ANNALES DE L'I. H. P., SECTION A

R. G. McLENAGHAN

Huygens' principle

Annales de l'I. H. P., section A, tome 37, nº 3 (1982), p. 211-236
<http://www.numdam.org/item?id=AIHPA_1982__37_3_211_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXVII, n° 3, 1982, p. 211-236. *Physique théorique.*

Huygens' Principle (*)

by

R. G. McLENAGHAN

Department of Applied Mathematics, University of Waterloo,
Waterloo, Ontario, Canada

ABSTRACT. — The known results on the question of the validity of Huygens' principle for second order linear partial differential equations of normal hyperbolic type in four independent variables are reviewed. A new family of space-times is given on which the self-adjoint differential equation satisfies certain necessary conditions for the validity of Huygen's principle.

RÉSUMÉ. — On décrit les résultats connus, sur la question de la validité du principe de Huygens, pour les équations aux dérivées partielles linéaires hyperboliques du second ordre à quatre variables indépendantes. On donne une nouvelle famille d'espaces-temps, sur lesquels les équations auto-adjointes satisfont certaines conditions nécessaires pour la validité du principe de Huygens.

1. INTRODUCTION

We shall be studying the general second order, homogeneous, linear, hyperbolic, partial differential equation in n independent variables. Such an equation can be written in coordinate invariant form as

$$\mathrm{F}(u) \equiv g^{ab}u_{;ab} + \mathrm{A}^{a}u_{,a} + \mathrm{C}u = 0\,, \tag{1.1}$$

(*) This paper was presented at the Journées Relativistes held at the Université de Caen, France, May 9-11, 1980. The work was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

where g^{ab} are the contravariant components of the metric tensor of a pseudo-Riemannian space V_n of signature $2-n$ and « , » and « ; » denote respectively the partial derivative and the covariant derivative with respect to the pseudo-Riemannian connection. The coefficients g^{ab}, A^a and C as well as V_n are assumed to be of class C^∞.

Cauchy's problem for the equation (1.1) is the problem of determining a solution which assumes given values of u and its normal derivative on a given space-like $(n-1)$-dimensional manifold S. These given values are called the Cauchy data. The first general solution to Cauchy's problem for (1.1) was given by Hadamard [*21*] in his Yale lectures. Alternate solutions have been presented by Mathisson [*30*], Sobolev [*41*], Bruhat [*3*] and Douglis [*10*].

The question of how the solution u of Cauchy's problem at a point $x_0 \in V_n$, depends on the Cauchy data is of considerable interest. Hadamard shows that in general $u(x_0)$ depends on the data in the interior of the intersection of the retrograde characteristic conoid $C^-(x_0)$ with the initial surface S. If the solution depends only on the data in an arbitrarily small neighbourhood of $S \cap C^-(x_0)$ for every Cauchy problem and for every $x_0 \in V_n$ we say that the equation satisfies *Huygens' principle* or is a *Huygens' differential equation.* Examples of such equations are the ordinary wave equations

$$\frac{\partial^2 u}{\partial x^{1^2}} - \sum_{i=2}^{2m} \frac{\partial^2 u}{\partial x^{i^2}} = 0 \tag{1.2}$$

in an even number of variables $n = 2m \geq 4$ (see for example Courant and Hilbert [*8*], p. 690).

Hadamard showed, using his solution formula, that in order that Huygens' principle be valid it is necessary that $n \geq 4$ be even. He then posed the problem of determining all the Huygens' differential equations. Since none other than the equations (1.2) were known he suggested that as a first step one should attempt to prove that every Huygens' differential equation is *equivalent* to some equation (1.2). We recall that two equations (1.1) are said to be equivalent if they are related by one or a combination of the following transformations called *trivial transformations* which preserve the Huygens' character of the differential equation:

a) a transformation of coordinates,

b) multiplication of both sides of the equation by a non-vanishing factor $e^{-2\phi}$, where $\phi(x)$ is a function of position (this transformation induces a conformal transformation of the metric),

c) replacement of the unknown u by λu, where $\lambda(x)$ is a non-vanishing function of position.

Hadamard's suggestion is often referred to as « Hadamard's conjecture » in the literature (see for example Ref. [*8*], p. 765).

The conjecture has been proven in the case $n = 4$, g^{ab} constant by Mathisson [31], Hadamard [22] and Asgeirsson [1]. However, it is known not to be true in general. The first counter examples were given by Stellmacher [42] for the case $n = 6$. Later [43] he provided examples in all even dimensions $n \geq 6$. These examples are given by the equation

$$\frac{\partial^2 u}{\partial x^{12}} - \sum_{i=2}^{2m} \frac{\partial^2 u}{\partial x^{i2}} + \left(\frac{\lambda_1}{(x^1)^2} - \sum_{i=2}^{2m} \frac{\lambda_i}{(x^i)^2}\right) u = 0, \tag{1.3}$$

where

1) $$-\lambda_i = v_i(v_i + 1), \qquad v_i = 0, 1, 2, \ldots$$

2) $$\sum_{i=1}^{2m} v_i \leq m - 2.$$

For example when $n = 3$. one possibility is

$$\frac{\partial^2 u}{\partial x^{12}} - \sum_{i=2}^{6} \frac{\partial^2 u}{\partial x^{i2}} - \frac{2u}{(x^1)^2} = 0, \tag{1.4}$$

which is one of the first examples given by Stellmacher. In order to see that these equations are not equivalent to the wave equation (1.2) one notes that necessary and sufficient conditions for equivalence are [7] [17]

$$\mathrm{C}_{abcd} = 0, \tag{1.5}$$

$$\mathrm{H}_{ab} \equiv \mathrm{A}_{[a,b]} = 0, \tag{1.6}$$

$$\mathrm{C} \equiv \mathrm{C} - \frac{1}{2}\mathrm{A}^a{}_{;a} - \frac{1}{4}\mathrm{A}_a\mathrm{A}^a - \frac{(n-2)}{4(n-1)}\mathrm{R} = 0, \tag{1.7}$$

where C_{abcd} denotes the Weyl conformal curvature tensor and R the curvature scalar associated to the metric g_{ab}. The result then follows since the conditions (1.5) and (1.6) hold for the equation (1.3), while (1.7) does not since $\mathrm{C} \neq 0$.

Counter examples for $n = 4$ have been given by Günther [18] (essentially the same examples were rediscovered by Ibragimov and Mamontov [25]). These examples arise from the wave equation

$$\square u \equiv g^{ab}u_{;ab} = 0 \tag{1.8}$$

on the maximum mobility Lorentzian spaces of type T_2 previously studied by Petrov [36] with metric

$$ds^2 = 2dx^1dx^2 - a_{\alpha\beta}dx^\alpha dx^\beta, \qquad (\alpha, \beta = 3,4), \tag{1.9}$$

where the symmetric matrix $(a_{\alpha\beta})$ is positive definite with components which are functions only of x^1. The above metric may be interpreted in the framework of the General Theory of Relativity as an exact plane wave solution of the vacuum or Einstein-Maxwell field equations. It has been studied in this context by Ehlers and Kundt [13] in a different coordinate system in which it has the form

$$ds^2 = 2dv[du + (Dz^2 + \bar{D}\bar{z}^2 + ez\bar{z})dv] - 2dzd\bar{z}, \tag{1.10}$$

where $D = D(v)$ and $e = e(v) = \bar{e}$. In addition it has been shown by Künzle [28] using the form (1.10) that Huygens' principle is satisfied by Maxwell's equations

$$dF = 0, \qquad \delta F = 0 \tag{1.11}$$

(according to a criterion given by Günther [19]) for a differential form F of all degrees in a plane wave space-time but is not satisfied by the wave equation on p-forms for $0 < p < 4$. These results have been generalized by Schimming [39] who shows that in a Lorentzian space with metric (1.9) where $\alpha, \beta = 3, 4, \ldots, n$, n even, Huygens' principle holds for Maxwell's equations for differential forms of all degrees $1 \leq p \leq n - 1$, and for the wave equation on p-forms of all degrees $0 \leq p \leq n$ except $n = 4, p = 1, 2, 3$. (See also Ibragimov and Mamontov [27] for the scalar case). Recently Wünsch [51] has shown that Huygens' principle is also valid for the Weyl equation

$$\nabla_{A'}{}^{B}\phi_B = 0 \tag{1.12}$$

and the wave equation

$$\nabla_a\nabla^a\phi_A = 0 \tag{1.13}$$

for a one-index two-spinor ϕ_A on the plane wave space-time with metric (1.9) or (1.10).

It has been shown by the present author [32] that a Huygens' differential equation on a conformally empty space-time is equivalent to Eq. (1.8) on the plane wave space-time which are the only known examples of Huygens' differential equations for $n = 4$. These results have been generalized to a certain extent by Wünsch [51]. However, it seems that the general problem of determining all the Huygens' differential equations still remains open. One of the purposes of the present work is to present a possible new class of Huygens' differential equations which are not equivalent to the ones on the plane-wave space-time mentioned above.

2. ELEMENTARY SOLUTIONS

In order to proceed with the discussion we shall need to examine Hadamard's necessary and sufficient condition for the validity of Huygens' principle. This condition may be expressed in terms of the *elementary solutions* of Eq. (1.1) which are distributions $E^{\pm}_{x_0}(x)$ which satisfy the equation

$$G(E^{\pm}_{x_0}(x)) = \delta_{x_0}(x), \tag{2.1}$$

where

$$G(v) \equiv g^{ab}v_{;ab} - (A^a v)_{;a} + Cv \tag{2.2}$$

is the adjoint differential operator to $F(u)$ and $\delta_{x_0}(x)$ is the Dirac delta distribution. Lichnerowicz [29] has shown these elementary solutions exist and are unique for C^∞ equations. Furthermore for $n = 4$ they decompose as follows (see for example Friedlander [14]):

$$E^{\pm}_{x_0}(x) = V(x_0, x)\delta^{\pm}(\Gamma(x_0, x)) + \mathscr{V}^{\pm}(x_0, x)\Delta^{\pm}(x_0, x), \tag{2.3}$$

when x and x_0 belong to some simple convex set Ω of V_4. The function V in Eq. (2.3) is defined by

$$V(x_0, x) = \frac{1}{2\pi} \exp\left\{ -\frac{1}{4}\int_0^{s(x)} (g^{ab}\Gamma_{;ab} - 8 - A^a\Gamma_{,a})\frac{dt}{t} \right\}, \tag{2.4}$$

where integration is along the geodesic joining x_0 and x, $\Gamma(x_0, x)$ is up to a sign the square of the geodesic distance from x_0 to x and s is an affine parameter. The functions $\mathscr{V}^{\pm}(x_0, x)$ are defined on the closures of the sets $D^{\pm}(x_0)$, which denote the respective interiors of the future and past pointing characteristic conoids $C^{\pm}(x_0)$, as follows:

$$G(\mathscr{V}^{\pm})(x_0, x) = 0, \qquad \text{when} \quad x \in D^{\pm}(x_0) \tag{2.5a}$$

$$\mathscr{V}^{\pm}(x_0, x) = \frac{V(x_0, x)}{s(x)} \int_0^{s(x)} \frac{G(V)}{V}\,dt, \qquad \text{when} \quad x \in C^{\pm}(x_0), \tag{2.5b}$$

Finally we have

$$\delta^{\pm}(\Gamma(x_0, x)) = \begin{cases} \delta(\Gamma(x_0, x)), & x \in C^{\pm}(x_0) \\ 0, & x \in C^{\mp}(x_0) \end{cases} \tag{2.6}$$

while $\Delta^{\pm}(x_0, x)$ denote the characteristic functions on $D^{\pm}(x_0)$.

In terms of the functions involved in the definition of the elementary solutions, Hadamards' necessary and sufficient condition for the validity of Huygens' principle takes the form

$$\mathscr{V}^{\pm}(x_0, x) = 0, \qquad \forall x_0 \quad \text{and} \quad \forall x \in D^{\pm}(x_0). \tag{2.7}$$

From Eq. (2.3) we see that this is equivalent to the elementary solutions having support only on the characteristic semi-conoids $C^{\pm}(x_0)$. For purposes of calculation a more useful form of the condition (2.4), first given by Hadamard [22], is

$$[G(V)(x_0, x)] = 0, \qquad \forall x_0 \in V_4, \tag{2.8}$$

where the brackets [ ] signify the restriction of the enclosed function to the set

$$C(x_0) = C^+(x_0) \cup C^-(x_0). \tag{2.9}$$

The convenience of the condition (2.7) results in part from the fact that the function V defined by Eq. (2.4) can be expressed as

$$V(x_0, x) = \frac{1}{2\pi}(\rho(x_0, x))^{-\frac{1}{2}} \exp\left\{\frac{1}{4}\int_0^{s(x)} A^a\Gamma_{,a}\frac{dt}{t}\right\}, \tag{2.10}$$

where

$$\rho = 8(g(x)g(x_0))^{\frac{1}{2}}\left[\det\left(\frac{\partial^2\Gamma}{\partial x^a \partial x_0^b}\right)\right]^{-1} \tag{2.11}$$

is the so called discriminant function and $g(x) = \det(g_{ab}(x))$.

It should be mentioned that the condition (2.8) is also valid when the coefficients of Eq. (1.1) are merely sufficiently differentiable (see Chevalier [*6*] and Douglis [*11*]).

3. NECESSARY CONDITIONS FOR THE VALIDITY OF HUYGENS' PRINCIPLE

The necessary and sufficient condition of Hadamard (2.7) gives an answer to the question of which equations (1.1) satisfy Huygens' principle. However, it does not satisfactorily resolve the problem since as has been explained by Hadamard (on p. 236 of ref. [*21*]) « We have said that we give *an* answer and not *the* answer, to our question: for it is clear that we can wish it to be « plus resolu » then it has been in the above. We have enunciated the necessary and sufficient condition, but we do not know how equations satisfying it can be found ». To circumvent this difficulty workers on the problem have derived a series of necessary conditions that apply directly to the coefficients of the Eq. (1.1). The first five of these conditions are

I $$C = \frac{1}{2}A^a{}_{;a} + \frac{1}{4}A_aA^a + \frac{1}{6}R,$$

II $$H_{ak;}{}^k = 0,$$

III $$S_{abk;}{}^k - \frac{1}{2}C^k{}_{ab}{}^l L_{kl} = -5\left(H_{ak}H_b{}^k - \frac{1}{4}g_{ab}H_{kl}H^{kl}\right),$$

IV $$\mathrm{TS}(S_{abk}H^k{}_c + C^k{}_{ab}{}^l H_{ck;l}) = 0,$$

V $$\begin{aligned}\mathrm{TS}(3C_{kabl;m}C^k{}_{cd}{}^{l}{}_{;}{}^{m} &+ 8C^k{}_{ab}{}^l{}_{;c}S_{kld} + 40S_{ab}{}^kS_{cdk} \\ &- 8C^k{}_{ab}{}^lS_{klc;d} - 24C^k{}_{ab}{}^lS_{cdk;l} + 4C^k{}_{ab}{}^lC_l{}^m{}_{ck}L_{dm} \\ &+ 12C^k{}_{ab}{}^lC^m{}_{cdl}L_{km} + 12H_{ka;bc}H^k{}_d - 16H_{ka;b}H^k{}_{c;d} \\ &- 84H^k{}_aC_{kbcl}H^l{}_d - 18H_{ka}H^k{}_bL_{cd}) = 0.\end{aligned}$$

In the above conditions

$$L_{ab} = -R_{ab} + \frac{1}{6}g_{ab}R, \tag{3.1}$$

$$S_{abc} = L_{a[b;c]}, \tag{3.2}$$

$$C_{abcd} = R_{abcd} - 2g_{[a[d}L_{b]c]}, \tag{3.3}$$

where R_{abcd} denotes the Riemannian curvature tensor, $R_{bc} = g^{ad}R_{abcd}$ the Ricci tensor and $R = g^{bc}R_{bc}$ the curvature scalar. The notation TS( ) denotes the trace-free symmetric part of the enclosed tensor.

The history of these conditions is as follows: Hölder [*24*] found Condition I in the case $A^a = C = 0$. Mathisson [*31*], Hadamard [*22*], and Asgeirsson [*1*] obtained the Conditions I, II and III in the case g^{ab} constant. The Conditions I to IV in the general case were given by Günther [*16*] and independently by Chevalier [*6*] for $A^a = C = 0$. McLenaghan [*32*] obtained Condition V when $R_{ab} = 0$. Subsequently Wünsch [*47*] gave it when $A^a = 0$. Condition V in the general case was found by McLenaghan [*33*]. Recently Vandercappellen [*46*] has obtained a further necessary condition (Condition VI) which is too complicated to be presented here (1).

Before discussing the consequences of Conditions I to V we shall indicate how they may be derived. To this end we shall need the transformation laws for the coefficients of the equation under the trivial transformations (*b*) and (*bc*) which is a combination of (*b*) and (*c*) defined as follows:

(*bc*) Replacement of the function u in (1.1) by λu ($\lambda \neq 0$) and simultaneous multiplication of the equation by λ^{-1}.

(The transformation (*bc*) has the property of leaving invariant the pseudo-Riemannian metric g^{ab}). The transformations (*b*) and (*bc*) transform the differential operator $F(u)$ into an operator $\overline{F}(u)$ of the same form but with different coefficients $\tilde{g}^{ab}$, $\overline{A}^a$ and $\overline{C}$. Explicitly

$$\overline{F}(u) \equiv \tilde{g}^{ab}u_{\tilde{;}ab} + \overline{A}^a u_{,a} + \overline{C}u = \lambda^{-1}e^{-2\phi}F(\lambda u), \tag{3.4}$$

where

$$\tilde{g}^{ab} = e^{-2\phi}g^{ab}, \qquad \tilde{g}_{ab} = e^{2\phi}g_{ab}, \tag{3.5}$$

$$\overline{A}_a = A_a + 2(\log\lambda)_{,a} - (n-2)\phi_{,a}, \qquad \overline{A}^a = \tilde{g}^{ab}\overline{A}_a, \tag{3.6}$$

$$\overline{C} = e^{-2\phi}(C + \lambda^{-1}\Box\lambda + A^a(\log\lambda)_{,a}). \tag{3.7}$$

Under the trivial transformations $C^a{}_{bcd}$, H_{ab} and $\mathscr{C}$ transform as follows:

$$\tilde{C}^a{}_{bcd} = \tilde{C}^a{}_{bcd}, \tag{3.8}$$

$$\overline{H}_{ab} = H_{ab}, \tag{3.9}$$

$$\overline{\mathscr{C}} = e^{-2\phi}\mathscr{C}. \tag{3.10}$$

The transformation laws for the adjoint differential operator and the elementary solutions are respectively [*33*]

$$\overline{G}(v) = \lambda e^{-n\phi}G(\lambda^{-1}e^{(n-2)\phi}v), \tag{3.11}$$

(1) This problem has also been studied by Goldoni [*15*] who finds (see Theorem 2 of his paper as modified in the note added in proof) that a necessary condition for the equation (1.1) to satisfy Huygen's principle when $n = 4$ is that space-time be conformal to an Einstein space. However, the present author has been unable to rederive this result.

and

$$\overline{E}^{\pm}_{x_0}(x) = \lambda\lambda_0^{-1}e^{(2-n)\phi}E^{\pm}_{x_0}(x), \tag{3.12}$$

where $\lambda_0 = \lambda(x_0)$.

In particular when $n = 4$

$$\overline{E}^{\pm}_{x_0}(x) = \lambda\lambda_0 e^{-2\phi}E^{\pm}_{x_0}(x). \tag{3.13}$$

From this it follows that the transformation laws for $V(x_0, x)$ and $\mathscr{V}(x_0, x)$ are given by

$$[\overline{V}] = \lambda_0^{-1}a_1[\lambda e^{-2\phi}V], \tag{3.14}$$

$$\overline{\mathscr{V}}^{\pm} = \lambda_0^{-1}\lambda e^{-2\phi}\mathscr{V}^{\pm}, \tag{3.15}$$

where

$$a_1 = \frac{1}{s}\int_0^{s(x)} e^{2\phi}dt. \tag{3.16}$$

Finally we have

$$[G(\overline{V})] = \lambda_0^{-1}a_1[\lambda e^{-4\phi}G(v)], \tag{3.17}$$

from which it follows that the condition (2.8) is (as it should be) invariant under trivial transformations.

If one considers only the transformation (*bc*), then it follows from Eqs (2.4) and (3.6) that

$$\overline{V} = \lambda_0^{-1}\lambda V. \tag{3.18}$$

In contrast to Eq. (3.14) the above transformation holds at every point in some normal neighbourhood of x_0, not just on $C(x_0)$.

We now proceed with a sketch of the derivation of the necessary conditions. The procedure to be described was first outlined by Mathisson and later perfected by Hadamard [*22*], Günther [*16*] and the author [*32*] [*33*]. One begins by choosing an arbitrary point $x_0 \in V_4$. Then a trivial transformation (*b*) is made such that

$$\overset{\circ}{\tilde{L}}_{ab} = \overset{\circ}{\tilde{L}}_{(ab;c)} = \overset{\circ}{\tilde{L}}_{(ab;cd)} = \ldots = 0, \tag{3.19}$$

where the small $^\circ$ over a tensor denotes evaluation at x_0. We next specify the transformation (*bc*) by

$$\lambda(x) = \exp\left\{-\frac{1}{4}\int_0^{s(x)} A^a\Gamma_{,a}\frac{dt}{t}\right\}, \tag{3.20}$$

where the tildes have been dropped. It follows that $\lambda_0 = 1$ and that

$$\overline{V}(x_0, x) = \frac{1}{2\pi}\rho^{-\frac{1}{2}}. \tag{3.21}$$

The choice (3.20) is equivalent to the requirement that

$$\bar{A}^a\Gamma_{,a} = 0. \tag{3.22}$$

Finally the trivial transformation (a) is specified by choosing a system of normal coordinates (x^a) with origin x_0. We recall that these conditions are defined by the condition [38]

$$g_{ab}x^b = \overset{\circ}{g}_{ab}x^b. \tag{3.23}$$

In view of the above choices for the trivial transformations the function V has the especially simple form

$$V = \frac{1}{2\pi}\left(\frac{\overset{\circ}{g}}{g}\right)^{\frac{1}{4}}, \tag{3.24}$$

from which it follows that the necessary and sufficient condition (2.8) at x_0 can be expressed as

$$[\sigma(x_0, x)] = 0, \tag{3.25}$$

where (dropping the tildes and bars)

$$\sigma = \gamma + A^a g^{bc} g_{bc,a} + 4A^a{}_{,a} - 4C, \tag{3.26}$$

and where

$$\gamma = (g^{ab}g^{cd}g_{cd,a})_{,b} + \frac{1}{4}g^{ab}g_{ab,c}g^{cd}g^{ef}g_{ef,d}. \tag{3.27}$$

Since σ must vanish on $C(x_0)$ the following conditions must be satisfied by the derivatives of σ at the point x_0:

$$\overset{\circ}{\sigma} = 0, \tag{3.28a}$$
$$\overset{\circ}{\sigma}_{,a} = 0, \tag{3.28b}$$
$$TS(\overset{\circ}{\sigma}_{;ab}) = 0, \tag{3.28c}$$
$$TS(\overset{\circ}{\sigma}_{;abc}) = 0, \tag{3.28d}$$
$$TS(\overset{\circ}{\sigma}_{;abcd}) = 0, \tag{3.28e}$$
$$\vdots$$

The derivatives of σ at x_0 are calculated in a systematic way from Taylor expansions about x_0 of the tensors g_{ab}, g^{ab}, A^a and the function C. To obtain the derivatives appearing in the condition (3.28e) it is necessary to expand g_{ab} to sixth order, A_a to fifth order and g^{ab} and C to fourth order. This has been carried out by the author in Refs [32] and [33] using the methods of Herglotz [23] and Günther [16]. For the purposes of illustration we shall give these expansions only to second order which is sufficient to enable us to give the derivation of Condition I. One has

$$g_{ab} = \overset{\circ}{g}_{ab} + \frac{1}{3}\overset{\circ}{R}_{acdb}x^{cd}, \tag{3.29}$$

$$g_{ab} = \overset{\circ}{g}_{ab} - \frac{1}{3}\overset{\circ}{R}{}^{a}{}_{cd}{}^{b}x^{cd}, \tag{3.30}$$

$$A_a = \overset{\circ}{H}_{ab}x^b + \frac{2}{3}\overset{\circ}{H}_{ab;c}x^{bc}, \tag{3.31}$$

$$C = \overset{\circ}{C} + \overset{\circ}{C}_{;a}x^a + \overset{\circ}{C}_{;ab}x^{ab}, \tag{3.32}$$

where $x^{cd} = x^c x^d$. It follows from the above and Eqs (3.26) and (3.27) that

$$\overset{\circ}{\sigma} = 4\overset{\circ}{C}. \tag{3.33}$$

Thus, in view of (3.28*a*), the first condition is

$$\overset{\circ}{C} = 0 \tag{3.34}$$

with our special choice of the trivial transformations. In order to express this condition in a form invariant under the trivial transformations it is necessary to find an invariant which reduces to $\overset{\circ}{C}$ when our special choice of the trivial transformations are made. Such a quantity is the Cotton invariant defined by Eq. (1.7) which obeys the transformation law (3.10). Thus the general form of the condition (3.34) at x_0 is

$$\overset{\circ}{\mathscr{C}} \equiv \overset{\circ}{C} - \frac{1}{2}\overset{\circ}{A}{}^{a}{}_{;a} - \frac{1}{4}\overset{\circ}{A}_a\overset{\circ}{A}{}^{a} - \frac{1}{6}\overset{\circ}{R} = 0.$$

Since x_0 was chosen arbitrarily we must have at every point of V_4

$$C - \frac{1}{2}A^a{}_{;a} - \frac{1}{4}A_aA^a - \frac{1}{6}R = 0, \tag{3.35}$$

which is our first necessary condition for a Huygens' differential equation. The subsequent conditions are obtained by similar procedures (for details see [*33*]) and by using the preceeding conditions to simplify the following ones.

Each necessary condition must be expressed by the vanishing of a tensor (necessarily trace-free and symmetric) which is invariant under the trivial transformations [*33*]. In the case of the self-adjoint equation ($A^a = 0$) this involves the study of conformally invariant tensors which are functions of the metric tensor and its partial derivatives up to a certain order. In particular the necessary conditions I to V reduce to

$$C = \frac{1}{6} R, \tag{3.36}$$

$$B_{ab} = 0, \tag{3.37}$$

$$\underset{1}{\mathscr{H}}_{abcd} = 0, \tag{3.38}$$

where

$$B_{ab} = S_{abk;}{}^{k} - \frac{1}{2} C^{k}{}_{ab}{}^{l} L_{kl} \tag{3.39}$$

is the so called Bach tensor [2] [40], and where

$$\underset{1}{\mathscr{H}}_{abcd} = \mathrm{TS}(3C_{kabl;m}C^{k}{}_{cd}{}^{l}{}_{;}{}^{m} + 8C^{k}{}_{ab}{}^{l}{}_{;c}S_{kld} + 40S_{ab}{}^{k}S_{cdk} - 8C^{k}{}_{ab}{}^{l}S_{klc;d} - 24C^{k}{}_{ab}{}^{l}S_{cdk;l} + 4C^{k}{}_{ab}{}^{l}C_{l}{}^{m}{}_{ck}L_{dm} + 12C^{k}{}_{ab}{}^{l}C^{m}{}_{cdl}L_{km}). \tag{3.40}$$

A tensor T is said to be conformally invariant of weight w if and only if

$$\tilde{T} = e^{2w\phi}T. \tag{3.41}$$

Under this definition it can be shown that both B_{ab} and $\underset{1}{\mathscr{H}}_{abcd}$ are conformally invariant tensors of weight -1. A proof of this fact for the Bach tensor may be found in Schouten [40]. The corresponding proof for the tensor $\underset{1}{\mathscr{H}}_{abcd}$ has been given by the author [33] and independently by Wünsch [49]. In [33] a second fourth rank, trace-free, symmetric, conformally invariant tensor of weight-1, defined as follows, is given:

$$\underset{2}{\mathscr{H}}_{abcd} = \mathrm{TS}(C_{kabl;m}C^{k}{}_{cd}{}^{l;}{}_{m} + 4C^{k}{}_{ab}{}^{l}{}_{;c}S_{kld} + 12S_{ab}{}^{k}S_{cdk} - 4C^{k}{}_{ab}{}^{l}S_{klc;d} - 8C^{k}{}_{ab}{}^{l}S_{cdk;l} + 2C^{k}{}_{ab}{}^{l}C_{k}{}^{m}{}_{cl}L_{dm} + 4C^{k}{}_{ab}{}^{l}C^{m}{}_{idk}L_{lm}). \tag{3.42}$$

The tensors B_{ab}, $\underset{1}{\mathscr{H}}_{abcd}$ and $\underset{2}{\mathscr{H}}_{abcd}$ are examples of the conformally invariant, rational integral, metric differential concomitants studied by Szekeres [45], du Plessis [12], and Wünsch [49]. Wünsch shows that the Bach tensor up to a constant factor is the only trace-free symmetric, second rank, conformally invariant tensor of this type of weight-1, and that any conformally invariant scalar, vector or trace-free symmetric third rank tensor of this weight is identically zero. It also follows from his work that every fourth rank, trace-free, symmetric, conformally invariant tensor of weight-1 can be expressed in a unique way as a linear combination of $\underset{1}{\mathscr{H}}_{abcd}$, $\underset{2}{\mathscr{H}}_{abcd}$ and the tensor

$$\mathrm{TS}(C_{kabl}C^{m}{}_{cd}{}^{n}C^{k}{}_{mn}{}^{l}). \tag{3.43}$$

Necessary conditions for the validity of Huygens' principle for Maxwell's equations may also be expressed in terms of the above tensors. The two known conditions obtained respectively by Günther [19] and Wünsch [48] are (see also Günther and Wünsch [20] for a treatment of the special case $R_{ab} = 0$).

$$B_{ab} = 0, \tag{3.44}$$

$$14\underset{2}{\mathcal{H}}_{abcd} - 3\underset{1}{\mathcal{H}}_{abcd} = 0. \tag{3.45}$$

Recently Wünsch [51] has obtained analogous necessary conditions for the Weyl equation (1.12).

4. CONSEQUENCES OF THE NECESSARY CONDITIONS

If $R_{abcd} = 0$ (Minkowski space) and if $A^a = 0$, Condition I implies that $C = 0$. For this reason no counter examples of the form $\Box u + Cu = 0$ can be constructed in Minkowski space.

If $R_{abcd} = 0$, the Conditions I, II and III imply that a Huygens' equation is equivalent to Eq. (1.2) with $m = 2$. This is essentially the result of Mathisson [31], Hadamard [22], and Asgeirsson [1]. The proof depends on the following lemma [16] (see also Ref. [26]).

LEMMA 4.1. — *If*

$$H_{ak}H_b{}^k - \frac{1}{4}g_{ab}H_{kl}H^{kl} = 0, \tag{4.1}$$

then Eq. (1.1) *is equivalent by a transformation* (*bc*) *to the self-adjoint equation*

$$g^{ab}u_{;ab} + \frac{1}{6}Ru = 0 \tag{4.2}$$

Proof. — The left hand side of (4.1) may be interpreted as the energy momentum tensor of the « Maxwell Field » H_{ab}. It is known that the vanishing of the energy momentum tensor implies the vanishing of the corresponding Maxwell field (see for example Mathisson [31]).

Thus it follows that

$$H_{ab} = A_{[a,b]} = 0, \tag{4.3}$$

that is that the differential one-form $A = A_a dx^a$ is closed. Thus there exists locally a function g such that $A = dg$. It follows that for the transformation (*bc*) defined by $\lambda = \exp(-g/2)$ one has $\overline{A}_a = 0$, from which it follows by Condition I that $\overline{C} = R/6$.

Hadamard's problem is solved in the case $R_{ab} = 0$ (empty space-time) by the following result due to the author [32]:

THEOREM 4.1. — *The Eq.* (4.2) *satisfies Huygens' principle on an empty space-time iff the space-time is flat or a plane-wave space-time which admits a coordinate system in which the metric takes the form of* (1.10) *with* $e = 0$.

This result follows from Condition V which under the hypotheses of the theorem reduces to

$$\mathrm{TS}(C_{kabl;m}C^{k}{}_{cd}{}^{;}{}^{l\,m}) = 0\,. \tag{4.4}$$

We note that the other conditions are satisfied identically in this case.

The solution of (4.4) may be achieved by introducing a two-component spinor formalism [35] [37]. We recall that in this formalism spinors and tensors are related by the complex connecting quantities $\sigma_a{}^{AB'}$ ($a = 1, 2, 3, 4$; A, $B' = 1, 2$) which are Hermitian in the spinor indices A and B' and satisfy the conditions

$$\sigma_{aAB'}\sigma_b{}^{AB'} = g_{ab}\,. \tag{4.5}$$

In Eq. (4.5) the spinor indices have been lowered by the skew symmetric spinors ε_{AB} and $\varepsilon_{A'B'}$ defined by $\varepsilon_{12} = \varepsilon_{1'2'} = 1$. The respective inverses of these spinors ε^{AB} and $\varepsilon^{A'B'}$ are used to raise spinor indices, the convention being $K^A = \varepsilon^{AB}K_B$.

The spinor form of Eq. (4.4) is [32]

$$\Psi_{ABCD;IJ'}\overline{\Psi}_{E'F'G'H';}{}^{IJ'} = 0\,, \tag{4.6}$$

where Ψ_{ABCD} is a completely symmetric four-indix spinor corresponding to the Weyl tensor, defined by the equation

$$C_{abcd}\sigma^a{}_{AE'}\sigma^b{}_{BF'}\sigma^c{}_{CG'}\sigma^d{}_{DH'} = \Psi_{ABCD}\varepsilon_{E'F'}\varepsilon_{G'H'} + \overline{\Psi}_{E'F'G'H'}\varepsilon_{AB}\varepsilon_{Cd}\,. \tag{4.7}$$

The proof of Theorem 4.1 depends on the following lemma which is somewhat stronger than required and which will be needed later.

LEMMA 4.2. — *In a space (not assumed empty) where*

$$S_{abc} = 0\,, \tag{4.8}$$

and where Eq. (4.6) *holds* (*assuming* $\Psi_{ABCD;EF'} \neq 0$), *there exists a one-index spinor* K_A *and a non-zero scalar* $\mathscr{A}$ *such that*

$$\Psi_{ABCD;EF'} = \mathscr{A}K_AK_BK_CK_DK_E\overline{K}_{F'} \tag{4.9}$$

Proof. — The proof this lemma is almost contained in the proof of Theorem 1 of Ref. [32]. One only has to note the additional fact that Eq. (4.8) (the defining equation of the C-spaces of Szekeres [44]) implies that the spinorial Bianchi identities have the form

$$\Psi_{ABCD;DF'} = 0\,, \tag{4.10}$$

that is the same form as in empty space-time.

It should be noted that the case of an empty symmetric space-time has to be given a separate treatment. This is because the defining equations of a *symmetric space-time* namely

$$R_{abcd;e} = 0\,.$$

imply that the Eqs (4.4) are *identically satisfied.* This case is studied in Ref. [32] where it is shown that Theorem 2.1 still holds with D a real constant and $e = 0$ in the metric (1.10). The same remark applies to the *conformal symmetric space-time* defined by the conditions $C_{abcd;e} = 0$, which also implies that the left hand side of Eq. (4.6) vanishes identically. In Ref. [34] it is shown that these space-times are necessarily plane-wave with metric (1.10) where $D = 1$. We thus conclude that the self-adjoint equation (4.2) satisfies Huygens' principle on a conformal symmetric space-time.

Hadamards problem on a *conformally empty space-time* is resolved by the following [32]:

COROLLARY 4.1. — *An Eq.* (1.1) *satisfies Huygens' principle on a conformally empty space-time iff it is equivalent to the self-adjoint Eq.* (4.2) *on a plane-wave space-time with metric* (1.10) *with* $e = 0$.

Proof. — A necessary condition that a space-time be conformal to an empty space-time is that the Bach tensor (3.3) vanish at every point [40]. It thus follows from Condition III and Lemma 4.1 that a Huygens' differential equation on such a space-time must be equivalent to the self-adjoint Eq. (4.2). The rest of the conclusion then follows from Theorem 2.1 once a conformal transformation to empty space-time has been made.

Wünsch [51] has extended Theorem 4.1 to the case when $R_{ab} = \lambda g_{ab}$ (Einstein space-time) and finds when $\lambda \neq 0$ that the validity of Huygens' principle for the self-adjoint equation (4.2) implies that the space-time is of constant curvature. Obviously Corollary 4.1 can now be generalized to read: An Eq. (1.1) satisfies Huygens' principle on space-time conformal to an Einstein space-time (with $\lambda \neq 0$) iff it is equivalent to the self-adjoint Eq. 4.2 on a space-time of constant curvature.

The case of a general symmetric space-time is covered by the following due to the author [33] (see also Wünsch [48]).

THEOREM 4.2. — *The self-adjoint Eq.* (4.2) *satisfies Huygens' principle on a symmetric space-time iff space-time is conformally flat or a symmetric plane-wave with metric given by Eq.* (1.10) *where e and* D *are real constants.*

The proof of this theorem depends on the classification of symmetric space-times given by Cahen and McLenaghan [5] in which it shown that such space-times are of the following types: *a*) the symmetric plane wave space-time with metric given by Eq. (1.10) where *e* and D are real constants; *b*) the Robonson-Bertotti space time with metric

$$ds^2 = 2\left[1 - \frac{1}{8}(R + \beta)uv\right]^{-2} dudv - 2\left[1 + \frac{1}{8}(R - \beta)z\bar{z}\right]^{-2} dzd\bar{z}, \quad (4.12)$$

where R and β are constant; *c*) three conformally flat space-times including the space-time of constant curvature. The Eq. (4.2) is known to satisfy Huygens' principle on the space-times *a*) and *c*), but it does so for the space-time *b*) iff the curvature scalar $R = 0$ that is when this space-time is conformally flat. This result, which follows from Conditions III and V, is a special case of the following theorem of Wünsch [*51*]:

THEOREM 4.3. — *A self-adjoint Eq.* 4.2 *satisfies Huygens' principle on a* (2×2)*-decomposable space-time, that is a space-time admitting a system of local coordinates in which the metric has the form*

$$ds^2 = g_{\alpha\beta}(x^1, x^2)dx^\alpha dx^\beta + g_{\mu\nu}(x^3, x^4)dx^\mu dx^\nu, \tag{4.13}$$

where $\alpha, \beta = 1, 2$ *and* $\mu, \nu = 3, 4$, *iff the space-time is conformally flat.*

Wünsch [*51*] has presented a number of further consequences of the Conditions I to V for the self-adjoint Eq. (4.2). The essential feature of all these results is the demonstration that the validity of Huygens' principle for the Eq. (4.2) on a space-time satisfying some supplementary condition (recurrent, conformally recurrent, etc.) implies that space-time is plane-wave or conformally flat, these being the only known space-times where the Eq. (4.2) satisfies Huygens' principle.

The purpose of the rest of this paper is to present a possible new class of Huygens' differential equations not equivalent to known ones that have just been mentioned. Our main result is the following:

THEOREM 4.4. — *On a space-time where the conditions* (4.4) *and* (4.8) *hold there exists a coordinate system* $(u, v, z, \bar{z})$ *in which the metric has the form*

$$\begin{aligned} ds^2 = 2dv(du + (a(z+\bar{z})u + Dz^2 + \bar{D}\bar{z}^2 + ez\bar{z} + Fz + \bar{F}\bar{z})dv) \\ - 2(dz + az^2dv)(d\bar{z} + a\bar{z}^2dv), \end{aligned} \tag{4.14}$$

where a and e are real valued and D *and* F *are complex valued functions only of* v.

Proof. — Since the conditions of Lemma 4.2 are satisfied there exist a complex scalar function $\mathscr{A}$ and a spinor field K_A such the Eq. (4.9) holds. The tensorial form of this equation is

$$\overset{+}{C}_{abcd;e} = \mathscr{A}\overset{+}{F}_{ab}F_{cd}k_e, \tag{4.15}$$

where

$$\overset{+}{C}_{abcd} = \frac{1}{2}(C_{abcd} - i{}^*C_{abcd}) \tag{4.16}$$

denotes the self-dual Weyl tensor (${}^*C_{abcd} = \frac{1}{2}\varepsilon_{abef}C^{ef}{}_{cd}$, where ε_{abef} are the components of the volume element), where

$$\overset{+}{F}_{ab} = \frac{1}{2}(F_{ab} - i{}^*F_{ab}) = K_A K_B \varepsilon_{C'D'} \sigma^{AC'}{}_a \sigma^{BD'}{}_b \tag{4.17}$$

is a self-fual bivector and where

$$k_a = \mathrm{K_A \overline{K}_{B'}} \sigma^{\mathrm{AB'}} a\,, \tag{4.18}$$

is a real null vector satisfying

$$\overset{+}{\mathrm{F}}_{ab} k^b = 0\,. \tag{4.19}$$

The remainder of the proof depends on the following lemma:

LEMMA 4.3. — *If the condition* (4.15) *is satisfied by a space-time, where* k_a *is a real principal null vector of the singular self-dual bivector* $\overset{+}{\mathrm{F}}_{ab}$, *one has*

$$\overset{+}{\mathrm{C}}_{abcd;e} = \overset{+}{\mathrm{C}}_{abcd} \mathrm{K}_e\,, \tag{4.20}$$

where

$$\mathrm{K}_e = \mathscr{A}' k_e\,, \tag{4.21}$$

and

$$\mathrm{C}_{abcd} k^d = 0\,. \tag{4.22}$$

In other words space-time is a Petrov type N *complex recurrent space-time* (because of Eq. (4.20)) the recurrence vector of which (K_a) is proportional to a principal null vector (k_a) of the Weyl tensor. Assuming the truth of this lemma, Theorem 4.4 follows immediately by the result of Sec. 7 of Ref. [*34*] where a coordinate system is constructed on a complex recurrent space-time with recurrence vector proportional to a principal null vector of the type N Weyl tensor, in which the metric takes the form (4.14). In the same reference it is shown that a complex recurrent space must be either Petrov type N or D and that when such a space-time is type D it is necessarily a (2×2)-decomposable space. Wünsch [*51*] has remarked that, in view of Theorem 4.3, a self-adjoint Huygens' equation on a complex recurrent space-time implies Petrov type N.

Proof of the lemma. — The proof is facilitated by the use of the vectorial formalism of Debever, Cahen and Defrise [*9*] [*4*] (see Appendix). Contraction of both sides of Eq. (4.15) by $\mathrm{Z}_\alpha{}^{ab} \mathrm{Z}_\beta{}^{cd} dx^e$ yields

$$\mathrm{DC}_{\alpha\beta} = \mathscr{A} \mathrm{F}_\alpha \mathrm{F}_\beta \mathrm{K}\,, \tag{4.23}$$

where D denotes the absolute covariant derivative of E. Cartan, where $\mathrm{C}_{\alpha\beta}$ is defined by (A.11), where

$$\mathrm{F}_\alpha = \overset{+}{\mathrm{F}}_{ab} \mathrm{Z}_\alpha{}^{ab} \tag{4.25}$$

are the vectorial components of the self-dual bivector $\overset{+}{\mathrm{F}}$, and where

$$\mathrm{K} = k_a dx^a \tag{4.26}$$

defines a null one-form. Integrability conditions may be obtained by taking the absolute covariant derivatives of both sides of Eq. (4.23) and by employing the identities

$$\mathrm{DDC}_{\alpha\beta} = - \mathrm{C}_{\gamma\beta}\Sigma^{\gamma}{}_{\alpha} - \mathrm{C}_{\alpha\gamma}\Sigma^{\gamma}{}_{\beta}, \tag{4.27}$$

which are a consequence of Eqs (A.8) and (A.13). One finds

$$\begin{aligned} - \mathrm{C}_{\gamma\beta}\Sigma^{\gamma}{}_{\alpha} - \mathrm{C}_{\alpha\gamma}\Sigma^{\gamma}{}_{\beta} = \mathrm{F}_{\alpha}\mathrm{F}_{\beta}\mathrm{D}\mathscr{A} \wedge \mathrm{K} + \mathscr{A}\mathrm{F}_{\beta}\mathrm{DF}_{\alpha} \wedge \mathrm{K} \\ + \mathscr{A}\mathrm{F}_{\alpha}\mathrm{DF}_{\beta} \wedge \mathrm{K} + \mathscr{A}\mathrm{F}_{\alpha}\mathrm{F}_{\beta}\mathrm{DK}. \end{aligned} \tag{4.28}$$

In order to examine the consequences of this condition it is advantageous to choose the null tetrad such that $\theta^0 = \mathrm{K}$. On account of Eq. (4.19) we have

$$\overset{+}{\mathrm{F}} = \mathrm{F}_{\alpha}\mathrm{Z}^{\alpha} = \mathrm{Z}^2 = \theta^0 \wedge \theta^1 \tag{4.29}$$

which implies that $\mathrm{F}_{\alpha} = \delta^2_{\alpha}$. When we take account of this and the fact that $\mathrm{D}\theta^0 = 0$, the condition (4.28) becomes

$$\mathrm{C}_{\gamma\beta}\Sigma^{\gamma}{}_{\alpha} + \mathrm{C}_{\alpha\gamma}\Sigma^{\gamma}{}_{\beta} = \delta^2_{\alpha}\delta^2_{\beta}\theta^0 \wedge d\mathscr{A} + \mathscr{A}(\delta^2_{\beta}\sigma^2{}_{\alpha} + \delta^2_{\alpha}\sigma^2{}_{\beta}) \wedge \theta^0, \tag{4.30}$$

and Eq. (4.23) reads

$$\mathrm{DC}_{\alpha\beta} = \mathscr{A}\delta^2_{\alpha}\delta^2_{\beta}\theta^0 . \tag{4.31}$$

When the latter equation is written out in expanded form using (A.13) we obtain

$$\mathrm{DC}_{11} = d\mathrm{C}_{11} + 2\mathrm{C}_{11}\sigma^3 + \mathrm{C}_{13}\sigma^2 = 0, \tag{4.32}$$

$$\mathrm{DC}_{12} = d\mathrm{C}_{12} - \frac{1}{2}\mathrm{C}_{13}\sigma^1 + \frac{1}{2}\mathrm{C}_{23}\sigma^2 = 0, \tag{4.33}$$

$$\mathrm{DC}_{13} = d\mathrm{C}_{13} - \mathrm{C}_{11}\sigma^1 + 3\mathrm{C}_{12}\sigma^2 + \mathrm{C}_{13}\sigma^3 = 0, \tag{4.34}$$

$$\mathrm{DC}_{22} = d\mathrm{C}_{22} - 2\mathrm{C}_{22}\sigma^3 - \mathrm{C}_{23}\sigma^1 = \mathscr{A}\theta^0, \tag{4.35}$$

$$\mathrm{DC}_{23} = d\mathrm{C}_{23} - \mathrm{C}_{23}\sigma^3 - 3\mathrm{C}_{12}\sigma^1 + \mathrm{C}_{22}\sigma^2 = 0. \tag{4.36}$$

Likewise we may obtain the expanded form of Eq. (4.30) by employing the Eqs (A.7), (A.9) and (A.10). After a considerable amount of calculation we find the equations

$$12\mathrm{C}_{11}\mathrm{C}_{12} - 3\mathrm{C}_{13}{}^2 + \mathrm{RC}_{11} = 0, \tag{4.37}$$

$$6\mathrm{C}_{11}\mathrm{C}_{23} - 6\mathrm{C}_{12}\mathrm{C}_{13} + \mathrm{RC}_{13} = 0, \tag{4.38}$$

$$2\mathrm{C}_{11}\mathrm{C}_{22} - 6\mathrm{C}_{12}{}^2 + \mathrm{C}_{13}\mathrm{C}_{23} + \mathrm{RC}_{12} = 0, \tag{4.39}$$

$$6\mathrm{C}_{13}\mathrm{C}_{22} - 6\mathrm{C}_{12}\mathrm{C}_{23} + \mathrm{RC}_{23} = 0, \tag{4.40}$$

$$6\mathrm{C}_{12}\mathrm{C}_{23} - 6\mathrm{C}_{13}\mathrm{C}_{22} - \mathrm{RC}_{23} - 6\mathscr{A}\sigma^2{}_3 = 0, \tag{4.41}$$

$$12\mathrm{C}_{12}\mathrm{C}_{22} - 3\mathrm{C}_{23}^2 + \mathrm{RC}_{22} - 6\mathscr{A}\sigma^2{}_1 = 0. \tag{4.42}$$

We shall now show that θ^0 is a quadruply repeated principal null form of the Weyl tensor which is thus necessarily of Petrov type N. We begin by assuming that θ^0 is not a principle null form of the Weyl tensor that is that $\mathrm{C}_{11} \neq 0$. (See Ref. [9] for a discussion of the Petrov classification in the vectorial formalism) .By means of a tetrad rotation (A.14) which preserves the direction of θ^0 we may set $C'_{11} = 1$ and $C'_{13} = 0$ (on account of the formulas (19.20) of Ref. [9]). Dropping the primes we note that Eqs. (4.37), (4.38) and (4.39) imply respectively that $C_{12} = -R/12$, $C_{23} = 0$, and $C_{22} = R^2/16$. Now Eq. (4.32) implies $\sigma^3 = 0$ while Eq. (4.33) gives $dC_{12} = 0$ from which it follows that $dR = dC_{22} = 0$. When we take all this into account in Eq. (4.35) we obtain $\mathscr{A}\theta^0 = 0$ which implies $\mathscr{A} = 0$ which is impossible. Thus we must have $C_{11} = 0$. It follows immediately from Eq. (4.37) that $C_{13} = 0$ while Eqs (4.39) and (4.40) imply $C_{12}(6C_{12} - R) = 0$ and $C_{23}(6C_{12} - R) = 0$. We have two possibilities *i)* $6C_{12} - R \neq 0$ or *ii)* $R = 6C_{12}$. Consider first the Case (*i*): We have $C_{12} = C_{23} = 0$, $R \neq 0$ so the Weyl tensor is of type N. It now follows from Eq. (4.36) that $C_{22}\sigma^2 = 0$ which implies $\sigma^2 = 0$ since space-time is not assumed conformally flat. When this is taken account of in Eq. (4.42) one finds $RC_{22} = 0$ which is a contradiction. We now study Case (*ii*) where $R = 6C_{12} \neq 0$. This implies that θ^0 is at most a doubly repeated principal null form. Thus we may use a tetrad rotation (A.14) to align $\theta^{3'}$ with one of the other principal null forms. This makes $C_{22'} = 0$. Dropping primes we remark that Eq. (4.34) yields $\sigma^2 = 0$. It thus follows from Eq. (4.42) that $C_{23} = 0$ which together with Eq. (4.35) implies $\mathscr{A}\theta^0 = 0$, a contradiction. Consequently we must have $R = C_{12} = 0$ which on account of Eq. (4.42) implies $C_{23} = 0$. Thus the space-time is of Petrov type N with θ^0 a quadruply repeated principal null form. Finally it follows from Eq. (4.36) that $C_{22}\sigma^2 = 0$ which implies $\sigma^2 = 0$. Thus all our equations are satisfied and we have

$$DC_{\alpha\beta} = C_{\alpha\beta}K', \tag{4.43}$$

where

$$C_{\alpha\beta} = C_{22}\delta^2_\alpha\delta^2_\beta, \tag{4.44}$$

and

$$K' = \mathscr{A}C_{22}^{-1}\theta^0. \tag{4.45}$$

When Eqs (4.43) and (4.45) are translated back to tensorial form one obtains Eqs (4.20) and (4.22) respectively where $\mathscr{A}' = \mathscr{A}C_{22}^{-1}$. The Eq. (4.22) follows from the fact that θ^0 is a principal null form of the type N Weyl tensor. This completes the proof of Lemma 4.3 and hence of Theorem 4.4.

We shall now discuss some of the main properties of the metric (4.14) of the general type N complex recurrent space-time with recurrence vector proportional to a principal null vector of the Weyl tensor. We call this space-time the *generalized plane-wave space-time* since the metric (4.14) reduces, modulo a coordinate transformation, to the plane-wave metric (1.10) when one sets $a = 0$. One of the most important properties of this space-time from the point of view of the present study is given in the following theorem:

THEOREM 4.5. — *On a generalized plane-wave space-time with metric* (4.14) *one has*

$$B_{ab} = 0\,, \tag{4.46}$$

$$\underset{1}{\mathcal{H}}_{abcd} = 0\,, \tag{4.47}$$

$$\underset{2}{\mathcal{H}}_{abcd} = 0\,,$$

where B_{ab}, the Bach tensor, is defined by Eq. (3.38) and where $\underset{1}{\mathcal{H}}_{abcd}$ and $\underset{2}{\mathcal{H}}_{abcd}$ are defined by the Eqs (3.39) and (3.40) respectively.

Proof. — In view of Eqs (4.4) and (4.8) the tensors B_{ab}, $\underset{1}{\mathcal{H}}_{abcd}$, and $\underset{2}{\mathcal{H}}_{abcd}$ reduce to

$$B_{ab} = -\frac{1}{2}C^{k}{}_{ab}{}^{l}L_{kl}\,, \tag{4.49}$$

$$\underset{1}{\mathcal{H}}_{abcd} = 4\mathrm{TS}(C^{k}{}_{ab}{}^{l}C_{k}{}^{m}{}_{cl}L_{dm} + 3C^{k}{}_{ab}{}^{l}C^{m}{}_{cdl}L_{km})\,, \tag{4.50}$$

$$\underset{2}{\mathcal{H}}_{abcd} = 2\mathrm{TS}(C^{k}{}_{ab}{}^{l}C_{k}{}^{m}{}_{cl}L_{dm} + 2C^{k}{}_{ab}{}^{l}C^{m}{}_{cdl}L_{km})\,. \tag{4.51}$$

To show that these tensors indeed vanish for the metric (4.14) we introduce the following null tetrad:

$$\theta^0 = k_a dx^a = dv\,,\qquad \theta^1 = m_a dx^a = dz + az^2 dv\,,\qquad \theta^2 = \bar{\theta}^1$$
$$\theta^3 = n_a dx^a = du + (a(z+\bar{z})u + Dz^2 + \bar{D}\bar{z}^2 + ez\bar{z} + Fz + \bar{F}\bar{z})dv\,. \tag{4.52}$$

The only non-vanishing Debever curvature components in this tetrad are [34]

$$\left.\begin{array}{ll} C_{22} = -4D\,, & E_{2\bar{3}} = -4a \\ E_{2\bar{2}} = -2av(z+\bar{z}) + 2a^2(z^2+\bar{z}^2) - e & \end{array}\right\} \tag{4.53}$$

It follows that

$$L_{ab} = R_{ab} = E_{2\bar{2}}k_a k_b + E_{2\bar{3}}k_{(a}m_{b)} + E_{3\bar{2}}k_{(a}\bar{m}_{b)}\,, \tag{4.54}$$

$$C_{abcd} = C_{22}k_{[a}m_{b]}k_{[c}m_{d]} + \bar{C}_{22}k_{[a}\bar{m}_{b]}k_{[c}\bar{m}_{d]}\,, \tag{4.55}$$

from which it is easily verified that

$$C_{abcd}k^d = 0\,. \tag{4.56}$$

In view of Eqs (4.54) and (4.56) it can be shown without much difficulty that the right hand side of Eq. (4.49) and the second term on the right hand side of Eq. (4.50) (and (4.51)) both vanish. To show that the first term on the right hand side of Eq. (4.50) (and 4.51) also vanishes we note that Eqs (4.54) and (4.56) imply

$$C^{k}{}_{ab}{}^{l}C_{k}{}^{m}{}_{cl}L_{dm} = \frac{1}{2}C^{k}{}_{ab}{}^{l}C_{k}{}^{m}{}_{cl}k_d(E_{2\bar{3}}m_m + E_{3\bar{2}}\bar{m}_m) \tag{4.57}$$

The result follows when we observe that Eq. (4.55) implies

$$C^{k}{}_{ab}{}^{l}C_{k}{}^{m}{}_{cl} = \frac{1}{8} C_{22}\bar{C}_{22}k_a k_b k^m k_c \,, \tag{4.58}$$

which completes the proof of the theorem.

COROLLARY 4.2. — *The self-adjoint equation* (4.2) *on the generalized plane-wave space-time with metric* (4.14) *satisfies the necessary Conditions I to V for the validity of Huygens' principle.*

Proof. — Since $R = A^a = 0$ we only have to consider the Conditions III and V which are satisfied in view of Eqs (4.46) and (4.47) of Theorem 4.5.

COROLLARY 4.3. — *Maxwells equations* (1.11) *on a generalized plane-wave space-time with metric* (4.14) *satisfies the necessary conditions* (3.42) *for the validity of Huygens' principle.*

Proof. — The result follows from Eqs (4.46), (4.47) and (4.48) of Theorem 4.5.

It is worth remarking that on account of a necessary condition derived by Wünsch [*51*] a result analogous to Corollary 4.2 is also valid for the Weyl equation on the generalized plane-wave space-time in view of Theorem 4.5.

Another key property of the generalized plane-wave space-time is given in the following theorem:

THEOREM 4.6. — *The generalized plane-wave space-time with metric* (4.14) *where $a \neq 0$ is not conformally related to the plane-wave space-time with metric* (1.10).

Proof. — We first remark that the plane-wave space-time has the property that

$$S_{abc} = 0 \,. \tag{4.59}$$

This follows from the fact that this space-time satisfies the conditions (4.20), (4.21) and (4.22) that is it is a type N complex recurrent space-time with recurrence vector proportional to a principal null vector of the Weyl tensor. (Recall that we may get the plane-wave metric (1.10) by setting $a = 0$ in the metric (4.14) and performing a suitable coordinate transformation). The Eq. (4.59) is obtained by contracting both sides of Eq. (4.20) by g^{ae} using Eqs (4.16), (4.21), (4.22) and Bianchi's identities

$$C^{d}{}_{abc;d} = - S_{abc} \,. \tag{4.60}$$

Now under a general conformal transformation of the metric given by Eq. (3.5) one finds that

$$\tilde{S}_{abc} = S_{abc} - \phi_{,k}C^{k}{}_{abc} \,. \tag{4.61}$$

Thus a conformal transformation that maps the generalized plane-wave metric (4.14) onto the plane-wave metric must satisfy

$$\phi_{,k}C^{k}{}_{abc} = 0 \tag{4.62}$$

Since the Weyl tensor is of Petrov type N with k_a a principal null vector satisfying Eq. (4.22) it follows that

$$\phi_{,a} = \alpha k_a \,. \tag{4.63}$$

In the coordinates system $(u, v, z, \bar{z})$ of the metric (4.14) and the tetrad (4.52) one has $k_a = \delta_a^2$. Thus Eq. (4.63) implies that $\phi = \phi(v)$. We now carry out the conformal transformation $e^{2\phi(v)}$ on the metric (4.14). The resulting metric has the form

$$\begin{aligned} d\tilde{s}^2 = e^{2\phi(v)}ds^2 = 2dv'(du' + (\tilde{a}(z + \bar{z}') + \tilde{\mathrm{D}}(z')^2 + \tilde{\bar{\mathrm{D}}}(\bar{z}')^2 + \tilde{e}z'\bar{z}' \\ + \tilde{\mathrm{F}}z' + \tilde{\bar{\mathrm{F}}}\bar{z})dv') - 2(dz' + \tilde{a}(z')^2dv)(d\bar{z}' + \tilde{a}(\bar{z}')2dv) \end{aligned} \tag{4.64}$$

where

$$\begin{aligned} &v' = \int e^{\phi}dv\,, \qquad z = e^{-\phi}z' + (2a)^{-1}\dot{\phi}e^{\phi}\,, \qquad \dot{\phi} = \frac{d\phi}{dv'} \\ &u = e^{-\phi}u' + e^{\phi}[(\dot{\phi}/2a)^{\dot{}} + 3\dot{\phi}^2/(4a)](z' + \bar{z}') \\ &\quad - e^{-\phi}\int(\dot{\phi}^2/(4a^2))e^{\phi}(\mathrm{D} + \bar{\mathrm{D}} + e)dv' - e^{-\phi}\int(\dot{\phi}/(2a))(\mathrm{F} + \bar{\mathrm{F}})dv' \\ &\quad - e^{-\phi}\int e^{3\phi}[(\dot{\phi}/(2a))^{\dot{}} + 3\dot{\phi}^2/(4a)]^2dv' \end{aligned} \tag{4.65}$$

$$\begin{aligned} &\tilde{a} = e^{-2\phi}a\,, \qquad \tilde{\mathrm{D}} = e^{2\phi}\mathrm{D} \\ &\tilde{e} = e^{2\phi}e + 2a[(\dot{\phi}/(2a))^{\dot{}} + 3\dot{\phi}^2/(4a)] \\ &\tilde{\mathrm{F}} = e^{-\phi}\mathrm{F} + (\dot{\phi}/(2a))(2\mathrm{D} + e) \\ &\qquad - ae^{-2\phi}\int \{ (\dot{\phi}^2/(4a^2))e^{\phi}(\mathrm{D} + \bar{\mathrm{D}} + e) + (\dot{\phi}/(2a))(\mathrm{F} + \bar{\mathrm{F}}) \\ &\qquad + e^{3\phi}[(\dot{\phi}/(2a))^{\dot{}} + 3\dot{\phi}^2/(4a)]^2 \}\, dv' \end{aligned} \tag{4.66}$$

It is remarkable that the transformed metric can be put into the same form as the metric (4.14) by a suitable change of coordinates. If one now introduces a null tetrad for the metric (4.64) which has the same form as the null tetrad (4.52) already defined for the metric (4.14) one finds expressions for the non-vanishing curvature components of the same form as Eq. (4.53). In particular

$$\tilde{\mathrm{E}}_{23} = -4\tilde{a} = -4e^{-2\phi}a = e^{-2\phi}\mathrm{E}_{23} \tag{4.67}$$

From this we deduce that $\tilde{\mathrm{E}}_{23} \neq 0$ since we are assuming that $\mathrm{E}_{2\bar{3}} \neq 0$. Thus we are able to conclude that the generalized plane-wave space-time is not conformal to a plane-wave space-time for which $\tilde{\mathrm{E}}_{2\bar{3}} = 0$. This completes the proof of the theorem.

It is worth noting that the above argument can also be used to show that the generalized plane-wave space-time is not conformally empty. This is so because $\tilde{S}_{abc} = 0$ and $\tilde{E}_{\alpha\bar{\beta}} = 0$ also hold in empty space-time. On the other hand the plane-wave space-time *is* conformally empty. This follows from an examination of the formulae (4.53) and (4.66) valid when $a = 0$, which show that $\tilde{E}_{22}$, the only non-vanishing tetrad component of the Ricci tensor, can be made to vanish by a suitable choice of ϕ. It is also worth remarking that the generalized plane-wave space-time is an example of a space-time satisfying the conditions of Theorem 3.6 of Ref. [*49*] (namely $B_{ab} = 0$ and $\underset{1}{\mathscr{H}}_{abcd} - 3\underset{2}{\mathscr{H}}_{abcd} = 0$ in our notation) which is *not* conformally related to an empty space-time.

CONCLUSION

In view of Corollary 4.2 and Theorem 4.6 the self-adjoint equation (4.2) satisfies the Conditions I to V for the validity of Huygens' principle on the generalized plane-wave space-time with metric (4.14) which is not conformally related to the plane-wave space-time with metric (1.10). Analogous results hold for Maxwell's equations (1.11) and the Weyl equation (1.12). This raises the possibility of a new class of space-times on which Huygens' principle is satisfied by the self-adjoint equation (4.2), Maxwell's equations and the Weyl equation. However, it is not yet known whether or not Huygens' principle is actually satisfied by any of these equations on the generalized plane-wave space-time.

APPENDIX

Following Ref. [9] we introduce a covariant null tetrad of one-forms $\theta^i = h^i{}_a dx^a$ $(i = 0, 1, 2, 3)$ in which the metric has the form

$$ds^2 = 2\theta^0\theta^3 - 2\theta^1\theta^2 . \tag{A.1}$$

The one-forms θ^0 and θ^3 are real while θ^1 and θ^2 are complex conjugate. A basis for the space of complex self-dual two-forms is provided by

$$Z^1 = \theta^2 \wedge \theta^3 , \qquad Z^2 = \theta^0 \wedge \theta^1 , \qquad Z^3 = \frac{1}{2}(\theta^0 \wedge \theta^3 - \theta^1 \wedge \theta^2) \tag{A.2}$$

The components of the metric in this space are

$$\gamma^{\alpha\beta} = (Z^\alpha, Z^\beta) = 2\delta^\alpha_{(1}\delta^\beta{}_{2)} - \frac{1}{2}\,\delta^\alpha_3\delta^\beta_3 \tag{A.3}$$

The complex connection one-forms $\sigma^\alpha{}_\beta$ are defined by the equation

$$dZ^\alpha + \sigma^\alpha{}_\beta \wedge Z^\beta = 0 \tag{A.4}$$

which is equivalent to the first Cartan structure equation

$$D\theta^a \equiv d\theta^a + \omega^a b \wedge \theta^b = 0 \tag{A.5}$$

that expresses the absence of torsion of the pseudo-Riemannian connection. The vectorial connection one-form is defined by

$$\sigma^\alpha = \frac{1}{2}\,\varepsilon^{\alpha\beta\gamma}\gamma_{\beta\delta}\sigma^\delta\gamma , \tag{A.6}$$

where $\varepsilon^{\alpha\beta\gamma}$ is the three-dimensional permutation symbol. The tetrad components $\sigma^\alpha{}_a$ (rotation coefficients) of σ^α are defined by

$$\sigma^\alpha = \sigma^\alpha{}_i\theta^i . \tag{A.7}$$

The complex curvature two-forms $\Sigma^\alpha{}_\beta$ are defined by

$$\Sigma^\alpha{}_\beta = d\sigma^\alpha{}_\beta + \sigma^\alpha{}_\gamma \wedge \sigma^\gamma{}_\beta \tag{A.8}$$

and the vectorial curvature two-form by

$$\Sigma_\alpha = \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma}\gamma^{\gamma\delta}\Sigma^\beta{}_\delta . \tag{A.9}$$

The vectorial curvature components are obtained by expanding Σ_α in the basis $\{ Z^\alpha, \overline{Z}^\alpha \}$: one obtains

$$\Sigma_\alpha = \left(C_{\alpha\beta} - \frac{1}{6}\,R\gamma_{\alpha\beta}\right)Z^\beta + E_{\alpha\bar\beta}\overline{Z}^\beta , \tag{A.10}$$

where $C_{\alpha\beta}$ is a trace-free symmetric tensor corresponding to the Weyl tensor and $E_{\alpha\bar\beta}$ is a Hermitian tensor corresponding to the trace-free Ricci tensor. The Weyl tensor and $C_{\alpha\beta}$ are related by

$$C_{\alpha\beta} = \overset{+}{C}_{abcd}Z_\alpha{}^{ab}Z_\beta{}^{cd} , \tag{A.11}$$

where the connecting quantities $Z_\alpha{}^{ab}$ are defined by

$$\theta^a \wedge \theta^b - i{*}(\theta^a \wedge \theta^b) = 2Z_\alpha{}^{ab}Z^\alpha . \tag{A.12}$$

We also note that the absolute covariant derivative of $C_{\alpha\beta}$ is given by

$$DC_{\alpha\beta} = dC_{\alpha\beta} - \sigma^{\gamma}{}_{\alpha}C_{\gamma\beta} - \sigma^{\gamma}{}_{\beta}C_{\alpha\gamma}. \tag{A.13}$$

Finally we remark that the subgroup of the connected component of the identity of the homogeneous Lorentz group which preserves the direction of θ^0 is given by

$$\left.\begin{aligned} \theta^{0'} &= e^{a}\theta^{0}, \\ \theta^{1'} &= e^{ib}(\theta^{1} + \bar{\gamma}\theta^{0}), \\ \theta^{3'} &= e^{-a}(\theta^{3} + \gamma\theta^{1} + \bar{\gamma}\theta^{2} + \bar{\gamma}\gamma\theta^{0}). \end{aligned}\right\} \tag{A.14}$$

REFERENCES

[1] L. ASGEIRSSON, Some hints on Huygen's principle and Hadamard's conjecture. *Comm. Pure Appl. Math.*, t. **9**, 1956, p. 307-326.

[2] R. BACH, Zur Weylshcen Relativitats theorie. *Math. Zeitscher*, t. **9**, 1921, p. 110-135.

[3] Y. BRUHAT, Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.*, t. **88**, 1952, p. 141-225.

[4] M. CAHEN, R. DEBEVER and L. DEFRISE, A Complex vectorial formalism in general relativity, *J. Math. Mech.*, t. **16**, 1967, p. 761-786.

[5] M. CAHEN and R. McLENAGHAN, Métriques des espaces lorentziens symétriques à quatre dimensions. *C. R. Acad. Sci. Paris*, t. **266**, 1968, p. 1125-1128.

[6] M. CHEVALIER, Sur le noyau de diffusion de l'opérateur laplacien. *C. R. Acad. Sci. Paris*, T. **264**, 1967, p. 380-382.

[7] E. COTTON, Sur les invariants différentials de quelques équations linéaires aux dérivées partielles du second ordre. *Ann. Sc. Ec. Norm. Superieure*, t. **17**, 1900, p. 211-244.

[8] R. COURANT and D. HILBERT, *Methods of mathematical physics*, vol. 2, Interscience, New York, 1962.

[9] R. DEBEVER, Le rayonnement gravitationnel, le tenseur de Riemann en relativité générale. *Cah. Phys.*, t. **168-169,** 1964, p. 303-349.

[10] A. DOUGLIS, The problem of Cauchy for linear hyperbolic equations of second order. *Comm. Pure Appl. Math.*, t. **7**, 1954, p. 271-295.

[11] A. DOUGLIS, A criterion for the validity of Huygens' principle. *Comm. Pure Appl. Math.*, t. **9**, 1956, p. 391-402.

[12] J. C. DU PLESSIS, Polynomial conformal tensors. *Proc. Cambridge Philos. Soc.*, t. **68**, 1970, p. 329-344.

[13] J. EHLERS and K. KUNDT, Exact solutions of the gravitational Field equations. Article in *Gravitation an introduction to current research;* edited by L. Witten, Wiley, New York, 1964.

[14] F. G. FRIEDLANDER, *The wave equation in a curved space-time.* Cambridge University Press, Cambridge, 1975.

[15] R. GOLDONI. A necessary condition for the validity of Huygen's principle on a curved space-time, *J. Math. Phys.*, **18**, 1977, p. 2125-2128.

[16] P. GÜNTHER, Zur Gültigkeit des Huygensschen Princips bei partiellen Differentialgleichungen von normalen hyperbolischen Typus. S.-B. Sächs. *Akad. Wiss. Leipzig Math.-Natur. Kl.*, t. **100**, 1952, p. 1-43.

[17] P. GÜNTHER, Über einige spezielle Probleme aus der Theorie der Linearen partiellen Differentialgleichungen zweiter Ordnung. S.-B. Sächs. *Akad. Wiss. Leipzig Math.-Natur. Kl.*, t. **102**, 1957, p. 1-50.

[18] P. GÜNTHER, Ein Beispiel einer nichttrivalen Huygensschen Differentialgleichungen mit vier unabhängigen Variablen. *Arch. Rational Mech. Anal.*, t. **18**, 1965, p. 103-106.
[19] P. GÜNTHER, Einige Sätze über huygenssche Differentialgleichungen. *Wiss. Zeitschr. Karl Marx Univ.. Math.-natu. Reihe Leipzig*, t. **14**, 1965, p. 497-507.
[20] P. GÜNTHER and V. WÜNSCH, Maxwellsche Gleichungen und Huygensshes. *Prinzip I. Math. Nach.*, *63*, 1974, p. 97-121.
[21] J. HADAMARD, *Lectures on Cauchy's problem in linear partial differential equations.* Yale University Press, New Haven, 1923.
[22] J. HADAMARD, The problem of diffusion of waves. *Ann. of Math.*, t. **43**, 1942, p. 510-522.
[23] G. HERGLOTZ, Über die Bestimmung eines Linienelementes in normal Koorinaten aus dem Riemannschen Krümmgstensor, *Math. Ann.*, t. **93**, 1925, p. 46-53.
[24] E. HÖLDER, Poissonsche Wellenformel in nicht euclidischen Raumen. *Ber. Verh. Sachs. Akad. Wiss. Leipzig*, t. **99**, 1938, p. 53-66.
[25] N. H. IBRAGIMOV and E. V. MAMONTOV, Sur le problème de J. Hadamard relatif à la diffusion des ondes. *C. R. Acad. Sc. Paris*, t. **270**, 1970, p. 456-458.
[26] N. H. IBRAGIMOV, Huygens' principle. *Amer. Math. Soc. Transl.* (2), t. **104**, 1976, p. 141-151.
[27] N. H. IBRAGIMOV and E. V. MAMONTOV, On the Cauchy problem for the equation $U_{tt} - U_{xx} - \sum_{i,j=1}^{n-1} a_{ij}(x - t)U_{y_i y_j} = 0$. *Math. Sbornik Tom*, t. **102** (144), 1977, N° 3, p. 347-363.
[28] H. P. KÜNZLE, Maxwell Fields satisfying Huygens' principle. *Proc. Cambridge Philos. Soc.*, t. **64**, 1968, p. 779-785.
[29] A. LICHNEROWICZ, Propagateurs et commutateurs en relativité générale. *Publ. Math. I. H. E. S.*, t. **10**, 1961, p. 293-344.
[30] M. MATHISSON, Eine Lösungsmethode fur Differentialgleichungen vom normalen hyperbolischen Typus. *Math. Ann.*, t. **107**, 1932, p. 400-419.
[31] M. MATHISSON, Le problème de M. Hadamard relatif à la diffusion des ondes. *Acta Math.*, t. **71**, 1939, p. 249-282.
[32] R. G. MCLENAGHAN, An explicit determination of the empty space-times on which the wave equation satisfies Huygens' principle. *Proc. Cambridge Philos. Soc.*, t. **65**, 1969, p. 139-155.
[33] R. G. MCLENAGHAN, On the validity of Huygens' principle for second order partial differential equations with four independent variables. Part I: Derivation of necessary conditions. *Ann. Inst. Henri Poincaré*, t. A **20**, 1974, p. 153-188.
[34] R. G. MCLENAGHAN and J. LEROY, Complex recurrent space-times. *Proc. Roy. Soc. London*, t. A **327**, 1972, p. 229-249.
[35] R. PENROSE, A spinor approach to general relativity, *Ann. Physics*, t. **10**, 1960, p. 171-201.
[36] A. Z. PETROV, *Einstein-Raume*, Akademie Verlag, Berlin, 1964.
[37] F. A. E. PIRANI, Introduction to gravitational radiation theory. Article in *Lectures on General Relativity*, edited by S. Deser and W. Ford, Brandeis Summer Institute in Theoretical Physics, t. **1**, 1964. Prentice-Hall. New York.
[38] H. S. RUSE, A. G. WALKER and T. J. WILLMORE, *Harmonic Spaces*. Edizioni Cremonese, Rome, 1961.
[39] R. SCHIMMING, Zur Gültigkeit des huygensschen Prinzips bei einer speziellen Metrik, *Z. A. M. M.*, t. **51**, 1971, p. 201-208.
[40] J. A. SCHOUTEN, *Ricci-Calculus*, Springer-Verlag, Berlin, 1954.
[41] S. L. SOBOLEV, Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales. *Mat. Sb. (N. S.)*, t. **1**, 1936, p. 39-70.

[42] K. L. STELLMACHER, Ein Beispiel einer Huygensschen Differentialgleichung. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.*, II, t. **10**, 1953, p. 133-138.

[43] K. L. STELLMACHER, Eine Klasse huygenscher Differentialgleichungen und ihre Integration. *Math. Ann.*, t. **130**, 1955, p. 219-233.

[44] P. SZEKERES, Spaces conformal to a class of spaces in general relativity. *Proc. Roy. Soc. London*, t. A **274**, 1963, p. 206-212.

[45] P. SZEKERES, Conformal Tensors. *Proc. Roy. Soc. London*, t. A **304**, 1968, p. 113-122.

[46] G. VANDERCAPPELLEN, *Contributions à l'étude du principle d'Huygens' en espace temps courbe*. Mémoire de Licence, Université de l'État à Mons, 1980.

[47] V. WÜNSCH, Über selbstadjungierte Huygenssche Differentialgleichungen mit vier unabhängigen Variablen. *Math. Nachr.*, t. **47**, 1970, p. 131-154.

[48] V. WÜNSCH, Maxwellsche Gleichungen und Huygensches II. *Math. Nach.*, t. **73**, 1976, p. 19-36.

[49] V. WÜNSCH, Über eine Klasse Konforminvarianter Tensoren. *Math. Nach.*, t. **73**, 1976, p. 37-58.

[50] V. WÜNSCH, Cauchy-Problem und Huygenssches Prinzip bei einigen Klassen spinorieller Feldgleichungen I. *Beitr. zur Analysis*, t. **12**, 1978, p. 47-76.

[51] V. WÜNSCH, Cauchy-Problem und Huygenssches Prinzip bei einigen Klassen spinorieller Feldgleichunger II. *Beitr. zur Analysis*, t. **13**, 1979, p. 147-177.

(Manuscrit reçu le 30 avril 1981)