A :
Physique théorique.

The De Broglie Fusion method applied to quarks at the ends of strings

by

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ABSTRACT. — Wave equations are found for diquarks restricted to a world line traced out by the end of a relativistic string. The method used involves combining two restricted Dirac equations using a variant of the de Broglie fusion method. The equation describing the spin 0 diquark is the Klein-Gordon equation restricted to the world line. However, the spin 1 diquark equation differs from the restricted Proca-de Broglie equation.

RÉSUMÉ. — Nous trouvons les équations d'ondes pour un diquark lié à la courbe décrite par l'extrémité d'une corde relativiste. La méthode consiste à combiner deux équations de Dirac restreintes, en utilisant une variante de la « méthode de fusion » de de Broglie. L'équation qui décrit le diquark de spin 0 est celle de Klein-Gordon, restreinte à la courbe. Cependant, l'équation pour le diquark de spin 1 n'est pas la même que l'équation de Proca-de Broglie restreinte.

1. INTRODUCTION

In 1943 a method was devised by de Broglie [*1*] to couple two or more free Dirac fields to produce higher spin fields. By coupling two free Dirac

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fields, de Broglie's « fusion method » produces the scalar Klein-Gordon and vector Proca-de Broglie equations.

In this paper we apply the de Broglie fusion method to two Dirac fields which have been restricted to a world line. The procedure of restricting free fields to world lines was developed by Bars and Hanson [*2*] to enable them to model fermionic quarks following the world lines traced out by the ends of a relativistic string. By using the fusion method one can extend the string model to include hadrons in which strings terminate in diquarks [*3*].

In § 2 we review previous work concerning the restriction of the Dirac field to a world line, emphasising in particular the case of the world line of a particle executing uniform circular motion. In § 3 the fusion method is applied to two Dirac fields to determine the equations describing the field of a diquark restricted to an arbitrary world line. The particular case of uniform circular motion is dealt with in § 4. The equations are interpreted as those of spin 0 and spin 1 diquarks.

Our conventions regarding notation are explained in Appendix 1.

2. THE DIRAC FIELD ON A WORLD LINE

By demanding that the field derivatives normal to a given submanifold of Minkowski 4-space vanish, Bars and Hanson [*2*] give a recipe for restricting a free classical field, described by a lagrangian density, to that submanifold. In particular, if a field with lagrangian density $\mathscr{L}(\psi, \psi_{,\mu})$ is to be restricted to the world line $x^\mu(\tau)$, then the following algorithm gives a lagrangian $\mathrm{L}(\psi, \psi_\tau)$ defined only on the world line:

$$\left.\begin{array}{ll} i) & \text{replace } \psi(x) \text{ by } \psi(\tau) \\ ii) & \text{replace } \partial_\mu \psi \text{ by } x_{\mu\tau}(x_\tau^2)^{-1}\psi_\tau \\ iii) & \text{multiply by an overall factor } (x_\tau^2)^{\frac{1}{2}} \end{array}\right\} \tag{1}$$

The subscript τ stands for $\partial/\partial\tau$.

By applying this recipe to the free field Dirac equation, Bars and Hanson determine the contribution to the action for quarks at the extremities of a relativistic string. For each quark, varying the action with respect to the field for that quark gives the restricted Dirac field equation

$$\dot{\psi}(s) + \left(\frac{1}{2}\,\dot{\not{x}}(s)\ddot{\not{x}}(s) + im\dot{\not{x}}(s)\right)\psi(s) = 0. \tag{2}$$

In (2) the parameter τ has been set equal to the proper time s along the world line. Dots indicate differentiation with respect to s, and the slash has its usual meaning, viz. $\not{a} = a^\mu \gamma_\mu$.

These results have also been obtained by Kikkawa and Sato [4] using an approach from lattice gauge theory. In their treatment the quark field ψ is a Grassmann algebra valued $\underline{a}$ number. When the full colour- flavour symmetry is included, the Grassmann algebra formalism serves the purpose of ensuring that only totally antisymmetric states survive. Since we are only concerned with the spin states, it is sufficient for our purposes to consider complex valued solutions of (2).

Equation (2) has been solved by Kikkawa *et al.* [5] for the particular case when the world line is that traced out by a point executing uniform circular motion about the origin in the $x - y$ plane. Setting τ equal to the co-ordinate time x^0, this world line is given by

$$x^\mu(\tau) = (\tau, \rho \cos \omega\tau, \rho \sin \omega\tau, 0) \tag{3}$$

where ρ is the radius of orbit and ω the angular frequency. By defining α such that $\rho\omega = \sin \alpha$, the four linearly independent solutions can be written as (using a similar notation to reference [2] with $s = \tau \cos \alpha =$ proper time)

$$\psi_i(s) = \exp\left(- \frac{1}{2} i\omega\tau\sigma_{12} \right)\tilde{\psi}_i(s) \tag{4}$$

where

$$\left.\begin{aligned} \tilde{\psi}_{1,2}(s) &= u_{1,2} \exp\left[- i(m - \mathrm{S}\omega \sec^2 \alpha)s\right] \\ &= u_{1,2} \exp(- ik_{1,2}s) \end{aligned}\right\} \tag{5a}$$

are positive energy spinors and

$$\left.\begin{aligned} \tilde{\psi}_{3,4}(s) &= v_{1,2} \exp\left[i(m + \mathrm{S}\omega \sec^2 \alpha)s\right] \\ &= v_{1,2} \exp(- ik_{3,4}s) \end{aligned}\right\}. \tag{5b}$$

are negative energy spinors. The spin S takes values $\pm \frac{1}{2}$ corresponding to the subscripts 1 and 2 respectively. The spin matrix is defined by

$$\mathrm{S}_{\mu\nu} = \frac{1}{4} \bar{\psi} \{ \sigma_{\mu\nu}, \not{\dot{x}} \} \psi \tag{6}$$

and the third component of spin, $\mathrm{S}^3 = \mathrm{S}_{12}$, takes the values $\mathrm{S} \sec \alpha$.

The four component spinors u_1, u_2, v_1 and v_2 are given by

$$u_1 = \frac{1}{2}\,[\cos\alpha(1-\cos\alpha)]^{-\frac{1}{2}} \begin{bmatrix} \sin\alpha \\ i(\cos\alpha - 1) \\ \sin\alpha \\ i(1-\cos\alpha) \end{bmatrix} \tag{7a}$$

$$u_2 = \frac{1}{2}\,[\cos\alpha(1-\cos\alpha)]^{-\frac{1}{2}} \begin{bmatrix} 1-\cos\alpha \\ -i\sin\alpha \\ \cos\alpha - 1 \\ -i\sin\alpha \end{bmatrix} \tag{7b}$$

$$v_1 = \frac{1}{2}\,[\cos\alpha(1-\cos\alpha)]^{-\frac{1}{2}} \begin{bmatrix} i\sin\alpha \\ 1-\cos\alpha \\ -i\sin\alpha \\ 1-\cos\alpha \end{bmatrix} \tag{7c}$$

$$v_2 = \frac{1}{2}\,[\cos\alpha(1-\cos\alpha)]^{-\frac{1}{2}} \begin{bmatrix} i(\cos\alpha - 1) \\ -\sin\alpha \\ i(\cos\alpha - 1) \\ \sin\alpha \end{bmatrix} \tag{7d}$$

3. THE FUSION METHOD ON AN ARBITRARY WORLD LINE

We return to the restricted Dirac equation (2) where $x^\mu(s)$ is any arbitrarily specified world line. In practice, if we are dealing with a string hadron with quarks or antiquarks at the free ends of the string segments, the world line traced out by each quark will be determined by the condition that the rate of change of momentum of that quark is equal to the string tension at the end of the string. In general, then, we can think of $x^\mu(s)$ as being arbitrarily specified, keeping in mind that energy and momentum are not conserved for the individual quark field.

Consider the tensor product of the space of solutions to the restricted Dirac equation with itself. Any element of this tensor product space can be realized as a 4×4 matrix of the form

$$\Psi(s) = \sum_{i,j} \lambda_{ij}\psi^{(i)}(s)\psi^{(j)}(s)^{\mathrm{T}} \tag{8}$$

where the $\psi^{(i)}$ form a complete linearly independent set of solutions to the linear equation (2). We split Ψ into symmetric and antisymmetric parts Ψ_S and Ψ_A respectively and expand in terms of the 16 independent products of Dirac matrices I, γ_μ, $\sigma_{\mu\nu}$, $\gamma_\mu\gamma_5$ and γ_5, and the charge conjugate matrix C as follows:

$$\Psi_A(s) = \phi(s)C + \chi^\mu(s)\gamma_\mu\gamma_5 C + \chi(s)\gamma_5 C \tag{9a}$$

$$\Psi_S(s) = \phi^\mu(s)\gamma_\mu C + \frac{1}{2}\phi^{\mu\nu}(s)\sigma_{\mu\nu}C. \tag{9b}$$

The quantities $\phi(s)$, $\chi^\mu(s)$, etc. are to be thought of as the state vectors of the two coupled Dirac particles.

If $\psi^{(1)}$ and $\psi^{(2)}$ are any two solutions of (2) then we have

$$\left.\begin{aligned} &\frac{d}{ds}(\psi^{(1)}\psi^{(2)T}) + \left(\frac{1}{2}\dot{\not{x}}\ddot{\not{x}} + im\dot{\not{x}}\right)\psi^{(1)}\psi^{(2)} \\ &\qquad + \psi^{(1)}\psi^{(2)T}\left(\frac{1}{2}\dot{\not{x}}\ddot{\not{x}} + im\dot{\not{x}}\right)^T = 0. \end{aligned}\right\} \tag{10}$$

Since (10) is linear we can replace $\psi^{(1)}\psi^{(2)T}$ by Ψ from (8). Writing the expansions (9) more compactly as $\Psi = \phi^a\gamma_a C$ where a is a generic label for the indices μ, $\mu\nu$, etc., we have

$$\dot{\phi}^a\gamma_a C + \left(\frac{1}{2}\dot{\not{x}}\ddot{\not{x}} + im\dot{\not{x}}\right)\phi^a\gamma_a C + \phi^a\gamma_a C\left(\frac{1}{2}\dot{\not{x}}\ddot{\not{x}} + im\dot{\not{x}}\right)^T = 0. \tag{11}$$

Using the properties of the charge conjugate matrix and multiplying on the right by C^{-1} gives

$$\dot{\phi}^a\gamma_a - \frac{i}{2}\dot{x}^\mu\ddot{x}^\nu\phi^a[\sigma_{\mu\nu}, \gamma_a] + im\dot{x}^\mu\phi^a[\gamma_\mu, \gamma_a] = 0 \tag{12}$$

Setting the coefficient of each γ_a equal to zero gives a set of first order coupled differential equations. From the antisymmetric part of Ψ we have

$$\dot{\phi} = 0 \tag{13a}$$

$$\dot{\chi}^\mu + (\dot{x}^\mu\ddot{x}^\nu - \dot{x}^\nu\ddot{x}^\mu)\chi_\nu + 2im\dot{x}^\mu\chi = 0 \tag{13b}$$

$$\dot{\chi} + 2im\dot{x}^\mu\chi_\mu = 0 \tag{13c}$$

and from the symmetric part

$$\dot{\phi}^\mu + (\dot{x}^\mu\ddot{x}^\nu - \dot{x}^\nu\ddot{x}^\mu)\phi_\nu + 2m\phi^{\mu\nu}\dot{x}_\nu = 0 \tag{14a}$$

$$\dot{\phi}^{\mu\nu} + \phi^{\mu\rho}(\ddot{x}_\rho\dot{x}^\nu - \ddot{x}^\nu\dot{x}_\rho) + (\dot{x}^\mu\ddot{x}_\rho - \dot{x}_\rho\ddot{x}^\mu)\phi^{\rho\nu} + 2m(\dot{x}^\mu\phi^\nu - \dot{x}^\nu\phi^\mu) = 0 \tag{14b}$$

We shall deal first with the antisymmetric part. In general there will be four linearly independent solutions to (2). Concealed within (13) is the information pertaining to the six possible linearly independent antisymmetric tensor products of these solutions. To extract this information it is useful to divide χ^μ into its parts parallel to and normal to the world line. We define

$$\chi_{||} = \dot{x}_\mu \chi^\mu = P_{||\mu}\chi^\mu \tag{15a}$$
$$\chi_\perp^\mu = (\delta_\nu^\mu - \dot{x}^\mu \dot{x}_\nu)\chi^\nu = P_{\perp\nu}^\mu \chi^\nu \tag{15b}$$

where $P_{||\mu}$ and $P_{\perp\nu}^\mu$ are operators projecting onto the parallel and normal subspaces to the world line at each point.

Writing the field equations (13) in terms of ϕ, $\chi_{||}$, $\chi_\perp^\mu$ and χ we can eliminate $\chi_{||}$ entirely leaving the three uncoupled equations

$$\dot{\phi} = 0 \tag{16a}$$
$$\dot{\chi}_\perp^\mu + \dot{x}^\mu \ddot{x}_\nu \chi_\perp^\nu = 0 \tag{16b}$$
$$\ddot{\chi} + 4m^2\chi = 0. \tag{16c}$$

Equation (16*c*) can be recognised as the Klein Gordon equation, restricted by the recipe (1), to an arbitrary world line. We therefore assume that the field χ is to be associated with antisymmetric spin zero states. In the next section we consider the particular world line described by (3) and show that, for this world line, χ is in fact the spin zero diquark and antidiquark field. We shall also interpret the remaining field equations (16*a*) and (16*b*) as descriptions of bound quark-antiquark states.

Consider next the symmetric states, which are described by the field equations (14). These equations contain information pertaining to the ten linearly independent symmetric bound states. As for the antisymmetric case it is useful to define an alternate set of fields parallel to and normal to the world line. We set

$$\phi_{||} = \dot{x}_\mu \phi^\mu = P_{||\mu}\phi^\mu \tag{17a}$$
$$\phi_\perp^\mu = (\delta_\nu^\mu - \dot{x}^\mu \dot{x}_\nu)\phi^\nu = P_{\perp\nu}^\mu \phi^\nu \tag{17b}$$
$$\theta^{\mu\nu} = (\delta_\rho^\mu - \dot{x}^\mu \dot{x}_\rho)\phi^{\rho\sigma}(\delta_\sigma^\nu - \dot{x}_\sigma \dot{x}^\nu) = P_{\perp\rho}^\mu \phi^{\rho\sigma} P_{\perp\sigma}^\nu \tag{17c}$$
$$\theta_\perp^\mu = \phi^{\mu\nu}\dot{x}_\nu = \phi^{\mu\nu}P_{||\nu} \tag{17d}$$

Because $\phi^{\mu\nu}$ is antisymmetric in its indices μ and ν it is easy to see that $\theta_\perp^\mu$ is in fact normal to the world line.

Equations (14) can now be written in terms of the new fields (17) to give

$$\dot{\phi}_{||} = 0 \tag{18a}$$
$$\dot{\phi}_\perp^\mu + \dot{x}^\mu \ddot{x}_\nu \phi_\perp^\nu + 2m\theta_\perp^\nu = 0 \tag{18b}$$
$$\dot{\theta}_\perp^\mu + \dot{x}^\mu \ddot{x}_\nu \theta_\perp^\nu - 2m\phi_\perp^\mu = 0 \tag{18c}$$
$$\dot{\theta}^{\mu\nu} + \dot{x}^\mu \ddot{x}_\rho \theta^{\rho\nu} + \theta^{\mu\rho}\ddot{x}_\rho \dot{x}^\nu = 0 \tag{18d}$$

Equations (18*a*) and (18*b*) are obtained by contracting (14*a*) on the left by the operators $P_{||}$ and $P_\perp$ respectively. Equation (18*c*) is obtained by applying $P_{||}$ to (14*b*) on the right, and (18*d*) by applying $P_\perp$ to (14*b*) on both the left and right hand sides.

We can eliminate $\theta^\mu_\perp$ from (18b) and (18c) and multiply on the left by the normal projection operator $P_\perp$ to give a single second order equation in $\phi^\mu_\perp$ only:

$$(\delta^\mu_\nu - \dot{x}^\mu \dot{x}_\nu)\ddot{\phi}^\nu_\perp - \ddot{x}^\mu \dot{x}_\nu \dot{\phi}^\nu_\perp + 4m^2 \phi^\mu_\perp = 0 \qquad (19)$$

The symmetric coupled states are now fully described by the uncoupled equations (18a), (18d) and (19). In the next section we shall interpret these equations for the world line described by (3). For this particular world line the field $\phi^\mu_\perp$ will be seen to represent the symmetric spin 1 diquark and antidiquark states, while (18a) and (18d) will be interpreted as descriptions of symmetric $q\bar{q}$ states.

4. SPECIFIC SOLUTIONS

We return to the case where the world line is that of a point executing uniform circular motion, given by (3). Combining the four independent states (4) two at a time one forms 10 symmetric and 6 antisymmetric bound states. In this section we associate these bound states with the solutions to the field equations found in § 3. In particular we show that the field equations (16c) and (19) describe fully the spin 0 and spin 1 diquark and antidiquark states.

For any particular state, the functions ϕ^a can be found by inverting the expansion (9) to give

$$\phi^a(s) = \frac{1}{4} \sum_{i,j} \lambda_{ij} \psi^{(j)\mathrm{T}}(s) C^{-1} \gamma^a \psi^{(i)} \qquad (20)$$

where $\gamma^a = (\gamma_a)^{-1}$. Substituting the solutions (4) into (20) gives

$$\phi^a(s) = \frac{1}{4} \sum_{i,j} \lambda_{ij} u^{(j)\mathrm{T}} C^{-1} e^{\frac{i\omega\tau}{2}\sigma_{12}} \gamma^a e^{-\frac{i\omega\tau}{2}\sigma_{12}} u^{(i)} e^{-i(k_i + k_j)s}$$

$$= \frac{1}{4} \Lambda^a_b(s) \sum_{i,j} \lambda_{ij} u^{(j)\mathrm{T}} C^{-1} \gamma^b u^{(i)} e^{-i(k_1 + k_j)s} \qquad (21)$$

where $\Lambda^a_b(s)$ is a rotation of $\omega\tau$ about the z-axis in the representation of the Lorentz group appropriate to the indices a and b.

Because of the presence of the factor Λ^a_b, any solution ϕ^a with a certain orientation to the world line at $s = 0$ will maintain that orientation to the world line for all s. Therefore, to determine which states correspond to which of the equations found in section 3 it is sufficient to consider the functions given by (21) at $s = 0$, i. e.

$$\phi^a(0) = \frac{1}{4} \sum_{i,j} \lambda_{ij} u^{(j)\mathrm{T}} C^{-1} \gamma^a u^{(i)} \qquad (22)$$

and examine their orientation to the world line.

In order to evaluate the expression (22) for all cases of interest more easily we transform to the rest frame at $s = 0$. We show in Appendix 2 that the four spinors given by the expressions (7*a*) to (7*d*) become

$$u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}; \quad u_2 = \frac{i}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}; \quad v_1 = \frac{i}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}; \quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} \tag{23}$$

in this frame. In tables 1 and 2 we list the results of substituting these spinors into (22) for all possible antisymmetric and symmetric combinations of solutions to the restricted Dirac equation. Since we are working in the rest frame, for which $\dot{x}^\mu = (1, 0, 0, 0)$, it is easy to recognise alignment of states parallel to or normal to the world line.

Consider first the antisymmetric states in table 1, which we compare

TABLE 1. — *The quantities* $\phi(0)$, $\chi(0)$ *and* $\chi^\mu(0)$ *calculated from the formula* (22) *in the rest frame for antisymmetric combinations of solutions to the restricted Dirac equation*[A].

State[B]	$\phi(0)$	$\chi(0)$	$\chi^\mu(0)$
Diquark states, qq and $\bar{q}\bar{q}$:			
$\frac{1}{\sqrt{2}}(\uparrow_+\downarrow_+ - \downarrow_+\uparrow_+)$ } $\frac{1}{\sqrt{2}}(\uparrow_-\downarrow_- - \downarrow_-\uparrow_-)$	0	1	(1, 0, 0, 0)
$q\bar{q}$ states:			
$\frac{1}{\sqrt{2}}(\uparrow_+\uparrow_- - \uparrow_-\uparrow_+)$	0	0	(0, 1, i, 0)
$\frac{1}{\sqrt{2}}(\uparrow_+\downarrow_- - \downarrow_-\uparrow_+)$ } $\frac{1}{\sqrt{2}}(\uparrow_-\downarrow_+ - \downarrow_+\uparrow_-)$	1	0	(0, 0, 0, 1)
$\frac{1}{\sqrt{2}}(\downarrow_-\downarrow_+ - \downarrow_-\downarrow_+)$	0	0	(0, 1, $-i$, 0)

A. Up to phase factors and not normalised.
B. $\uparrow$ = spin up, $\downarrow$ = spin down; subscripts indicate the sign of the energy.

TABLE 2. — *The quantities $\phi^{\mu}(0)$ and $\phi^{\mu\nu}(0)$ calculated from the formula* (22) *in the rest frame for symmetric combinations of solutions to the restricted Dirac equation*[A].

State[B]	$\phi^{\mu}(0)$	$\phi^{\mu\nu}(0)$
Diquark states, qq and $\overline{qq}$:		
$\uparrow_+\uparrow_+$ and $\uparrow_-\uparrow_-$	$\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & i & 0 \\ -1 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$\left.\begin{matrix} \frac{1}{\sqrt{2}}(\uparrow_+\downarrow_+ + \downarrow_+\uparrow_+) \\ \frac{1}{\sqrt{2}}(\uparrow_-\downarrow_- + \downarrow_-\uparrow_-) \end{matrix}\right\}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$
$\downarrow_+\downarrow_+$ and $\downarrow_-\downarrow_-$	$\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -i & 0 \\ -1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$q\bar{q}$ states:		
$\frac{1}{\sqrt{2}}\uparrow_+\uparrow_- + \uparrow_-\uparrow_+)$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ 0 & -1 & -i & 0 \end{pmatrix}$
$\left.\begin{matrix} \frac{1}{\sqrt{2}}(\uparrow_+\downarrow_- + \downarrow_-\uparrow_+) \\ \frac{1}{\sqrt{2}}(\uparrow_-\downarrow_+ + \downarrow_+\uparrow_-) \end{matrix}\right\}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$\frac{1}{\sqrt{2}}(\downarrow_+\downarrow_- + \downarrow_-\downarrow_+)$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & -1 & i & 0 \end{pmatrix}$

A. Up to phase factors and not normalised.
B. $\uparrow$ = spin up, $\downarrow$ = spin down; subscripts indicate the sign of the energy.

with the solutions of the field equations (16*a*), (16*b*) and (16*c*) from § 3. For the diquark and antidiquark spin zero singlets only the fields χ and $\chi_{||}$ are non-zero. We can therefore associate these states with the two independent solutions of (16*c*), which involves only χ. It follows from (13) and (15) that $\chi_{||}$ satisfies an identical equation. The remaining antisymmetric states, namely $q\bar{q}$ states, are described fully by the projection $\chi_\perp^\mu$ of χ^μ normal to the world line. We therefore identify these states with the solutions of (16*b*). Equation (16*a*) is redundant.

Next we consider the symmetric states, which are listed in table 2. These states are to be associated with solutions to the field equations (18*a*), (18*d*) and (19). Examining table 2, we see that the spin 1 diquark and antidiquark triplets are equally well described by the projection $\phi_\perp^\mu$ of ϕ^μ normal to the world line or the field $\theta_\perp^\mu$, defined by (17*d*) from $\phi^{\mu\nu}$. We therefore associated these states with solutions of the field equation (19). If one chooses to eliminate the field $\phi_\perp^\mu$ from (18*b*) and (18*c*) instead of $\theta_\perp^\mu$, an equation identical to (19) results with $\phi_\perp^\mu$ replaced by $\theta_\perp^\mu$.

Finally we note that the symmetric $q - \bar{q}$ states in table 2 are fully described by the fields ϕ^{ij} $(i, j = 1, 2, 3)$. Because we are working in the instantaneous rest frame, these are just the fields $\theta^{\mu\nu}$ defined by (17*c*). We therefore associate these states with the field equation (18*d*) and note that the remaining equation, (18*a*) is redundant.

So far we have determined that the bound $q - q$ states are solutions to particular field equations found in § 3 when the world line is that given by (3). Conversely, we can find a complete set of solutions for the equations found in § 3 for this particular world line in the following manner: guided by the formula (21) we seek solutions of the form

$$\phi^a(s) = \Lambda^a_b(s) e^{-i\mathrm{K}s} \phi^b(0) . \tag{24}$$

Substitution of this function into the differential equation will give a matrix equation in which $\phi^a(0)$ and K are unknown but can be solved for. We apply this technique to the spin 1 diquark equation (19) to show that there are no solutions other than the three positive energy and three negative energy solutions which have been attributed to this equation.

Substituting the appropriate version of (24) into (19) and multiplying through on the left by $\Lambda_\nu^{-1\mu}(s)$ gives the matrix equation

$$\left.\begin{array}{l} \{ (\mathrm{I} - \dot{x}(0)\dot{x}(0)) [(\Lambda^{-1}(0)\dot{\Lambda}(0))^2 - 2i\mathrm{K}\Lambda^{-1}(0)\dot{\Lambda}(0) - \mathrm{K}^2] \\ - \ddot{x}(0)\dot{x}(0)(\Lambda^{-1}(0)\dot{\Lambda}(0) - i\mathrm{K}) + 4m^2 \}^\mu{}_\nu \phi_\perp^\nu(0) = 0 \end{array}\right\} \tag{25}$$

In the rest frame at $s = 0$ we have

$$\left.\begin{array}{l} (\Lambda^{-1}\dot{\Lambda})^\mu_\nu = \omega \sec\alpha \begin{bmatrix} 0 & -\tan\alpha & 0 & 0 \\ -\tan\alpha & 0 & -\sec\alpha & 0 \\ 0 & \sec\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \dot{x}^\mu(0) = (1, 0, 0, 0) ; \ \ddot{x}^\mu(0) = (0, -1, 0, 0)\omega \sin\alpha \sec^2\alpha \end{array}\right\} \tag{26}$$

giving the matrix equation (25) as

$$\begin{bmatrix} 4m^2 & 0 & 0 & 0 \\ \dfrac{i\mathrm{K}\sin\alpha}{\cos^2\alpha} & 4m^2 - \dfrac{\omega^2}{\cos^4\alpha} - \mathrm{K}^2 & \dfrac{2i\mathrm{K}\omega}{\cos^2\alpha} & 0 \\ -\dfrac{\omega^2\sin\alpha}{\cos^4\alpha} & -\dfrac{2i\mathrm{K}\omega}{\cos^2\alpha} & 4m^2 - \dfrac{\omega^2}{\cos^4\alpha} - \mathrm{K}^2 & 0 \\ 0 & 0 & 0 & 4m^2 - \mathrm{K}^2 \end{bmatrix} \begin{bmatrix} \phi_\perp^0(0) \\ \phi_\perp^1(0) \\ \phi_\perp^2(0) \\ \phi_\perp^3(0) \end{bmatrix} = 0 \tag{27}$$

The only solutions to this matrix equation are given by the vectors $\varphi^\mu(0)$ in table 2 corresponding to diquark states. The corresponding values for the frequencies K agree with those found by adding the frequencies k_i of the constituent quarks.

We note in passing that the spin 1 diquark equation (19) obtained from the fusion method differs from the equation obtained by restricting the free field Proca de Broglie field to the world line. The free field lagrangian density for the vector field Φ^μ of mass $2m$ is

$$\mathscr{L}_{\mathrm{P}-de\,\mathrm{B}}(\Phi, \partial_\mu\Phi) = -\frac{1}{4}(\partial_\mu\Phi_\nu^* - \partial_\nu\Phi_\mu^*)(\partial^\mu\Phi^\nu - \partial^\nu\Phi^\mu) + \frac{1}{2}(2m)^2\Phi_\mu^*\Phi^\mu . \tag{28}$$

Restricting to the world line $x^\mu(s)$ according to the formula (1) gives the wave equation

$$(\delta_\nu^\mu - \dot{x}^\mu\dot{x}_\nu)\ddot{\Phi}^\nu - (\dot{x}^\mu\ddot{x}_\nu + \ddot{x}^\mu\dot{x}_\nu)\dot{\Phi}^\nu + 4m^2\Phi^\mu = 0, \tag{29}$$

which we compare with (19).

CONCLUSION

We have found field equations governing the evolution of bound states of two Dirac fields restricted to a world line. As a particular application, we are interested in describing bound states of two quarks at the extremities of a relativistic string.

When the world line is that of a point executing uniform circular motion, the spin 0 diquark at the end of a string is described by (16*c*). We recognise this equation as the Klein-Gordon equation restricted to a world line. The spin 1 diquark for this world line is described by the field equation (19). This equation differs from that obtained by restricting a free Proca-de Broglie field to a world line.

Bound states of a quark and an antiquark are described by equations (18*d*) for spin-symmetric states and (16*b*) for spin-antisymmetric states. From the point of view of the string model there is nothing to preclude the possibility of a string terminating in a $q\bar{q}$ state, provided the state belongs to a colour octet. If, for instance, the other end of the string terminates in the conjugate state, we have an exotic meson.

Although we have interpreted the field equations in terms of qq, $q\bar{q}$ and $\overline{qq}$ states for a particular world line, the field equations apply for arbitrary world lines as descriptions of bound states of solutions to the restricted Dirac equation.

APPENDIX 1

NOTATION

We use the metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and set $\hbar = c = 1$. Where a particular representation of the Dirac matrices is required, we take

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}; \qquad \gamma_k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}, \qquad k = 1, 2, 3.$$

We also have

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]; \qquad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3.$$

The charge conjugate matrix, $C = i\gamma^2\gamma^0$, satisfies

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T$$

where T = transpose.

APPENDIX 2

THE SPINORS (7) IN THE INITIAL REST FRAME

Transforming from the « laboratory » frame to the rest frame at $s = 0$ is effected by the matrix

$$L^{\mu}_{\nu} = \begin{bmatrix} \sec\alpha & 0 & -\tan\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\tan\alpha & 0 & \sec\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (\mathrm{A}\,1)$$

that is, $L^{\mu}_{\nu}\dot{x}^{\nu}(0) = (1, 0, 0, 0)$. It is required to find the Dirac spinor representation (i. e. the $\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right)$ representation) of this transformation.

The matrix (A 1) can be written as

$$L^{\mu}_{\nu} = (\exp\,[iv\mathscr{S}_{02}])^{\mu}_{\nu} \quad (\mathrm{A}\,2)$$

where $(i\mathscr{S}_{\mu\nu})_{\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}$ are the generators of the homogeneous Lorentz group in the vector $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation and v is defined by

$$\cosh v = \sec\alpha\,; \qquad \sinh v = \tan\alpha. \quad (\mathrm{A}\,3)$$

The equivalent generators of the Dirac representation of the homogeneous Lorentz group are

$$i\mathscr{S}_{\mu\nu} = -\frac{i}{2}\sigma_{\mu\nu}, \quad (\mathrm{A}\,4)$$

so the Dirac representative of the matrix (A 1) is

$$\begin{aligned} L &= \exp\left[-\frac{i}{2}v\sigma_{02}\right] \\ &= \cosh\frac{1}{2}v - i\sigma_{02}\sinh\frac{1}{2}v \\ &= \frac{1}{(2\cos\alpha)^{\frac{1}{2}}}\begin{bmatrix} (1+\cos\alpha)^{\frac{1}{2}} & -i(1-\cos\alpha)^{\frac{1}{2}} & 0 & 0 \\ i(1-\cos\alpha)^{\frac{1}{2}} & (1+\cos\alpha)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & (1+\cos\alpha)^{\frac{1}{2}} & i(1-\cos\alpha)^{\frac{1}{2}} \\ 0 & 0 & -i(1-\cos\alpha)^{\frac{1}{2}} & (1+\cos\alpha)^{\frac{1}{2}} \end{bmatrix} \end{aligned} \quad (\mathrm{A}\,5)$$

Applying this matrix to the laboratory frame four-spinors (7) gives the equivalent four-spinors (23) in the rest frame at $s = 0$.

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