ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 37, nº 2 (1982), p. 201-209

<http://www.numdam.org/item?id=AIHPA_1982__37_2_201_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXXVII, n° 2, 1982, p. 201-209. *Physique théorique.*

Curvature identifies on Hermite manifolds

by

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SUMMARY. — In this paper, identities in the Riemann-Christoffel curvature tensor and Ricci tensor on Hermite manifolds are obtained. Such identities were already obtained on Kähler and nearly Kähler manifolds and had found useful applications.

RÉSUMÉ. — Dans ce travail on démontre certaines identités satisfaites par le tenseur de courbure de Riemann-Christoffel et le tenseur de Ricci sur les variétés hermitiennes.

Des identités analogues avaient déjà été obtenues sur les variétés kählérienne et aussi sur les variétés prékählériennes. Ces identités ont eu des applications utiles.

1. INTRODUCTION

Let V_{2n} be an almost complex manifold, that is a manifold of differentiability class C^{r+1} having a tensor field F of the type (1,1) and class C^r and satisfies

$$F^2 + I_{2n} = 0, \tag{1.1}$$

where I_{2n} is the Kronecker tensor of order $2n$.

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It is known [1] [2] that the manifold V_{2n} can be endowed with a (Hermitian) metric g such that:

$$g(\overline{X}, \overline{Y}) = g(X, Y), \qquad \overline{X} \overset{\text{def}}{=} FX \tag{1.2}$$

for arbitrary vectors X and Y. Then V_{2n} is called *Almost Hermite Manifold.* On such a manifold, we have:

$$g(\overline{X}, Y) + g(X, \overline{Y}) = 0 \tag{1.3}$$

Suppose D is a Riemannian connexion on V_{2n}, then

$$D_X Y - D_Y X = [X, Y], \tag{1.4}$$

$$D_X g = 0, \tag{1.5}$$

where [X, Y] is the Lie-bracket of X and Y. If in addition we suppose that D satisfies on an almost Hermite manifold V_{2n}:

$$(D_X F)Y = 0, \tag{1.6}$$

$$(D_X F)X = 0, \tag{1.7}$$

$$(D_X F)Y = (D_{\overline{X}} F)\overline{Y}, \tag{1.8}$$

then the manifold is said to be *Kähler*, *Nearly Kähler* or *Hermite* respectively.

The relation (1.8) is equivalent to

$$(D_{\overline{X}} F)Y + (D_X F)\overline{Y} = 0; \tag{1.9}$$

and it can be shown [1], that the vanishing of the Nijenhuis tensor N(X, Y) defined by:

$$N(X, Y) \overset{\text{def}}{=} [\overline{X}, \overline{Y}] + \overline{\overline{[X, Y]}} - \overline{[X, \overline{Y}]} - \overline{[\overline{X}, Y]} \tag{1.10}$$

is a necessary and sufficient condition for V_{2n} to be Hermite. This is also the condition for the almost complex structure defined by the tensor F to be integrable; or equivalently, to be induced by a complex structure [1] [2] [3].

The tensor

$$K_{XY} \overset{\text{def}}{=} [D_X, D_Y] - D_{[X,Y]} \tag{1.11}$$

is the *curvature operator*, and K given by

$$K(X, Y, Z) \overset{\text{def}}{=} K_{XY} Z \tag{1.12}$$

is the *curvature tensor* of V_{2n} with respect to the connexion D; and as D is Riemannian, K is the *Riemann-Christoffel curvature tensor of the second kind.*

The Riemann-Christoffel tensor of the first kind is defined by:

$$'K(X, Y, Z, T) \overset{\text{def}}{=} g(K(X, Y, Z), T) \tag{1.13.1}$$

and it satisfies the following properties [1]:

(1.13.2) $'K(X,Y,Z,T) = -'K(Y,X,Z,T) = -'K(X,Y,T,Z) = 'K(Z,T,X,Y)$,

and the Bianchi's identities:

(1.13.3) $'K(X, Y, Z, T) + 'K(Y, Z, X, T) + 'K(Z, X, Y, T) = 0$.

The fundamental 2-form of V_{2n} is denoted by $'F$ and defined by:

(1.14) $$'F(X, Y) \overset{\text{def}}{=} g(FX, Y) = g(\overline{X}, Y).$$

This tensor is skew-symmetric and hybrid.

It is well known [1] [4] that on a *Kähler* manifold

(1.15) $$'K(X, Y, \overline{Z}, \overline{T}) - 'K(X, Y, Z, T) = 0,$$

and on a *nearly Kähler* manifold

(1.16) $$'K(X, Y, \overline{Z}, \overline{T}) - 'K(X, Y, Z, T) = g((D_XF)Y, (D_ZF)T).$$

Here, on Hermite manifolds, the corresponding curvature identity to (1.16) is found.

In a subsequent paper, equivalent curvature identities, together with some relations in the Ricci tensor, will be used in the study of the holomorphic sectional curvature and other properties of the Hermite manifolds.

2. SOME USEFUL LEMMAS

LEMMA (2.1). — *On almost Hermite manifolds, we have* [5] [6]:

(2.1.1) $$(D_XF)\overline{Y} + \overline{(D_XF)Y} = 0,$$

(2.1.2) $$g((D_XF)\overline{Y}, (D_ZF)\overline{T}) = g((D_XF)Y, (D_ZF)T),$$

(2.1.3) $$(D_X{'F})(Y, Z) = g((D_XF)Y, Z),$$

(2.1.4) $$(D_X{'F})(\overline{Y}, Z) = (D'_XF)(Y, Z).$$

LEMMA (2.2). — *On almost Hermite manifolds, the curvature operator* K_{XY} *satisfies the following relations:*

(2.2.1) $$'K(X, Y, \overline{Z}, T) + 'K(X, Y, Z, \overline{T}) = g((K_{XY}F)Z, T)$$

(2.2.2) $$= (K_{XY}{'F})(Z, T)$$

(2.2.3) $$= A(X, Y, Z, T) - A(Y, X, Z, T),$$

where

(2.2.4) $$A(X, Y, Z, T) \overset{\text{def}}{=} (D_XD_Y{'F} - D_{D_XY}{'F})(Z, T).$$

Proof. — In consequence of (1.13) and using the relation

$$K_{XY}(FZ) = (K_{XY}F)Z + FK_{XY}Z,$$

which can be easily verified, we have respectively:

$$\begin{aligned}'K(X, Y, \overline{Z}, T) + 'K(X, Y, Z, \overline{T}) &= g(K_{XY}\overline{Z}, T) + g(K_{XY}Z, \overline{T}) \\ &= g((K_{XY}F)Z + \overline{K_{XY}Z}, T) + g(K_{XY}Z, \overline{T}) \\ &= g((K_{XY}F)Z, T) + g(\overline{K_{XY}Z}, T) + g(K_{XY}Z, \overline{T}) \\ &= g((K_{XY}Z, T),\end{aligned}$$

by virtue of (1.3).

Now, a covariant differentiation of (2.1.3), written in the form

$$(D_Y{}'F)(Z, T) = g((D_YF)Z, T)$$

gives

$$(D_XD_Y{}'F)(Z, T) = g((D_XD_YF)Z, T);$$

and by a straightforward calculation based on the last two equations, we get:

$$\begin{aligned}&(D_XD_Y{}'F - D_YD_X{}'F - D_{[X,Y]}{}'F)(Z, T) \\ &\qquad = g((D_XD_YF - D_YD_XF - D_{[X,Y]}F)Z, T),\end{aligned}$$

which can be put in the following form with the help of (1.11):

$$(K_{XY}{}'F)(Z, T) = g((K_{XY}F)Z, T).$$

Finally, using (1.4) and (2.2.4), we have:

$$\begin{aligned}(K_{XY}{}'F)(Z, T) &= (D_XD_Y{}'F - D_{D_XY}{}'F)(Z, T) - (D_YD_X{}'F - D_{D_YX}{}'F)(Z, T) \\ &= A(X, Y, Z, T) - A(Y, X, Z, T).\end{aligned}$$

Note (2.1). — In view of the skew-symmetry of F and (2.2.4), we have:

$$A(X, Y, Z, T) = - A(X, Y, T, Z). \tag{2.2.5}$$

Note (2.2). — The relation (2.2.3) can evidently be put in the following equivalent form:

$$'K(X, Y, \overline{Z}, \overline{T}) - 'K(X, Y, Z, T) = A(X, Y, Z, \overline{T}) - A(Y, X, Z, \overline{T}). \tag{2.2.6}$$

LEMMA (2.3). — *An almost Hermite manifold is Hermite, iff one of the following (equivalent) conditions is satisfied on that manifold.*

$$(D_{\overline{X}}F)Y = \overline{(D_XF)Y}, \tag{2.3.1}$$

$$'F((D_XF)Y, Z) = g((D_{\overline{X}}F)Y, Z), \tag{2.3.2}$$

$$(D_X{}'F)(\overline{Y}, Z) + (D_{\overline{X}}{}'F)(Y, Z) = 0, \tag{2.3.3}$$

$$(D_X{}'F)(Y, \overline{Z}) + (D_{\overline{X}}{}'F)(Y, Z) = 0. \tag{2.3.4}$$

Proof. — The statement related to (2.3.1) is an immediate consequence of (2.1.1) and (1.9).

With $'F((D_XF)Y, Z) \overset{\text{def}}{=} g(\overline{(D_XF)Y}, Z)$, the statement related to (2.3.2) follows directly from (2.3.1). This was also proved in another manner by Mishra and Ram Hit [*6*].

Barring Y and X in (2.1.3), we get respectively:

$$(D_X{}'F)(\overline{Y}, Z) = g((D_XF)\overline{Y}, Z),$$
$$(D_{\overline{X}}{}'F)(Y, Z) = g((D_{\overline{X}}F)Y, Z).$$

Adding the last two equations side by side, we obtain:

$$\begin{aligned}(D_X{}'F)(\overline{Y}, Z) + (D_{\overline{X}}{}'F)(Y, Z) &= g((D_XF)\overline{Y}, Z) + g((D_{\overline{X}}F)Y, Z)\\ &= g((D_XF)\overline{Y} + (D_{\overline{X}}F)Y, Z).\end{aligned}$$

From the latter, which is valid on almost Hermite manifolds, the equivalence between (1.9) and (2.3.3) can be easily concluded.

The relation (2.3.4) is equivalent to (2.3.3) by virtue of (2.1.4).

LEMMA (2.4). — *On Hermite manifolds, the following relation holds:*

$$\begin{aligned}(2.4.1)\quad & A(X, Y, Z, \overline{T}) + A(X, \overline{Y}, Z, T) - A(Y, X, Z, \overline{T}) - A(Y, \overline{X}, Z, T)\\ &= g((D_XF)Z, (D_YF)T) - g((D_YF)Z, (D_XF)T) - (D_{(D_XF)Y-(D_YF)X}{}'F)(Z, T)\end{aligned}$$

Proof. — On Hermite manifolds, the identity (2.3.4)

$$(D_Y{}'F)(Z, \overline{T}) = -(D_{\overline{Y}}{}'F)(Z, T)$$

yields

$$(D_XD_Y{}'F)(Z, \overline{T}) = -(D_Y{}'F)(Z, (D_XF)T) - (D_XD_{\overline{Y}}{}'F)(Z, T),$$

or, using (2.1.3),

$$(2.4.2)\quad (D_XD_Y{}'F)(Z, \overline{T}) = -g((D_YF)Z, (D_XF)T) - (D_XD_{\overline{Y}}{}'F)(Z, T).$$

Similarly, we have:

$$(2.4.3)\quad -(D_YD_X{}'F)(Z, \overline{T}) = g((D_XF)Z, (D_YF)T) + (D_YD_{\overline{X}}{}'F)(Z, T).$$

Also:

$$\begin{aligned}(2.4.4)\quad -(D_{[X,Y]}{}'F)(Z, \overline{T}) &= (D_{\overline{[X,Y]}}{}'F)(Z, T)\\ &= (D_{D_X\overline{Y}}{}'F - D_{D_Y\overline{X}}{}'F - D_{(D_XF)Y-(D_YF)X}{}'F)(Z, T).\end{aligned}$$

Adding (2.4.2), (2.4.3) and (2.4.4) side by side, and taking into account (2.2.2), (2.2.6) and (2.2.4), we obtain (2.4.1).

3. CURVATURE IDENTITIES

Following Mishra [1] and taking into account (2.2.4) and (2.4.1), we conclude that A(X, Y, Z, T) is a linear combination of terms of the type:

$$'F((D_PF)Q, (D_RF)S), \qquad g((D_PF)Q, (D_RF)S),$$
$$(D_{(D_PF)Q}'F)(R, S), \qquad (D_{(D_PF)Q}'F)(R, \overline{S});$$

where P, Q, R, S are some combination of X, Y, Z, T.

Also, A(X, Y, Z, T) must be skew-symmetric in the last two slots in view of (2.2.5).

Writing A(X, Y, Z, T) as indicated above, substituting this value in (2.4.1) and comparing the coefficients, A(X, Y, Z, T) reduces to the following form, after a somewhat lengthy but straightforward calculation:

$$\begin{aligned} A(X, Y, Z, T) = \frac{1}{2}\{ &'F((D_XF)Z, (D_YF)T) - 'F((D_XF)T, (D_YF)Z) \} \\ &+ \alpha\{ 'F((D_TF)X, (D_YF)Z) - 'F((D_ZF)X, (D_YF)T) \} \\ &+ \beta\{ 'F((D_XF)Z, (D_TF)Y) - 'F((D_XF)T, (D_ZF)Y) \} \\ &+ \frac{1}{2}(D_{(D_XF)Y}'F)(Z, \overline{T}). \end{aligned}$$

where α, β are parameters.

Substituting this value in (2.2.6) and putting $\lambda = \alpha + \beta$ we obtain :

$$\begin{aligned} (3.1.1)\quad &'K(X, Y, \overline{Z}, \overline{T}) - 'K(X, Y, Z, T) \\ &= \lambda\{ 'F((D_{\overline{T}}F)X, (D_YF)Z) - 'F((D_ZF)X, (D_YF)\overline{T}) \\ &+ 'F((D_XF)Z, (D_{\overline{T}}F)Y) - 'F((D_XF)\overline{T}, (D_ZF)Y) \} \\ &+ \frac{1}{2}(D_{(D_XF)Y-(D_YF)X}'F)(Z, T). \end{aligned}$$

Or, by virtue of (2.3.2) and (1.9):

$$\begin{aligned} (3.1.2)\quad &'K(X, Y, \overline{Z}, \overline{T}) - 'K(X, Y, Z, T) \\ &= \lambda\{ g((D_ZF)X, (D_YF)T) - g((D_TF)X, (D_YF)Z) \\ &+ g((D_TF)Y, (D_XF)Z) - g((D_ZF)Y, (D_XF)T) \} \\ &- \frac{1}{2}(D_{(D_XF)Y-(D_YF)X}'F)(Z, T). \end{aligned}$$

where λ is a parameter.

Barring T in (3.1.1), we get:

$$\begin{aligned} (3.1.3)\quad &'K(X, Y, \overline{Z}, T) + 'K(X, Y, Z, \overline{T}) \\ &= \lambda\{ 'F((D_TF)X, (D_YF)Z) - 'F((D_ZF)X, (D_YF)T) \\ &+ 'F((D_XF)Z, (D_TF)Y) - 'F((D_XF)T, (D_ZF)Y) \} \\ &+ \frac{1}{2}(D_{(D_XF)Y-(D_YF)X}'F)(Z, \overline{T}). \end{aligned}$$

Interchanging X, Y and Z, T in (3.1.2) and using (1.13.2), we will have at once:

$$(3.1.4)\quad 'K(\overline{X}, \overline{Y}, Z, T) - 'K(X, Y, Z, T) = \lambda \{ g((D_Z F)X, (D_Y F)T) - g((D_T F)X, (D_Y F)Z) + g((D_T F)Y, (D_X F)Z) - g((D_Z F)Y, (D_X F)T) \} - \frac{1}{2}(D_{(D_Z F)T - (D_T F)Z}'F)(X, Y).$$

Also, barring Y, T in (3.1.2) and using (1.8), (1.9), (2.1.2) as well as (2.1.1), (2.3.1), (2.3.4), we get:

$$(3.1.5)\quad 'K(X, \overline{Y}, \overline{Z}, T) + 'K(X, \overline{Y}, Z, \overline{T}) = \lambda \{ g((D_T F)X, (D_Y F)Z) - g((D_Z F)X, (D_Y F)T) + g((D_Z F)Y, (D_X F)T) - g((D_T F)Y, (D_X F)Z) \} - \frac{1}{2}(D_{(D_X F)Y + (D_Y F)X}'F)(Z, T).$$

Similarly, barring X, T in (3.1.2) yields:

$$(3.1.6)\quad 'K(\overline{X}, Y, \overline{Z}, T) + 'K(\overline{X}, Y, Z, \overline{T}) = \lambda \{ g((D_T F)X, (D_Y F)Z) - g((D_Z F)X, (D_Y F)T) + g((D_Z F)Y, (D_X F)T) - g((D_T F)Y, (D_X F)Z) \} + \frac{1}{2}(D_{(D_X F)Y + (D_Y F)X}'F)(Z, T).$$

Subtracting (3.1.2) from (3.1.4) side by side, we get:

$$(3.2.1)\quad 'K(\overline{X}, \overline{Y}, Z, T) - 'K(X, Y, \overline{Z}, \overline{T}) = \frac{1}{2}\{ (D_{(D_X F)Y - (D_Y F)X}'F)(Z, T) - (D_{(D_Z F)T - (D_T F)Z}'F)(X, Y) \}.$$

Barring Z, T in the last identity and using (2.1.4), (1.8), we obtain:

$$(3.2.2)\quad 'K(\overline{X}, \overline{Y}, \overline{Z}, \overline{T}) - 'K(X, Y, Z, T) = -\frac{1}{2}\{ (D_{(D_X F)Y - (D_Y F)X}'F)(Z, T) + (D_{(D_Z F)T - (D_T F)Z}'F)(X, Y) \}.$$

Similarly, barring X, Z in (3.2.2), we get:

$$(3.2.3)\quad 'K(X, \overline{Y}, Z, \overline{T}) - 'K(\overline{X}, Y, \overline{Z}, T) = -\frac{1}{2}\{ (D_{(D_X F)Y + (D_Y F)X}'F)(Z, T) - (D_{(D_Z F)T + (D_T F)Z}'F)(X, Y) \}.$$

The identity (3.2.1) can be written as:

$$(3.3.1)\quad K(\overline{X}, \overline{Y}, Z) + \overline{K(X, Y, \overline{Z})} = \frac{1}{2}\{(D_{(D_X F)Y-(D_Y F)X}{}'F)Z - (D_{(D_Z{}^{-1}GF)-({}^{-1}G\nabla F)Z}{}'F)(X, Y)\},$$

where ${}^{-1}G$ is the inverse map of the linear map G induced by g, i. e. $G(X)Y \overset{\text{def}}{=} g(X, Y)$, and where ∇ is the symbol of covariant differentiation.

Barring X, Y in (3.3.1) yields:

$$-\overline{K(\overline{X}, \overline{Y}, \overline{Z})} - K(X, Y, Z) = -\frac{1}{2}\{(D_{(D_X F)Y-(D_Y F)X}F)Z - (D_{(D_Z{}^{-1}GF)-({}^{-1}G\nabla F)Z}{}'F)(X, Y)\}.$$

Contracting this equation with respect to X, we immediately obtain:

$$(3.3.2)\quad \operatorname{Ric}(\overline{Y}, \overline{Z}) - \operatorname{Ric}(Y, Z) = -\frac{1}{2}\{e^i((D_{(D_{e_i}F)-(D_Y F)e_i}F)Z - e^i((D_{(D_Z{}^{-1}GF)-({}^{-1}G\nabla F)Z}{}'F)(e_i, Y)\}$$

where Ric is the *Ricci tensor* defined by

$$(3.3.3)\quad \operatorname{Ric}(Y, Z) \overset{\text{def}}{=} (CK)(Y, Z).$$

and $\{e_i\}$ is a basis of the tangent manifold and $\{e^i\}$ is the dual base at the point considered on the manifold.

Thus we see that while

$$'K(\overline{X}, \overline{Y}, Z, T) - {}'K(X, Y, Z, T)$$

is expressed in terms of a parameter λ, the following quantities:

$$\begin{gathered} 'K(\overline{X}, \overline{Y}, Z, T) - {}'K(X, Y, \overline{Z}, \overline{T}), \\ 'K(X, \overline{Y}, Z, \overline{T}) - {}'K(\overline{X}, Y, \overline{Z}, T), \\ 'K(\overline{X}, \overline{Y}, \overline{Z}, \overline{T}) - {}'K(X, Y, Z, T), \\ \operatorname{Ric}(\overline{Y}, \overline{Z}) - \operatorname{Ric}(Y, Z), \end{gathered}$$

are parameter independent. These quantities vanish *iff* the Hermite manifold reduces to Kahler manifold.

In the next paper we will consider some of the geometrical applications of the formulae obtained in this paper.

The author is grateful to Prof. R. S. Mishra for his guidance.

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(Manuscrit reçu le 22 décembre 1981)