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E. CALICETI

M. MAIOLI

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Odd anharmonic oscillators and shape resonances

by

E. CALICETI (*) and M. MAIOLI (*)

Istituto Matematico, Università di Modena,
41100 Modena, Italy

ABSTRACT. — By adapting a stability argument of Vock and Hunziker it is proved that the Borel sum of the Rayleigh-Schrödinger perturbation expansion of any odd anharmonic oscillator $p^2 + x^2 + gx^{2k+1}$, $k = 1, 2, \dots$, is the limit of a sequence of resonances in the standard sense of dilation analyticity. The same method yields, for a suitable class of potentials, an existence proof of shape resonances.

RÉSUMÉ. — En adaptant un argument de stabilité dû à Vock et Hunziker, on montre que la somme de Borel de la série de perturbations de Rayleigh Schrödinger pour un oscillateur anharmonique impair $p^2 + x^2 + gx^{2k+1}$, $k = 1, 2, \dots$, est la limite d'une suite de résonances au sens usuel de l'analyticité par dilatation. La même méthode fournit, pour une classe convenable de potentiels, une preuve d'existence des « résonances de forme ».

1. INTRODUCTION

The purpose of this paper is to complete an earlier investigation [2] on the spectral and perturbation theory of any odd anharmonic oscillator $p^2 + x^2 + gx^{2k+1}$, $k = 1, 2, \dots$. The aim is to justify the standard physical picture of these problems (see e. g. Davydov [4] and, for a more recent discussion of this order of ideas, Coleman [3]) given in terms of unstable states generated by the shape of the potential, which should be represented by the Rayleigh-Schrödinger perturbation expansion, in the framework of dilation analyticity and Borel summability.

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Now the differential expression $H = -d^2/dx^2 + x^2 + gx^{2k+1}$, $k \in \mathbb{N}$, $g > 0$ is, when defined on $C_0^\infty(\mathbb{R})$, a symmetric operator in $L^2(\mathbb{R})$ admitting infinitely many distinct self-adjoint extensions, each of them having discrete spectrum. Therefore the differential expression H was first realized as an operator in $L^2(\mathbb{R})$ for g complex, $0 < \arg(g) < \pi$, where it represents a holomorphic family of type A with compact resolvents. It turns out that H converges in the generalized sense to $p^2 + x^2$ as $|g| \rightarrow 0$, $0 < \arg(g) < \pi$, so that $\sigma_d(H) \neq \emptyset$ for $|g|$ suitably small, and that the (divergent) Rayleigh-Schrödinger perturbation expansion near any simple eigenvalue of $p^2 + x^2$ is Borel summable to the nearby eigenvalue of H . When analytically continued to $g \in \mathbb{R}$ the eigenvalues of H can be interpreted as resonances of the problem: in particular they are shown to be second sheet poles of a unique generalized resolvent (see e. g. [1] for a definition) of the symmetric operator H .

From this fact one can ask whether it is possible to approximate $x^2 + gx^{2k+1}$ by dilation analytic potentials so that such generalized resonances become the limit of resonances of self-adjoint operators in the standard sense of dilation analyticity.

This is just what we prove in Sect. 2 of this paper, by means of the following family of approximating potentials

$$V_\alpha(x) = (x^2 + gx^{2k+1})(\alpha^2 x^{4k+2} + 1)^{-1/2}, \quad \text{as } \alpha \rightarrow 0.$$

Taking into account the result in [2] on the discreteness of the spectrum of $H(0, \theta) = e^{-\theta} p^2 + e^\theta x^2 + ge^{(2k+1)\theta/2} x^{2k+1}$ for $\text{Im } \theta > 0$ and $g > 0$, the problem reduces to a stability result for the eigenvalues of $H(0, \theta)$ with respect to the family $H(\alpha, \theta) = e^{-\theta} p^2 + V_\alpha(e^{\theta/2} x)$ as $\alpha \rightarrow 0$. In this connection we apply the stability theory recently developed by Hunziker and Vock [6]: to this end we need to extend the operator class explicitly provided for applications ([6], Example 5) and prove the uniform boundedness of the resolvents which is not given by a control of the numerical ranges.

Since $V_\alpha(x)$ exhibits the typical shape of a barrier, this result exemplifies at the same time how in some cases the so-called shape resonances [3] can be proved to exist in the standard sense of dilation analyticity. This last result, obtained by adapting to $L^2(\mathbb{R}_+)$ our arguments valid on $L^2(\mathbb{R})$, is briefly described after proving the existence of resonances for $p^2 + V_\alpha(x)$ by the above mentioned stability argument.

2. STABILITY OF EIGENVALUES AND APPROXIMATING RESONANCES

The following theorem allows to enlarge the operator class given in [6], Example 5, in which our model is not included.

THEOREM 2.1. — Let $G \subset L^2_{loc}(\mathbb{R}^v)$ and $\{A(V)\}_{V \in G}$ be a class of closable operators in $L^2(\mathbb{R}^v)$ of the form $A(V) = p^2 + V$, with $C^\infty_0(\mathbb{R}^v)$ as a core and such that for any $V \in G$ there exists a function $\gamma_V: \mathbb{R}^v \rightarrow \mathbb{R}$ satisfying the following conditions:

- (1) $\operatorname{Re} \langle u \cos \gamma_V, p^2 u \rangle + \operatorname{Im} \langle u \sin \gamma_V, p^2 u \rangle \leq \xi [\operatorname{Re} \langle u \cos \gamma_V, A(V)u \rangle + \operatorname{Im} \langle u \sin \gamma_V, A(V)u \rangle + \eta \langle u, u \rangle]$, $\forall u \in C^\infty_0(\mathbb{R}^v)$
- (2) $\inf_{x \in \mathbb{R}^v} \cos \gamma_V(x) \geq C_0 > 0$
- (3) $\gamma_V \in H^1_{loc}(\mathbb{R}^v)$ and $\|\nabla \cos \gamma_V\|_\infty \leq C_1$, $\|\nabla \sin \gamma_V\|_\infty \leq C_1$,

where ξ, η, C_0 and C_1 are positive constants independent of V . Here $H^1_{loc}(\mathbb{R}^v)$ denotes, as usual, the space of locally square integrable functions which admit first order generalized derivatives belonging to $L^2_{loc}(\mathbb{R}^v)$;

$$p^2 = - \sum_{k=1}^v \partial^2 / \partial x_k^2.$$

Then there is $b > 0$ independent of $V \in G$ such that $\forall u \in C^\infty_0(\mathbb{R}^v)$

$$\|(1 + p^2)^{1/2} u\| \leq b(\|u\| + \|A(V)u\|) \tag{2.1}$$

and there exists a sequence of multiplication operators $\{M_n\}_{n \in \mathbb{N}}$ in $L^2(\mathbb{R}^v)$ satisfying (i) and (ii) of Hypothesis 3 in [6]. Such Hypothesis is completely fulfilled if, moreover, $\Delta \neq \emptyset$ (Δ as defined in [6]).

Proof. — It suffices to prove the following inequality

$$\langle u, p^2 u \rangle \leq \xi_1 [\operatorname{Re} \langle u \cos \gamma_V, A(V)u \rangle + \operatorname{Im} \langle u \sin \gamma_V, A(V)u \rangle + \eta_1 \langle u, u \rangle] \tag{2.2}$$

$\forall u \in C^\infty_0(\mathbb{R}^v)$, with ξ_1, η_1 independent of V , $\xi_1 > 0$, $\eta_1 > 0$.

In fact (2.1) easily follows from (2.2) by Schwarz' inequality. Without loss of generality we can assume $\|u\| = 1$. Then

$$\begin{aligned} & \operatorname{Re} \langle u \cos \gamma_V, p^2 u \rangle + \operatorname{Im} \langle u \sin \gamma_V, p^2 u \rangle \\ &= \sum_{k=1}^v \langle (\cos \gamma_V) p_k u, p_k u \rangle + \sum_{k=1}^v \operatorname{Re} \langle u p_k \cos \gamma_V, p_k u \rangle + \sum_{k=1}^v \operatorname{Im} \langle u p_k \sin \gamma_V, p_k u \rangle \\ &\geq C_0 \langle u, p^2 u \rangle - \sum_{k=1}^v |\langle u p_k \cos \gamma_V, p_k u \rangle| - \sum_{k=1}^v |\langle u p_k \sin \gamma_V, p_k u \rangle| \\ &\geq C_0 \langle u, p^2 u \rangle - 2C_1 \sum_{k=1}^v \langle u, p_k^2 u \rangle^{1/2} \\ &\geq C_0 \langle u, p^2 u \rangle - C_1 \lambda^{-1} \sum_{k=1}^v 2\lambda \langle u, p_k^2 u \rangle^{1/2}, \end{aligned}$$

for any $\lambda > 0$,

$$\geq C_0 \langle u, p^2 u \rangle - C_1 \lambda^{-1} \sum_{k=1}^v [\lambda^2 + \langle u, p_k^2 u \rangle] = (C_0 - C_1 \lambda^{-1}) \langle u, p^2 u \rangle - C_1 \lambda v$$

where the second inequality follows from Schwarz' inequality and assumption (3). Here, as usual, $p_k = -i\partial/\partial x_k$, $k=1, 2, \dots, v$. Now we choose $\lambda > 0$ such that $C_0 - C_1 \lambda^{-1} > 0$ and (2.2) follows from assumption (1).

In order to show that Hypothesis 3 in [6] is satisfied we proceed as in Lemma 3.2 of [6], choosing $M_n = 1 - \chi_n$, where $\chi_n(x) = \chi(x/n)$, $\chi \in C_0^\infty(\mathbb{R}^v)$, $\chi(x) = 1$ for $|x| < 1$, and the theorem is thus proved.

Consider in $L^2(\mathbb{R})$ the formal differential expression

$$H = p^2 + x^2 + gx^{2k+1}, \quad k = 1, 2, \dots; \quad g \in \mathbb{R} \setminus \{0\}$$

where, as usual, $p^2 = -d^2/dx^2$. Since specular arguments apply for $g < 0$, we can take $g > 0$ without loss of generality.

By \bar{H} we mean the closure of the symmetric operator defined by H on $C_0^\infty(\mathbb{R})$. Set

$$H(0, \theta) = e^{-\theta} p^2 + e^\theta x^2 + g e^{(2k+1)\theta/2} x^{2k+1}.$$

For $\theta \in \mathbb{R}$, $H(0, \theta)$ can be written as $U(\theta) H U(\theta)^{-1}$, where $U(\theta) f(x) = e^{\theta/4} f(e^{\theta/2} x)$ defines the group of unitary dilations $U(\theta)$ in $L^2(\mathbb{R})$.

It is well known ([8], [9]) that \bar{H} admits infinitely many distinct self-adjoint extensions. The following proposition is proved in [2].

THEOREM 2.2. — *i)* Each self-adjoint extension of \bar{H} has discrete spectrum.

ii) Let θ be complex, $0 < \text{Im } \theta < \min(\pi/4, 2\pi/(2k+3))$. Then $H(0, \theta)$ represents a holomorphic family of type A of compact resolvent operators in $L^2(\mathbb{R})$ if defined on $D(p^2) \cap D(x^{2k+1}) = H^2(\mathbb{R}) \cap L^2_{(2k+1)/2}(\mathbb{R})$ where $H^2(\mathbb{R})$ and $L^2_{(2k+1)/2}(\mathbb{R})$ denote the usual Sobolev and weighted L^2 spaces respectively.

iii) For any compact Γ contained in the strip

$$0 < \text{Im } \theta < \min(\pi/4, 2\pi/(2k+3)),$$

$\exists g(\Gamma) > 0$ such that, $\forall g < g(\Gamma)$, $H(0, \theta)$ admits eigenvalues (independent of θ) for $\theta \in \Gamma$.

iv) There is a dense set S of dilation analytic vectors

$$(\text{for } |\text{Im } \theta| < \min(\pi/4, 2\pi/(2k+3)))$$

and a generalized resolvent $\mathcal{R}(E)$ of the closed symmetric operator \bar{H} (see e. g. [1] for a definition) such that if $\psi \in S$ the function

$$f_\psi(E) = \langle \mathcal{R}(E)\psi, \psi \rangle$$

which is *a priori* defined as an analytic function in the upper half-plane $\{E \mid \text{Im } E > 0\}$ has a meromorphic continuation to the lower half-plane $\{E \mid \text{Im } E < 0\}$ across the real axis. The set of singularities $\{E \mid f_\psi \text{ has a pole at } E \text{ for some } \psi \in S\}$ coincides with

$$\bigcup_{\theta} \{ \sigma(H(0, \theta)) \mid 0 < \text{Im } \theta < \min(\pi/4, 2\pi/(2k + 3)) \},$$

and $\mathcal{R}(E)$ is uniquely determined by the identity

$$\langle \mathcal{R}(E)\phi, \psi \rangle = \langle [H(0, \theta) - E]^{-1}\phi(\theta), \psi(\bar{\theta}) \rangle, \\ 0 < \text{Im } \theta < \min(\pi/4, 2\pi/(2k + 3)); \phi, \psi \in S, \psi(\theta)(x) = e^{\theta/4}\psi(e^{\theta/2}x); \text{Im } E > 0.$$

For $\alpha > 0$ and θ complex let us define

$$H(\alpha, \theta) = e^{-\theta}[p^2 + V_\alpha(x, \theta)] = e^{-\theta} \left[p^2 + \frac{e^{2\theta}x^2 + ge^{(2k+3)\theta/2}x^{2k+1}}{(\alpha^2 e^{(2k+1)\theta}x^{4k+2} + 1)^{1/2}} \right] \quad (2.3)$$

on the domain $D(H(\alpha, \theta)) = H^2(\mathbb{R})$, as an operator in $L^2(\mathbb{R})$. It is easily seen that the complex-valued potential $V_\alpha(x, \theta)$ is an analytic function of θ in the strip $|\text{Im } \theta| < \pi/(4k + 2)$. Since $V_\alpha(x, \theta)$ is bounded, by standard arguments (see e. g. Kato [7]) $H(\alpha, \theta)$ is a holomorphic family of type A for $|\text{Im } \theta| < \pi/(4k + 2)$.

THEOREM 2.3. — Let $|\text{Im } \theta| < \pi/(4k + 2)$. Then

$$\sigma_{\text{ess}}(H(\alpha, \theta)) = \{z = -g\alpha^{-1} + \lambda e^{-\theta} \mid \lambda \geq 0\} \cup \{z = g\alpha^{-1} + \lambda e^{-\theta} \mid \lambda \geq 0\}.$$

Proof. — Set $D' = \{u \in H^2(\mathbb{R}) \mid \text{there is a compact subset } K(u) \text{ of } \mathbb{R} \text{ such that } \text{supp } u \subset K(u)\}$ and $H'(\alpha, \theta) = H(\alpha, \theta) \upharpoonright D'$. Now we restrict D' by imposing the additional conditions $u(0) = u'(0) = 0$; let D'' denote the manifold obtained by this restriction and $H''(\alpha, \theta) = H'(\alpha, \theta) \upharpoonright D''$. Then $H''(\alpha, \theta) = H'_1(\alpha, \theta) \oplus H'_2(\alpha, \theta)$ (for the « decomposition » method see [5] or [8]) where $H'_1(\alpha, \theta)$ and $H'_2(\alpha, \theta)$ are generated in $L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R}_+)$ respectively by the differential expression $e^{-\theta}[p^2 + V_\alpha(x, \theta)]$ in the same way as $H'(\alpha, \theta)$ was. Then $\sigma_{\text{ess}}(\overline{H''}(\alpha, \theta)) = \sigma_{\text{ess}}(\overline{H'_1}(\alpha, \theta)) \cup \sigma_{\text{ess}}(\overline{H'_2}(\alpha, \theta))$, where $\overline{H''}(\alpha, \theta)$ denotes any finite dimensional closed extension of $H''(\alpha, \theta)$ and similarly for $\overline{H'_1}$ and $\overline{H'_2}$ (again see [5]): in particular $\overline{H''}(\alpha, \theta)$ can be $H(\alpha, \theta)$. Now, $(e^{-\theta}V_\alpha(x, \theta) - g\alpha^{-1})$ is relatively compact with respect to $(e^{-\theta}p^2 + g\alpha^{-1})$ in $L^2(\mathbb{R}_+)$, since $(e^{-\theta}V_\alpha(x, \theta) - g\alpha^{-1}) \rightarrow 0$ as $x \rightarrow +\infty$ (see [7]), so that

$$\sigma_{\text{ess}}(\overline{H'_2}(\alpha, \theta)) = \{z = g\alpha^{-1} + \lambda e^{-\theta} \mid \lambda \geq 0\}.$$

An analogous argument works for $\overline{H'_1}(\alpha, \theta)$ and the theorem is thus proved.

From now on, θ will be fixed in the strip $0 < \text{Im } \theta < \pi/(4k + 2)$; for any $\alpha \geq 0$ we shall use the simplified notation $H(\alpha)$ to denote the differential operator defined by $H(\alpha) = e^\theta H(\alpha, \theta) = p^2 + V_\alpha$, $D(H(0)) = H^2(\mathbb{R}) \cap L^2_{(2k+1)/2}(\mathbb{R})$ and $D(H(\alpha)) = H^2(\mathbb{R}) \forall \alpha > 0$.

LEMMA 2.4. — The operator family $\{H(\alpha)\}_{\alpha \geq 0}$ satisfies assumptions (1), (2), (3) of Theorem 2.1.

Proof. — For simplicity we assume θ of the form $\theta = i\theta_0$, with

$$0 < \theta_0 = \text{Im } \theta < \pi/(4k + 2).$$

Let $\alpha \geq 0$ and set $f_\alpha(x) = (\alpha^2 e^{(2k+1)i\theta_0} x^{4k+2} + 1)^{1/2}$. Then

$$H(\alpha) = p^2 + (e^{2i\theta_0} x^2 + g e^{(2k+3)i\theta_0/2} x^{2k+1}) \bar{f}_\alpha(x) |f_\alpha(x)|^{-2}.$$

Now we proceed to calculate the terms of the right hand side of the inequality to be proved.

$$\begin{aligned} \text{Re} \langle u, \cos \gamma_\alpha H(\alpha) u \rangle &= \text{Re} \langle u, \cos \gamma_\alpha p^2 u \rangle \\ &+ \langle u, \cos \gamma_\alpha (x^2 \cos 2\theta_0 + g x^{2k+1} \cos (2k+3)\theta_0/2) |f_\alpha|^{-2} \text{Re } f_\alpha u \rangle \\ &+ \langle u, \cos \gamma_\alpha (x^2 \sin 2\theta_0 + g x^{2k+1} \sin (2k+3)\theta_0/2) |f_\alpha|^{-2} \text{Im } f_\alpha u \rangle. \end{aligned}$$

Similarly

$$\begin{aligned} \text{Im} \langle u, \sin \gamma_\alpha H(\alpha) u \rangle &= \text{Im} \langle u, \sin \gamma_\alpha p^2 u \rangle \\ &+ \langle u, \sin \gamma_\alpha (x^2 \cos 2\theta_0 + g x^{2k+1} \cos (2k+3)\theta_0/2) |f_\alpha|^{-2} \text{Im } f_\alpha u \rangle \\ &- \langle u, \sin \gamma_\alpha (x^2 \sin 2\theta_0 + g x^{2k+1} \sin (2k+3)\theta_0/2) |f_\alpha|^{-2} \text{Re } f_\alpha u \rangle. \end{aligned}$$

Then

$$\begin{aligned} &\text{Re} \langle u, \cos \gamma_\alpha H(\alpha) u \rangle + \text{Im} \langle u, \sin \gamma_\alpha H(\alpha) u \rangle \\ &= \text{Re} \langle u, \cos \gamma_\alpha p^2 u \rangle + \text{Im} \langle u, \sin \gamma_\alpha p^2 u \rangle \\ &+ \langle u, x^2 |f_\alpha|^{-2} [\text{Re } f_\alpha \cos (2\theta_0 + \gamma_\alpha) + \text{Im } f_\alpha \sin (2\theta_0 + \gamma_\alpha)] u \rangle \\ &+ \langle u, g x^{2k+1} |f_\alpha|^{-2} [\text{Re } f_\alpha \cos (\gamma_\alpha + (2k+3)\theta_0/2) + \text{Im } f_\alpha \sin (\gamma_\alpha + (2k+3)\theta_0/2)] u \rangle \end{aligned}$$

Let us construct $\gamma_\alpha(x)$ so that the second term of the right hand side of the last inequality is positive and the third one vanishes.

If $\gamma_\alpha(x) \neq -(2k+3)\theta_0/2$ then $\sin [\gamma_\alpha(x) + (2k+3)\theta_0/2] \neq 0$ and the condition $\text{Re } f_\alpha [\cos \gamma_\alpha + (2k+3)\theta_0/2] + \text{Im } f_\alpha \sin [\gamma_\alpha + (2k+3)\theta_0/2] = 0$ is equivalent to the following

$$-\tan \arg (f_\alpha) = \cot [\gamma_\alpha + (2k+3)\theta_0/2] = \tan [\pi/2 - \gamma_\alpha - (2k+3)\theta_0/2].$$

One solution to this equation is given by

$$\gamma_\alpha(x) = \pi/2 + \arg (f_\alpha(x)) - (2k+3)\theta_0/2 \quad (2.4)$$

Note that for any $\alpha > 0$ $\inf_x \arg (f_\alpha(x)) = 0$ and $\sup_x \arg (f_\alpha(x)) = (2k+1)\theta_0/2$.

On the other hand, $f_0(x) = 1$, for all x ; hence

$$\inf_x \cos \gamma_\alpha(x) \geq \cos (\pi/2 - \theta_0) = \sin \theta_0 = C_0 > 0,$$

for any $\alpha \geq 0$, and for this choice of $\gamma_\alpha(x)$ assumption (2) of Theorem 2.1 is satisfied.

Now we need to show that if $\gamma_\alpha(x)$ is given by (2.4) then

$$\operatorname{Re} f_\alpha(x) \cos (2\theta_0 + \gamma_\alpha(x)) + \operatorname{Im} f_\alpha(x) \sin (2\theta_0 + \gamma_\alpha(x)) \geq 0, \quad \forall x \quad (2.5)$$

We have $2\theta_0 + \gamma_\alpha(x) = \pi/2 + \arg (f_\alpha(x)) - (2k - 1)\theta_0/2$. Thus for $\alpha = 0$, (2.5) is trivially verified. For $\alpha > 0$,

$$\sup_x (\gamma_\alpha(x) + 2\theta_0) = \theta_0 + \pi/2 \quad \text{and} \quad \inf_x (\gamma_\alpha(x) + 2\theta_0) = \pi/2 - (2k - 1)\theta_0/2;$$

therefore $\sin (\gamma_\alpha(x) + 2\theta_0) > 0$ for all x and (2.5) is now equivalent to $\cot (2\theta_0 + \gamma_\alpha(x)) \geq -\tan \arg (f_\alpha(x))$, i. e.

$$\tan ((2k - 1)\theta_0/2 - \arg (f_\alpha(x))) \geq \tan (-\arg (f_\alpha(x)))$$

and this last inequality is certainly satisfied $\forall x$.

Now the inequality in assumption (1) of Theorem 2.1 holds with $\xi = 1$ and $\eta = 0$ for any $\alpha \geq 0$. In order to prove part (3) we need a more explicit expression for $\arg (f_\alpha(x))$, $\alpha > 0$: $\operatorname{Im} f_\alpha^2(x) = a^2 x^{4k+2} \sin (2k + 1)\theta_0$ and $\operatorname{Re} f_\alpha^2(x) = \alpha^2 x^{4k+2} \cos (2k + 1)\theta_0 + 1$. Thus,

$$\forall x \neq 0, \arg (f_\alpha(x)) = (1/2) \operatorname{arc} \cot [\cot (2k + 1)\theta_0 + (\alpha^2 x^{4k+2} \sin (2k + 1)\theta_0)^{-1}].$$

An easy calculation yields

$$\frac{d}{dx} (\cos \gamma_\alpha(x)) = -(2k + 1) \sin \gamma_\alpha(x) [\alpha^2 x^{4k+1} \sin (2k + 1)\theta_0] [h(x)]^{-1}$$

where $h(x) = (1 + a^2)\alpha^4 x^{8k+4} \sin^2 (2k + 1)\theta_0 + 2a\alpha^2 x^{4k+2} \sin (2k + 1)\theta_0 + 1$, with $a = \cot (2k + 1)\theta_0$. If we assume $\theta_0 < \pi/(8k + 4)$ then the rational term in the last equation is easily seen to be bounded by 1, uniformly in α . Since

$\frac{d}{dx} (\cos \gamma_0(x)) = 0, \forall x$, we conclude that

$$\left\| \frac{d}{dx} \cos \gamma_\alpha \right\|_\infty \leq (2k + 1) = C_1 < +\infty, \quad \forall \alpha \geq 0.$$

Similarly one can prove $\left\| \frac{d}{dx} \sin \gamma_\alpha \right\|_\infty \leq C_1$ and this completes the proof of Lemma 2.4.

Now set $\Delta = \{z \in \mathbb{C} \mid \text{there is } \bar{\alpha}(z) > 0 \text{ s. t. } z \notin \sigma(H(\alpha)) \text{ and } \|(H(\alpha) - z)^{-1}\| \text{ is uniformly bounded for } 0 \leq \alpha < \bar{\alpha}(z)\}$.

It is well known (see [7]) that Δ is open and, given any compact $\Gamma \subset \Delta$ there is $\bar{\alpha}(\Gamma) > 0$ such that $\|(H(\alpha) - z)^{-1}\|$ is uniformly bounded for $z \in \Gamma$ and $0 \leq \alpha < \bar{\alpha}(\Gamma)$.

Unlike the situation presented in [6], Example 5, we are dealing with operators whose numerical ranges invade the whole complex plane as $\alpha \rightarrow 0$. For this reason we need the following.

LEMMA 2.5. — The operator family $\{H(\alpha)\}_{\alpha \geq 0}$ satisfies $\Delta \neq \emptyset$.

Proof. — By Theorem 2.2 $\sigma_{ess}(H(0)) = \emptyset$ and by Theorem 2.3 for any $z \in \mathbb{C}$ there exists $\bar{\alpha}(z) > 0$ such that $z \notin \sigma_{ess}(H(\alpha))$ for $0 \leq \alpha < \bar{\alpha}(z)$. Thus, Lemma 5.1 of [6] applies and if $z \notin \sigma_d(H(0))$ then $z \in \Delta$, unless there exist two sequences $\{\alpha_n\} \subset \mathbb{R}_+$ and $\{u_n\} \subset L^2(\mathbb{R})$ such that $\alpha_n \rightarrow 0, u_n \in D(H(\alpha_n)), \|u_n\| \not\rightarrow 0, u_n \xrightarrow{w} 0$ and $\|(z - H(\alpha_n))u_n\| \rightarrow 0$. In order to exclude the second alternative we notice that u_n can be chosen in $C_0^\infty(\mathbb{R})$, since it is a core for $H(\alpha_n)$. Now fix $\chi \in C_0^\infty(\mathbb{R})$ and let $M_n = 1 - \chi_n$ be the sequence of multiplication operators specified in the proof of Theorem 2.1. If we define

$$M_n^+(x) = M^+(x/n) = \begin{cases} 0 & , \text{ if } x < 0 \\ 1 - \chi(x/n), & \text{ if } x \geq 0 \end{cases}$$

and

$$M_n^-(x) = M^-(x/n) = \begin{cases} 1 - \chi(x/n), & \text{ if } x \leq 0 \\ 0 & , \text{ if } x > 0, \end{cases}$$

then $M_n = M_n^+ + M_n^-$ and by (ii) of Hypothesis 3 in [6], which follows from Theorem 2.1 and Lemma 2.4, there exists $a > 0$ such that, for each n ,

$$\limsup_{m \rightarrow +\infty} \|M_n u_m\|^2 = \limsup_{m \rightarrow +\infty} (\|M_n^+ u_m\|^2 + \|M_n^- u_m\|^2) \geq a^2.$$

Thus, for each n

either
$$\limsup_m \|M_n^+ u_m\| \geq a/2 \tag{2.6a}$$

or
$$\limsup_m \|M_n^- u_m\| \geq a/2 \tag{2.6b}$$

Then the second alternative can be specified either with a sequence $v_n^+ = M_n^+ u_{m(n)}$, such that $v_n^+(x) = 0$ for $x < n$, or with a sequence $v_n^- = M_n^- u_{m(n)}$, such that $v_n^-(x) = 0$ for $x > -n$, by suitably choosing $m = m(n)$.

In fact, let (2.6a) hold. We have

$$\|(z - H(\alpha_m))M_n^+ u_m\| \leq \|(z - H(\alpha_m))u_m\| + \|[M_n^+, H(\alpha_m)]u_m\| \tag{2.7}$$

and, since

$$[M_n^+, H(\alpha)] = [M_n^+, p^2] = 2in^{-1} \frac{dM^+}{dx}(x/n) - n^{-2} \frac{d^2 M^+}{dx^2}(x/n),$$

it follows from (2.1) that

$$\|[M_n^+, H(\alpha)]u\| \leq cn^{-1}(\|H(\alpha)u\| + \|u\|) \tag{2.8}$$

where $c > 0$ is a constant independent of α and n . Now, combining (2.6a), (2.7) and (2.8), a suitable choice of $m = m(n)$ and normalization yield

$$\lim_n \|(z - H(\alpha_{m(n)})v_n^+\| = 0, \quad v_n^+ \xrightarrow{w} 0 \quad \text{and} \quad \|v_n^+\| = 1.$$

The analogous properties can be specified for the sequence $\{v_n^-\}$ if (2.6b) holds. Now the contradiction follows from

$$\|[H(\alpha_{m(n)}) - z]v_n^+\| \geq \text{dist}(z, E_n^+(\alpha_{m(n)}))$$

where $E_n^+(\alpha) = \{ \langle H(\alpha)v, v \rangle \mid v \in C_0^\infty(\mathbb{R}), \|v\| = 1, v(x) = 0 \text{ for } x < n \}$. In fact we have

$$\operatorname{Re} \langle (p^2 + V_\alpha)v, v \rangle \geq \langle (\operatorname{Re} V_\alpha)v, v \rangle \quad \text{and} \quad \lim_{\substack{x \rightarrow +\infty \\ \alpha \rightarrow 0}} \operatorname{Re} V_\alpha(x) = +\infty$$

i. e. there are not two sequences $\{x_k\} \rightarrow +\infty$ and $\{\alpha_k\} \rightarrow 0$ such that $\operatorname{Re} V_{x_k}(\alpha_k)$ is bounded from above. Thus, since $\alpha_{m(n)} \rightarrow 0$, for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\| [H(\alpha_{m(n)}) - z]v_n^+ \| \geq \delta > 0, \forall n \geq n_0$. Similarly we obtain a contradiction if (2.6b) holds, since $\lim_{\substack{x \rightarrow -\infty \\ \alpha \rightarrow 0}} \operatorname{Im} V_\alpha(x) = -\infty$, and this completes the proof of Lemma 2.5.

COROLLARY 2.6. — Every eigenvalue of $H(0)$ is stable (in the sense of Kato [7]) with respect to the family $\{H(\alpha)\}_{\alpha \geq 0}$.

Proof. — By Lemma 2.4 and 2.5 the family $\{H(\alpha)\}_{\alpha \geq 0}$ satisfies Hypothesis 3 of [6] with the operators M_n specified in the proof of Theorem 2.1. Moreover, by Theorems 2.2 and 2.3, for any fixed $\lambda \in \mathbb{C}$ and $\delta > 0$ there exists $\alpha_0 > 0$ such that $\operatorname{dist}(\lambda, \sigma_{\text{ess}}(H(\alpha))) \geq \delta$ for $0 \leq \alpha \leq \alpha_0$. Finally we can decompose $M_n = M_n^+ + M_n^-$, as specified in the proof of Lemma 2.5, where M_n^+ and M_n^- also satisfy (iii) of Hypothesis 3 in [6]. Since $\lim_{\substack{x \rightarrow +\infty \\ \alpha \rightarrow 0}} \operatorname{Re} V_\alpha(x) = +\infty$, proceeding as in the last part of the preceding lemma,

we find $\alpha_1 > 0$ and $n_0 \in \mathbb{N}$ such that

$$d_n^+(\lambda, \alpha) = \inf_{u \in D(H(\alpha)), \|M_n^+ u\| = 1} \|(\lambda - H(\alpha))M_n^+ u\| \geq \delta > 0$$

for $\alpha < \alpha_1$ and $n > n_0$. Similarly, since $\lim_{\substack{x \rightarrow -\infty \\ \alpha \rightarrow 0}} \operatorname{Im} V_\alpha(x) = -\infty, d_n^-(\lambda, \alpha) \geq \delta$

for $\alpha < \alpha_1$ and $n > n_0$. Therefore we can apply Theorem 5.8 of [6] and the stability of eigenvalues is proved.

As a consequence, we can now prove the result announced in the introduction. For convenience, we return to the more explicit notation of Theorem 2.2, by replacing $H(\alpha)$ with $e^\theta H(\alpha, \theta)$.

THEOREM 2.7. — Let $H(\alpha, \theta)$ be the holomorphic family of type A in $L^2(\mathbb{R})$ defined for $|\operatorname{Im} \theta| < \pi/(4k + 2)$ and $\alpha > 0$ by (2.3), self-adjoint for $\theta \in \mathbb{R}$. Then there exists $g_0 > 0$ such that for $0 < g < g_0$ there is $\alpha_g > 0$ with the property that $\forall \alpha \in (0, \alpha_g), H(\alpha, 0)$ admits second sheet poles of the resolvent in the following sense: if $S \subset L^2(\mathbb{R})$ is the dense set of all dilation analytic vectors for $|\operatorname{Im} \theta| < \pi/(4k + 2)$ and $\psi \in S$, the scalar product

$$g_\psi(E) = \langle [H(\alpha, 0) - E]^{-1} \psi, \psi \rangle,$$

which is *a priori* analytic in $\mathbb{C} \setminus \sigma(H(\alpha, 0))$, admits a meromorphic continua-

tion onto the second sheet across the cut $[-g\alpha^{-1}, +\infty) = \sigma_{\text{ess}}(\mathbf{H}(\alpha, 0))$ (from the upper half-plane of the first sheet) to the region

$$\{ E \mid -\tan(\text{Im } \theta)(\text{Re } E + g\alpha^{-1}) \leq \text{Im } E \leq 0 \} \setminus \sigma_{\text{ess}}(\mathbf{H}(\alpha, \theta)),$$

for any $\theta, 0 < \text{Im } \theta < \pi/(4k + 2)$. For $i \in \mathbb{N}$, let $\alpha(i) > 0$ be such that for $\alpha < \alpha(i)$ the eigenvalue $E_i(\alpha)$ of $\mathbf{H}(\alpha, \theta)$ exists by stability with respect to the i -th eigenvalue of $\mathbf{H}(0, \theta)$. Then for fixed $\bar{i} \in \mathbb{N}$ and $\alpha \leq \min(\alpha(1), \dots, \alpha(\bar{i}))$ the set $\{ E(\alpha) \mid E(\alpha) \text{ is a pole of } g_\psi(E) \text{ for some } \psi \in \mathbf{S} \}$ contains at least the finite set of eigenvalues $\{ E_i(\alpha) \mid i = 1, 2, \dots, \bar{i} \}$. For each $i \in \mathbb{N}$, $E_i(\alpha)$ tends to the corresponding second sheet pole of the generalized resolvent $\mathcal{R}(E)$ (specified in Theorem 2.2) of the closure of the symmetric operator $p^2 + x^2 + gx^{2k+1}$.

Proof. — It is a consequence of Corollary 2.6 since, by the usual analyticity arguments (see e. g. [10])

$$g_\psi(E) = \langle [\mathbf{H}(\alpha, \theta) - E]^{-1} \psi(\theta), \psi(\bar{\theta}) \rangle$$

identically, for $0 < \text{Im } \theta < \pi/(4k + 2), \text{Im } E > 0$, where $\psi(\theta)(x) = e^{\theta/4} \psi(e^{\theta/2} x)$ for $\psi \in \mathbf{S}$.

An analogous stability argument can be used to obtain an existence proof of shape resonances for a suitable class of Schrödinger operators in $L^2(\mathbb{R}_+)$. In this case the results of Vock and Hunziker [6] can be directly applied, having care to set the necessary Dirichlet condition at the origin. Explicitly, if we set

$$V_0(z) = z^2 - gz^{2k+1}, \quad k \in \mathbb{N}, \quad g > 0, \quad z \in \mathbb{C}, \quad (2.9)$$

then for the operator family defined in $L^2(\mathbb{R}_+)$ by the differential expression $A(0, \theta) = e^{-\theta} p^2 + V_0(e^{\theta/2} x) = e^{-\theta} p^2 + e^{2\theta} x^2 - ge^{(2k+3)\theta/2} x^{2k+1}$, ($0 < \text{Im } \theta < \min(\pi/4, 2\pi/(2k + 3))$) we can prove results analogous to Theorem 2.2. In particular $A(0, \theta)$ is a holomorphic family of type A in $L^2(\mathbb{R}_+)$ for $0 < \text{Im } \theta < \min(\pi/4, 2\pi/(2k + 3))$ when defined on

$$D = D(A(0, \theta)) = \{ u \mid u \in H^2(\mathbb{R}_+) \cap L^2_{(2k+1)/2}(\mathbb{R}_+), u(0) = 0 \} \quad (2.10)$$

and we can show the existence of eigenvalues $E_i = E_i(g)$ (for $0 < g < g_0$, for some $g_0 > 0$) with $\text{Im } E_i < 0, i \in \mathbb{N}$, which tend to the odd eigenvalues of the harmonic oscillator as $g \rightarrow 0$.

THEOREM 2.8. — Let $V_\alpha(x), \alpha > 0, x \in \mathbb{R}_+$ be a family of real-valued functions enjoying the following properties:

i) there is $\theta_0 \in (0, \min(\pi/4, 2\pi/(2k + 2)))$ such that, for any $\alpha > 0, V_\alpha(x)$ is the restriction to $z \in \mathbb{R}_+$ of a function $V_\alpha(z)$ holomorphic at least in the sector $\{ z \in \mathbb{C} \mid |\arg z| < \theta_0 \}$, and bounded near $z = 0$;

ii) for $\alpha > 0, |\text{Im } \theta| < \theta_0, V_\alpha(e^{\theta/2} x) \rightarrow c(\alpha)$ as $x \xrightarrow{+} +\infty$, for some $c(\alpha) \in \mathbb{R}$, and $\lim_{\alpha \rightarrow 0} c(\alpha) = -\infty$;

iii) for any fixed θ with $0 < \text{Im } \theta < \theta_0$, $V_\alpha(e^{\theta/2}x) \rightarrow V_0(e^{\theta/2}x)$ as $\alpha \rightarrow 0$ uniformly on the compact subsets of \mathbb{R}_+ ($V_0(z)$ as defined in (2.9));

iv) the family $\{e^\theta V_\alpha(e^{\theta/2}x)\}_{\alpha \geq 0}$ satisfies the hypotheses of Theorem 6.1 in [6], with $C_0^\infty(\mathbb{R})$ replaced by \mathbb{D} , as defined in (2.10). Then:

(1) for any $\alpha > 0$ the differential expression $e^{-\theta}p^2 + V_\alpha(e^{\theta/2}x)$ defines a holomorphic family of type A in $L^2(\mathbb{R}_+)$, denoted by $A(\alpha, \theta)$, for $|\text{Im } \theta| < \theta_0$, if defined on $\mathbb{D}(A(\alpha, \theta)) = \{u \mid u \in H^2(\mathbb{R}_+), u(0) = 0\}$, with \mathbb{D} as a core. Furthermore $\sigma_{\text{ess}}(A(\alpha, \theta)) = \{z = c(\alpha) + \lambda e^{-\theta} \mid \lambda \geq 0\}$.

Let E_0 be an eigenvalue of $A(0, \theta)$ (whence $\text{Im } E_0 < 0$); suppose that for some $\varepsilon > 0$, $n_0 \in \mathbb{N}$ and $\alpha_0 > 0$, $\text{dist}(E_0, F_n(\alpha)) \geq \varepsilon > 0, \forall n \geq n_0, 0 < \alpha < \alpha_0$, where $F_n(\alpha) = \{\langle A(\alpha, \theta)u, u \rangle \mid \|u\| = 1, u \in \mathbb{D}(A(\alpha, \theta)), \text{supp } u \subset [n, +\infty)\}$. Then:

(2) E_0 is a stable eigenvalue with respect to the family $\{A(\alpha, \theta)\}_{\alpha \geq 0}$; in particular there is $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha})$ there exists $E(\alpha) \in \sigma_d(A(\alpha, \theta))$ with $\text{Im } E(\alpha) < 0$ and $E(\alpha) \rightarrow E_0$ as $\alpha \rightarrow 0$;

(3) if S is the dense set of dilation analytic vectors for $|\text{Im } \theta| < \theta_0$, then for all $\alpha \in (0, \bar{\alpha})$ and $\psi \in S$ the function $h_\psi(E) = \langle [A(\alpha, 0) - E]^{-1}\psi, \psi \rangle$, which is *a priori* analytic in $\mathbb{C} \setminus \sigma(A(\alpha, 0))$, admits a meromorphic continuation onto the second sheet to an open connected region of the lower half plane (from the upper half plane of the first sheet) across the cut

$$\sigma_{\text{ess}}(A(\alpha, 0)) = [c(\alpha), +\infty).$$

Moreover there is $\psi \in S$ such that the continuation of h_ψ has a simple pole at $E(\alpha)$.

Natural examples are self-adjoint operators of the form

$$A(\alpha, 0) = -d^2/dx^2 + (x^2 + \alpha P^{(2k)}(x) - gx^{2k+1})(\alpha x^\nu + 1)^{1/\nu}.$$

Here k, ν and ρ are fixed ($k \in \mathbb{N}, \nu > 0$ and $\rho = 2k + 1$) $\alpha > 0, P^{(2k)}(x)$ is an arbitrary polynomial of order not larger than $2k$,

$$\mathbb{D}(A(\alpha, 0)) = \{u \mid u \in H^2(\mathbb{R}_+), u(0) = 0\}.$$

Since in this case $\lim_{\substack{\alpha \rightarrow 0 \\ x \rightarrow +\infty}} \text{Im } [e^\theta V_\alpha(e^{\theta/2}x)] = -\infty$, every eigenvalue E_i of $A(0, \theta)$ satisfies the required inequality $\text{dist}(E_i, F_n(\alpha)) \geq \varepsilon_i > 0$ for $n \geq n_i, 0 < \alpha < \alpha_i$. So, by the previous theorem, $A(\alpha, 0)$ provides an example of shape resonances in the standard sense of dilation analyticity.

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