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Section A :
Physique théorique.

Poincaré gauge field theory and gravitation

by

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ABSTRACT. — A gauge theory of gravitation based on a Riemann-Cartan space-time U_4 is presented. In addition to Einstein's equations describing the classical long range gravitational field a further set of differential source equations is introduced coupling a matter current to the torsion tensor. The relation of this current to a quantum mechanical wave function description of matter based on the Poincaré group as a gauge group is formulated using an old idea of Lurçat.

RÉSUMÉ. — Une théorie de jauge pour la gravitation basée sur un espace-temps de Riemann-Cartan U_4 est proposée. Outre les équations d'Einstein déterminant le champ classique de la gravitation, un autre système d'équations différentielles est introduit, liant un courant de matière au tenseur de torsion. La relation de ce courant à une représentation de la matière au moyen de fonctions d'ondes quantiques est formulée en traitant le groupe de Poincaré comme groupe de jauge et utilisant une idée de Lurçat.

I. INTRODUCTION

If gravitation can be given the form of a gauge theory it is a legitimate question to ask on what geometric structure gravity is in fact realized as a gauge interaction. It has repeatedly been suggested that the Poincaré group is the relevant group for a gauge description of the gravitational interaction [*1*]-[*5*]. However, previous proposals for a gauge formulation of gravity used a Lagrangian formulation extending in the well-known way the Lorentz and translational invariance of a Lagrangean referring to a flat-space theory to space-time dependent transformations interpreting the resulting theory as a description in a curved Riemannian or a Riemann-Cartan space-time. It is not apparent from this procedure what the differential geometric structure of the so obtained theory actually is. Only in ref. [*5*] certain notions of modern differential geometry have been used to partially clarify in what context one could give to the translations the meaning of gauge transformations. However, also in this paper only infinitesimal gauge translations were considered as is in fact the case for all the Lagrangean models discussed in the literature.

The geometrical structure needed to introduce the Poincaré group as a gauge group is the affine frame bundle and the affine tangent bundle over space-time [*6*] [*7*] [*8*]. The first is a principal fiber bundle over space-time possessing the Poincaré group, ISO(3, 1), as structural group, the second is an associated bundle possessing as fiber the Minkowski space – considered as an affine space – on which the Poincaré group acts as a group of motion. If the translational gauge degrees of freedom are indeed essential in a gauge description of gravitation it is apparent that these fiber bundles – or related ones which we shall briefly mention below and discuss in detail in Sect. III – must be of importance in any gauge theory based on the Poincaré group. It appears to us even more advantageous to start from a particular fiber bundle and formulate the theory in a geometrical manner than to begin with a certain Lagrangean and manipulate it to yield a gauge theory the content of which is not immediately transparent from the point of view of geometry.

There is one further essential point to be mentioned in this context: Matter in Einstein's theory of gravitation enters the field equations through the classical energy-momentum tensor being the source for the long range classical gravitational field. If one ever hopes to combine this classical metric description of gravity with a quantum mechanical description of matter by representing matter at small distances in the form of a generalized quantum mechanical wave function it is suggestive – if not compelling – to use these fiber bundles with structural group ISO(3, 1) as the geometric stratum on which such generalized matter wave functions are to be defined. We shall call these bundles, which play a prominent role in any Poincaré gauge field theory, the soldered H-bundles over space-time. They are characterized by two facts *i*) that their fibers are homogeneous spaces of the Poincaré group, i. e. $F \cong H = G/G'$, with $G = ISO(3, 1)$ and G' being a subgroup of the Poincaré group leaving a particular point of H fixed (stability subgroup of this point), and *ii*) that the homogeneous space H contains the Minkowski space M_4 through which the fiber (H) and the base (curved space-time) are soldered to each other [*9*] [*10*], i. e. have a first order contact at each point of the base manifold.

In using these H-bundles over space-time on each of which the Poincaré group acts as a transitive gauge group (group of motion in H) we invoke an old idea of Lurçat [*11*]. This author criticized in a paper of 1964 the undynamical way in which the spin degrees of freedom are treated in conventional Minkowski field theory. The canonical formalism of quantum field theory allows one to go « off mass shell » but not to go « off spin shell »: By the choice of the representation of the Lorentz group i. e. by the number of components of any field the spin content of these fields is fixed once and for all. Lurçat pointed out that mass (m) *and* spin (s) should be treated in a similar fashion in a dynamical theory which is capable of correlating states with different values of $[m, s]$ characterizing an elementary particle as a member of a whole spectrum of states. He suggested the use of the ten-parameter group space i. e. the ten-dimensional homogeneous space of the Poincaré group as reference space on which wave-functions are to be defined in a dynamical theory for mass and spin. In subsequent years other authors classified all the homogeneous spaces of the Poincaré group (1) [*13*] [*14*] and suggested particular ones with lower dimension than ten as reference spaces on which scalar fields describing dynamical objects in space-time possessing also internal motion are to be defined. The lowest dimensional space on which ISO(3, 1) acts transitively is clearly M_4, the Minkowski space-time, corresponding to the stability subgroup $G' = SO(3, 1)$. Carrying now Lurçat's idea over to a curved space-time manifold, i. e. embedding such a dynamical formulation of particle theory into a space perturbed by classical long range gravitational fields one would clearly have to introduce the affine tangent bundle (with four-dimensional fiber) or the other soldered H-bundles over space-time (with five-dimensional up to ten-dimensional fibers) which we shall discuss in Sect. III.

In this paper, however, we not only want to extend Lurçat's idea by giving it a gauge theoretical interpretation and connecting it with gravity. We also ask the question whether in this way one could establish a framework in which an extension of Einstein's theory of gravitation to small distances in the presence of matter described in terms of a generalized quantum mechanical wave function is indeed possible and would yield physically interesting results. Our first remark in this connection is that one has to go beyond a metric theory of gravitation and allow for the presence of torsion in the underlying geometry. Thus in the discussion presented in the following sections the base space of our bundle geometry will be a Riemann-Cartan space-time U_4 the geometry of which we develop in Sect. II using the language of alternating forms (2). We thus aim at

(1) Compare also the work of Finkelstein [*12*] in this context.

(2) Compare in this context refs. [*15*] and [*16*].

a joint description of phenomena based on geometry of which one aspect is the Einstein metric theory of gravitation coupled to a classical source distribution of energy and momentum, represented by the tensor $\overline{T}_{\mu\nu}$, yielding the familiar long range classical gravitational interaction; and of which the other aspect is a nonmetric theory of another interaction (possibly interpreted so far as of nongravitational origin) coupling a bilinear source current $J_{\varkappa\mu\nu}(\phi)$ associated with a generalized matter field ϕ to torsion. This second aspect describes in a geometrical manner the influence of matter on the geometry at small distances mediated through a current distribution connected with a generalized matter wave function ϕ. The formulation of this relation between a current and torsion – which, by the way, is of different type as in the Einstein-Cartan theory described by Trautman [*17*] and Hehl *et al.* [*18*] – is provided by an additional equation of Yang-Mills type correlating the covariant derivative of the contracted dual curvature tensor in the U_4 space to a dual current. We shall first investigate in subsection (*a*) of Sect. IV what freedom one actually has in a geometric theory based on a Riemann-Cartan space-time in demanding an additional source equation to be satisfied, i. e. by introducing a further feed-back mechanism between matter and geometry beyond the one expressed by Einstein's field equations in general relativity. Of course Einstein's theory for the metrical field generated by the energy-momentum distribution of distant matter described in classical terms should be one part of the picture, as mentioned above, providing the background metric (the long range classical field). In addition the proposed relation between a current and torsion when spealized to a totally antisymmetric torsion tensor as well as to a totally antisymmetric current tensor is expressed by a set of further nonlinear differential equations coupling in essence an axial vector source current (the dual current) to an axial vector (the dual of the torsion tensor) the latter characterizing the nonmetrical part of the geometry (the short distance field).

The resulting equations are similar to Yang-Mills-type equations and interesting as small distance modifications of the classical Einsteinian metrical theory of gravitation. They are obtained here by a procedure aimed at giving gravity the form of a gauge theory realized on a fiber bundle with structural group ISO(3, 1) including torsion besides the Riemannian curvature as an essential quality determining the geometry in the small. The question of what type of interaction is actually described in this manner through the current-torsion-equations we shall leave open in this paper. Except for some general remarks in the discussion (Sect. V) this question we hope to address in a separate investigation. However, we like to point out in closing this introduction that the coupling strength with which torsion is generated by the presence of a matter distribution described by a quantum mechanical source current need not be the gravitational constant $\varkappa$ as, in fact, is the case in the Einstein-Cartan theory [17] [18]. In that theory a spin current is only algebraically related to the torsion tensor. Here, in contradistinction, we obtain a set of nonlinear differential equations of second order – interesting as such – governing the law by which torsion is generated in the geometry through a matter current. Moreover, for dimensional reasons the definition of the source current in terms of the ϕ-field brings into the picture also an elementary length parameter R_0. This parameter together with the choice of a particular soldered H-bundle as the arena for the internal Poincaré gauge degrees of freedom described by the field ϕ will have to be determined by comparing the formalism presented in this paper with observation in certain limiting cases. We have to defer this part of the problem – although essential for the physical interpretation of our approach – to a later communication and now proceed setting up the formalism.

II. THE GEOMETRY OF A SPACE-TIME WITH TORSION

Cartan's structural equations for a Riemann-Cartan space-time U_4 are given by [19] [6] [7]

$$\nabla\theta^i = d\theta^i + \omega_k{}^i \wedge \theta^k = \Omega^i \tag{2.1}$$

$$d\omega_{ij} + \omega_{ik} \wedge \omega_j{}^k = \Omega_{ij} \tag{2.2}$$

Eq. (2.1) defines the torsion two-form (vector valued) and eq. (2.2) the curvature two-form (tensor valued) in terms of the connection one-form $\omega_{ik} = -\omega_{ki}$. The symbol d applied to a form denotes the exterior derivative, and ∇ stands for the exterior covariant derivative with respect to the connection ω_{ik} of the form which follows this symbol. A similar notation will be adopted below for tensors with the exterior product (denoted by $\wedge$) being replaced by the ordinary product in the corresponding formulae (see, for example, eqs. (2.22)-(2.25) below). The summation convention is adopted throughout.

We shall use latin indices running over 0, 1, 2, 3 to denote quantities referring to a local Lorentz frame, $e_i = e_i(x)$; $i = 0, 1, 2, 3$ [Cartan's repère mobile], providing a basis in T_x, the local tangent space of U_4 at x. The dual basis in T_x^* is given by the fundamental one-forms θ^i; $i=0, 1, 2, 3$. Apart from the latin indexed quantities referred to the so-called non-holonomic (i. e. local Lorentzian) system of axes we shall frequently use greek indexed quantities referring to an oblique set of axes defined in terms of the coordinates, x^μ, associated with an atlas $\{U_i, \phi_i\}$ covering U_4 [6] [7]. The corresponding holonomic or natural basis in T_x and T_x^* is denoted by $e_\mu = \partial_\mu$ and dx^μ, respectively. The transition between greek and latin indexed quantities is provided by the vierbein fields, $\lambda^i_\mu(x)$, and their inverse, $\lambda^\mu_i(x)$, obeying

$$g_{\mu\nu}(x) = \lambda^i_\mu \lambda^k_\nu \eta_{ik}\,, \tag{2.3}$$

$$\lambda^i_\mu \lambda^\mu_k = \delta^i_k\,; \qquad \lambda^i_\mu \lambda^\nu_i = \delta^\nu_\mu\,, \tag{2.4}$$

with

$$\eta_{ik} = \mathrm{diag}\,(1, -1, -1, -1)\,. \tag{2.5}$$

Local Lorentz indizes will be raised and lowered by the Minkowski tensor η^{ik} and η_{ik}, respectively (i. e. $\omega_k{}^i = \omega_{kj}\eta^{ji}$ in eqs. (2.1) and (2.2)), while greek indices will be lowered and raised by the metric tensor $g_{\mu\nu}(x)$ and its inverse $g^{\mu\nu}(x)$.

The relation between the fundamental one-forms θ^i and the greek indexed coordinate differentials dx^μ; $\mu = 0, 1, 2, 3$, are given by

$$\theta^i = \lambda^i_\mu dx^\mu\,. \tag{2.6}$$

The corresponding relation for the two bases in T_x is

$$e_i = \lambda^\mu_i \partial_\mu\,. \tag{2.6'}$$

Expanding the vectorial two-forms Ω^i and the tensorial two-forms $\Omega_{ij} = -\Omega_{ji}$ in terms of a basis of two-forms in $T^*_x \wedge T^*_x$ one has

$$\Omega^i = \frac{1}{2}\,\theta^k \wedge \theta^l \quad S_{kl}{}^i, \tag{2.7}$$

$$\Omega_{ij} = \frac{1}{2}\,\theta^k \wedge \theta^l \quad R_{klij}\,, \tag{2.8}$$

where the $S_{kl}{}^i$ and R_{klij} are the torsion and curvature tensors, respectively, which characterize the geometry in the Riemann-Cartan space-time (3). They obey the symmetry relations

$$S_{kl}{}^i = -S_{lk}{}^i\,, \tag{2.9}$$

$$R_{klij} = -R_{lkij} = -R_{klji}\,. \tag{2.10}$$

(3) Performing a Lorentz gauge transformation $\Lambda(x)$, i. e. going over in a smooth fashion to another system of local Lorentz frames $\hat{e}_i = [\Lambda^{-1}(x)]^k{}_i e_k = [\Lambda(x)]_i{}^k e_k$ at the point x and a certain neighborhood $U_i \subset U_4$ (change of the local cross section over U_i on the Lorentz frame bundle over space-time) one has the following transformation rule for Ω^i and Ω_{ij}:

$$\hat{\Omega}^i = [\Lambda(x)]^i{}_k \Omega^k\,, \tag{α}$$

$$\hat{\Omega}_{ij} = [\Lambda^{-1}(x)]^k{}_i [\Lambda^{-1}(x)]^l{}_j \Omega_{kl}\,, \tag{β}$$

and similarly for any other latin indexed tensorial quantity possessing a transformation rule determined by the position and number of indices. For the connection one has the inhomogeneous transformating law:

$$\hat{\omega}_l{}^k = [\Lambda(x)]^k{}_i [\Lambda^{-1}(x)]^j{}_l \omega_j{}^i + [\Lambda(x)]^k{}_j d\,[\Lambda^{-1}(x)]^j{}_l \tag{γ}$$

The connection on U_4 — or, more exactly, the linear connection on the bundle of orthonormal frames over U_4 — is composed of two parts, a Riemannian (metrical) or V_4 part denoted by $\overline{\omega}_{ik}$, and the torsion part denoted by $\tau_{ik} = -\tau_{ki}$. (Here and in the subsequent discussions we shall denote quantities given by the metric alone i. e. referring to a Riemannian space-time V_4 by a bar)

$$\omega_{ik} = \overline{\omega}_{ik} + \tau_{ik} \tag{2.11}$$

with

$$\overline{\omega}_{ik} = \theta^l \overline{\Gamma}_{lik} = -\theta^l \Omega_{\{\{lik\}\}}, \tag{2.12}$$

and

$$\tau_{ik} = \theta^l \mathrm{K}_{lik} = \theta^l \frac{1}{2} \mathrm{S}_{\{\{lik\}\}}. \tag{2.13}$$

The antisymmetry of the ω_{ik} implies the antisymmetry of the Ricci rotation coefficients $\overline{\Gamma}_{lik}$ in their last two indices, i. e. $\overline{\Gamma}_{lik} = -\overline{\Gamma}_{lki}$. Similarly for the torsion part one has $\mathrm{K}_{lik} = -\mathrm{K}_{lki}$. In eqs. (2.12) and (2.13) $\Omega_{\{\{lik\}\}}$ is a short-hand notation for the combination

$$\Omega_{\{\{lik\}\}} = \Omega_{lik} + \Omega_{kli} - \Omega_{ikl} = -\Omega_{\{\{lki\}\}} \tag{2.14}$$

defining an antisymmetric quantity in the last two indices i, k in terms of objects antisymmetric in the first two indices l, i. The quantities $\Omega_{lik} = \Omega_{li}{}^s \eta_{sk}$ are defined in terms of derivatives of the vierbein fields by

$$\Omega_{li}{}^s = \frac{1}{2} \lambda_l^\mu \lambda_i^\nu (\partial_\mu \lambda_\nu^s - \partial_\nu \lambda_\mu^s). \tag{2.15}$$

Similarly one proceeds for $\mathrm{S}_{lik} = \mathrm{S}_{li}{}^s \eta_{sk}$ defined in eqs. (2.7) and (2.9). Expressing eqs. (2.11)-(2.13) in greek indixed quantities one finds

$$\Gamma_{\mu\nu\rho} = \overline{\Gamma}_{\mu\nu\rho} + \frac{1}{2} \mathrm{S}_{\{\{\mu\nu\rho\}\}}, \tag{2.16}$$

with the metrical part being given by

$$\overline{\Gamma}_{\mu\nu\rho} = \frac{1}{2} \partial_{\{\{\mu} g_{\rho\nu\}\}} = \left\{ {}_{\mu\nu}^{\lambda} \right\} g_{\lambda\rho}. \tag{2.17}$$

Let us rewrite eq. (2.16) in a slightly different form introducing the abbreviation analogous to (2.14) and raising the last index with $g^{\rho\sigma}$, i. e.

$$\mathrm{K}_{\mu\nu}{}^\rho = \frac{1}{2} (\mathrm{S}_{\mu\nu}{}^\rho - \mathrm{S}_\nu{}^\rho{}_\mu + \mathrm{S}^\rho{}_{\mu\nu}), \tag{2.18}$$

then

$$\Gamma_{\mu\nu}{}^\rho = \overline{\Gamma}_{\mu\nu}{}^\rho + \mathrm{K}_{\mu\nu}{}^\rho. \tag{2.19}$$

with $\overline{\Gamma}_{\mu\nu}{}^\rho = \left\{ {}_\mu{}^\rho{}_\nu \right\} = \overline{\Gamma}_{\nu\mu}{}^\rho$ representing the in μ and ν symmetric metric part of the connection (the Christoffel symbols in a V_4) and with $\mathrm{K}_{\mu\nu}{}^\rho$ denoting the torsion contribution to the connection which, in general, possesses a symmetric and an antisymmetric part in the indices μ, ν. Thus for $K_{(\mu\nu)}{}^{\rho} = \frac{1}{2}(K_{\mu\nu}{}^{\rho} + K_{\nu\mu}{}^{\rho}) \neq 0$ one would obtain a deviation in the form of the equation for a geodesic in a U_4 compared to that in a V_4. In our later discussion in Sect. IV we shall specialize to a completely antisymmetric torsion tensor ([4]). However, in most of this chapter (until eq. (2.48)) we treat the general case with $K_{\mu\nu}{}^{\rho}$ defined by eq. (2.18). Eqs. (2.18) and (2.19) imply

$$\Gamma_{[\mu\nu]}{}^{\rho} = \frac{1}{2}(\Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}) = \frac{1}{2}S_{\mu\nu}{}^{\rho}. \tag{2.20}$$

We mention in passing that the transformation of the connection coefficients from greek to latin indices is neatly expressed by Cartan's equations

$$\nabla e_i = \omega_i{}^j e_j \tag{2.21}$$

where e_j is the jth basis vector with components $e_j = (\lambda_j^{\mu=0}, \lambda_j^{\mu=1}, \lambda_j^{\mu=2}, \lambda_j^{\mu=3})$, and ∇ stands for the covariant derivative of the greek index of λ_i^{μ}.

Formulated for an arbitrary greek indexed contravariant vector v^{μ} one has for the covariant differentiation the familiar formula:

$$\nabla v^{\mu} = dx^{\nu}\nabla_{\nu}v^{\mu} = dx^{\nu}(\partial_{\nu}v^{\mu} + \Gamma_{\nu\rho}{}^{\mu}v^{\rho}), \tag{2.22}$$

and for a covariant vector v_{μ}:

$$\nabla v_{\mu} = dx^{\nu}\nabla_{\nu}v_{\mu} = dx^{\nu}(\partial_{\nu}v_{\mu} - \Gamma_{\nu\mu}{}^{\rho}v_{\rho}). \tag{2.23}$$

Similar formulae hold for the latin indexed vector quantities, i. e.

$$\nabla v^i = dv^i + \omega_k{}^i v^k = \theta^l \nabla_l v^i = \theta^l(\partial_l v^i + \Gamma_{lk}{}^i v^k), \tag{2.24}$$

$$\nabla v_i = dv_i - \omega_i{}^k v_k = \theta^l \nabla_l v_i = \theta^l(\partial_l v_i - \Gamma_{li}{}^k v_k). \tag{2.25}$$

For higher order tensorial quantities a corresponding Γ-term appears for each latin or greek index in the formulae for the covariant differentiation as is well-known from Riemannian geometry ([5]).

Covariant derivatives with respect to the metric connection alone will be denoted by a bar. Because of the antisymmetry of the $K_{\mu\nu\rho}$ in the last two indices as shown by eq. (2.18) one finds

$$\nabla_{\rho}g_{\mu\nu} = \overline{\nabla}_{\rho}g_{\mu\nu} = 0. \tag{2.26}$$

The Bianchi identities following as integrability conditions from eqs. (2.1) and (2.2) by exterior differentiation read

$$\nabla\Omega^i = \Omega_k{}^i \wedge \theta^k, \tag{2.27}$$

$$\nabla\Omega_{ij} = 0. \tag{2.28}$$

([4]) The tensor $K_{\mu\nu\rho}$ is sometimes called contorsion tensor. We refer here to both tensors, $K_{\mu\nu\rho}$ and $S_{\mu\nu\rho}$ as to the torsion tensor. In the completely antisymmetric case discussed later they differ only by a factor 1/2 (compare eq. (2.18) and eq. (2.49) below).

([5]) Compare ref. [*20*] in this context.

Splitting eqs. (2.1) and (2.2) as well as eqs. (2.27) and (2.28) each into a V_4 i. e. metrical part and a torsion part one finds using eq. (2.11):

$$\overline{\nabla}\theta^i = d\theta^i + \overline{\omega}_k{}^i \wedge \theta^k = 0\,, \tag{2.29}$$

$$\tau_k{}^i \wedge \theta^k = \Omega^i\,, \tag{2.30}$$

$$\overline{\Omega}_{ij} + \Omega'_{ij} = \Omega_{ij}\,. \tag{2.31}$$

where

$$\overline{\Omega}_{ij} = d\overline{\omega}_{ij} + \overline{\omega}_{ik} \wedge \overline{\omega}_j{}^k \tag{2.32}$$

is the curvature form for a Riemannian V_4, and

$$\Omega'_{ij} = \overline{\nabla}\tau_{ij} + \tau_{ik} \wedge \tau_j{}^k \tag{2.33}$$

is the torsion contribution to the curvature tensor in the Riemann-Cartan space-time U_4. Moreover, one finds

$$\nabla\tau_{ij} = \overline{\nabla}\tau_{ij} + 2\tau_{ik} \wedge \tau_j{}^k. \tag{2.34}$$

It is straightforward to convert the above equations for forms into the corresponding equations for the latin or greek indexed tensorial quantities. We shall do this only occasionally for reference purpose in later sections. For example, eq. (2.33) implies with

$$\Omega'_{ij} = \frac{1}{2}\,\theta^k \wedge \theta^l \qquad P_{klij} \tag{2.35}$$

and τ_{ik} given by eq. (2.13):

$$P_{klij} = \overline{\nabla}_k K_{lij} - \overline{\nabla}_l K_{kij} + K_{kis}K_{lj}{}^s - K_{lis}K_{kj}{}^s\,, \tag{2.36}$$

with $K_{lij} = -K_{lji}$ and $P_{klij} = -P_{lkij} = -P_{klji}$.

Splitting the Bianchi identities in a similar manner into V_4 and torsion contributions one obtains from eq. (2.27)

$$\overline{\Omega}_k{}^i \wedge \theta^k = 0\,, \tag{2.37}$$

yielding the cyclic identity for the curvature tensor $\overline{R}_{slk}{}^i$ which is valid in Riemannian geometry, and

$$\nabla\Omega^i = \overline{\nabla}\Omega^i + \tau_k{}^i \wedge \Omega^k = \Omega'_k{}^i \wedge \theta^k\,. \tag{2.38}$$

Similarly, eq. (2.28) implies the Bianchi identities for the Riemannian part

$$\overline{\nabla}\overline{\Omega}_{ij} = 0 \tag{2.39}$$

as well as

$$\begin{aligned}\nabla\Omega'_{ij} &= \overline{\nabla}\Omega'_{ij} - \tau_i{}^k \wedge \Omega'_{kj} - \tau_j{}^k \wedge \Omega'_{ik} \\ &= \tau_i{}^k \wedge \overline{\Omega}_{kj} + \tau_j{}^k \wedge \overline{\Omega}_{ik}\end{aligned} \tag{2.40}$$

Let us now write down some of these relations for the components of the respective tensor quantities and derive certain contractions of them. These contracted Bianchi identities for a U_4 are essential for the discussion of field equations in a Riemann-Cartan space presented in Sect. IV. Eqs. (2.27) and (2.28) are equivalent to the equations:

$$R_{\{ijk\}}{}^{l} = \nabla_{\{i}S_{jk\}}{}^{l} - S_{\{ij}{}^{t}S_{k\}t}{}^{l}, \qquad (2.27')$$

$$\nabla_{\{s}R_{ij\}kl} = S_{\{si}{}^{t}R_{j\}tkl}. \qquad (2.28')$$

Here $\{ ijk \}$ denotes the cyclic sum of the indices enclosed in the brackets. Since two of the indices in the curly brackets in eqs. (2.27′) and (2.28′) already form an antisymmetric pair both sides of these equations are completely antisymmetric in the indices surrounded by the curly brackets. Thus eqs. (2.27′) represent $4 \cdot 4 = 16$ identities, and eqs. (2.28′) represent $4 \cdot 6 = 24$ identities connecting curvature and torsion in a U_4. Separating these equations into their V_4 and torsion parts yields (compare eqs. (2.37)-(2.40))

$$\overline{R}_{\{ijk\}}{}^{l} = 0, \qquad (2.37')$$

$$\begin{aligned} P_{\{ijk\}}{}^{l} &= \nabla_{\{i}S_{jk\}}{}^{l} - S_{\{ij}{}^{t}S_{k\}t}{}^{l} \\ &= \overline{\nabla}_{\{i}S_{jk\}}{}^{l} + S_{\{ij}{}^{t}K_{k\}t}{}^{l}, \end{aligned} \qquad (2.38')$$

$$\overline{\nabla}_{\{s}\overline{R}_{ij\}kl} = 0, \qquad (2.39')$$

and

$$\overline{\nabla}_{\{s}P_{ij\}kl} - K_{\{s|k}{}^{t}P_{ij\}tl} - K_{\{s|l}{}^{t}P_{ij\}kt} = K_{\{s|k}{}^{t}\overline{R}_{ij\}tl} + K_{\{s|l}{}^{t}\overline{R}_{ij\}kt}. \qquad (2.40')$$

Eqs. (2.37′) and (2.39′) are well-known from Riemannian geometry the former yielding the symmetry

$$\overline{R}_{ijkl} = \overline{R}_{klij} \qquad (2.41)$$

of the Riemannian curvature tensor which is not shared by the tensor P_{ijkl}.

Needless to say, all the tensor equations (2.37′)-(2.40′) are equally valid for the greek indexed quantities obtained by converting each latin index into a greek one with the help of the vierbein fields λ^{i}_{μ} according to

$$\overline{R}_{\mu\nu\varkappa\lambda} = \lambda^{i}_{\mu}\lambda^{j}_{\nu}\lambda^{k}_{\varkappa}\lambda^{l}_{\lambda}\overline{R}_{ijkl} \qquad (2.42)$$

and similarly for any other tensorial quantity.

Contracting now the indices i and l in eq. (2.38′) yields the identity

$$\frac{1}{2}P_{\{ijk\}}{}^{i} = - P_{[jk]} = \overline{\nabla}_{i}T_{jk}{}^{i} - K_{ij}{}^{s}T_{sk}{}^{i} + K_{ik}{}^{s}T_{sj}{}^{i}, \qquad (2.43)$$

Where $T_{jk}{}^{i}$ is defined by

$$T_{jk}{}^{i} = \frac{1}{2}\left(S_{jk}{}^{i} + \delta^{i}_{j}S_{kl}{}^{l} - \delta^{i}_{k}S_{jl}{}^{l}\right), \qquad (2.44)$$

and P_{ik} is given by

$$P_{ik} = P_{ijkl}\eta^{jl}. \qquad (2.45)$$

Since the Ricci tensor $\overline{R}_{ik} = \overline{R}_{ijkl}\eta^{jl}$ is symmetrical in its indices one could on the left-hand side (l.-h. s.) of (2.43) replace $P_{[jk]}$ by $R_{[jk]}$ and $P_{\{ijk\}}{}^{i}$ by

$R_{\{ijk\}}{}^{i}$ (compare eqs. (2.31) and (2.37′)). Similarly the contraction of eq. (2.40′) leads to

$$\begin{aligned}\overline{\nabla}_{\{s}R_{ij\}k}{}^{s} = \overline{\nabla}_{\{s}P_{ij\}k}{}^{s} = \overline{\nabla}_{s}P_{ijk}{}^{s} + \overline{\nabla}_{i}P_{jk} - \overline{\nabla}_{j}P_{ik} \\ = R_{ij}{}^{st}K_{stk} - \left(R_{is}K_{jk}{}^{s} - R_{js}K_{ik}{}^{s}\right) \\ + R_{ijk}{}^{t}K_{s}{}^{s}{}_{t} - \left(R^{s}{}_{i}{}^{t}{}_{k}K_{jst} - R^{s}{}_{j}{}^{t}{}_{k}K_{ist}\right). \quad (2.46)\end{aligned}$$

A further contraction results in the equations:

$$\overline{\nabla}^{s}\left(P_{ls} - \frac{1}{2}\eta_{ls}P\right) = R_{l}{}^{its}K_{sti} + R_{ls}K_{i}{}^{is} - R_{[is]}K_{l}{}^{is}, \quad (2.47)$$

with

$$P = P_{jk}\eta^{jk}. \quad (2.48)$$

The l.-h. s. of eq. (2.47) could also be written as $\overline{\nabla}^{s}\left(R_{ls} - \frac{1}{2}\eta_{ls}R\right)$ with $R_{ls} = \overline{R}_{ls} + P_{ls}$ and $R = \overline{R} + P$, where $\overline{R} = \overline{R}_{ik}\eta^{ik}$ is the curvature scalar in Riemannian geometry. As is well-known, the reason is that because of the contracted Bianchi identities for a V_4 (compare eq. (2.39′)) the expression $\overline{\nabla}^{s}\left(\overline{R}_{ls} - \frac{1}{2}\eta_{ls}\overline{R}\right)$ is identically zero guaranteeing the covariant energy-momentum conservation in general relativity.

We conclude this section by spezializing some of the above given formulae to the case of a *totally antisymmetric torsion tensor* determined by four independent components. Switching over for later convenience to greek indices one has

$$K_{\mu\nu\rho} = \frac{1}{2}S_{\mu\nu\rho} \quad (2.49)$$

obeying

$$K_{\{\mu\nu\rho\}} = 3K_{\mu\nu\rho}. \quad (2.50)$$

We first observe that the torsion-torsion terms appearing on the r.-h. s. of eq. (2.38′) now vanish i. e.

$$K_{\{\mu\nu}{}^{\sigma}K_{\varkappa\}\sigma\lambda} = 0 \quad (2.51)$$

for totally antisymmetric $K_{\mu\nu\rho}$ in a U_4. As a consequence the eqs. (2.43) now assume the simple form

$$R_{[\mu\nu]} = P_{[\mu\nu]} = -\nabla^{\rho}K_{\mu\nu\rho} = -\overline{\nabla}^{\rho}K_{\mu\nu\rho}. \quad (2.52)$$

Furthermore, one finds in this case

$$P_{\rho\varkappa\sigma\tau} = \overline{\nabla}_{\rho}K_{\varkappa\sigma\tau} - \overline{\nabla}_{\varkappa}K_{\rho\sigma\tau} + K_{\rho\varkappa}{}^{\lambda}K_{\sigma\tau\lambda}, \quad (2.53)$$

$$\nabla_{\rho}K_{\varkappa\sigma\tau} = \overline{\nabla}_{\rho}K_{\varkappa\sigma\tau}, \quad (2.54)$$

$$P_{(\mu\nu)} = K_{\mu}{}^{\rho\lambda}K_{\nu\rho\lambda}, \quad (2.55)$$

and

$$P = K^{\sigma\rho\lambda}K_{\sigma\rho\lambda}. \quad (2.56)$$

Eqs. (2.53) and (2.54) are an immediate consequence of (2.51). The cyclic identities (2.37′) moreover imply

$$\overline{R}_{\sigma\tau\varkappa\lambda}K^{\sigma\tau\varkappa} = 0\,. \tag{2.57}$$

Using these relations eq. (2.47) is seen to reduce for totally antisymmetric $K_{\mu\nu\rho}$ to the form

$$\overline{\nabla}^{\rho}\left(P_{\lambda\rho} - \frac{1}{2}\,g_{\lambda\rho}P\right) = -\,P_{\lambda\sigma\tau\varkappa}K^{\sigma\tau\varkappa} - P_{[\sigma\tau]}K_{\lambda}{}^{\sigma\tau}\,. \tag{2.58}$$

This ends our review of the geometry in a Riemann-Cartan space-time. We now turn to the discussion of certain fiber bundles raised over such a space as base space.

III. SOLDERED H-BUNDLES OVER U_4

In the previous section we introduced the one-forms ω_{ik} determining a connection on the Lorentz frame bundle over space-time being a principal fiber bundle over U_4 with structural group SO(3, 1) with the latter being a short-hand notation for $O(3,1)^{++}$. Let us now extend this bundle to the bundle of *affine* frames on which the Poincaré group acts as the structural or gauge group. Following Lichnerowicz [6] we write the transformations of the inhomogeneous Lorentz group in 5 × 5 matrix form as

$$[B]^{A}_{B} = [B(\Lambda, a)]^{A}_{B} = \begin{pmatrix} 1 & 0 \\ a^i & \Lambda^i{}_k \end{pmatrix} \tag{3.1}$$

obeying the multiplication law

$$B_1B_2 = \begin{pmatrix} 1 & 0 \\ a_{(1)} & \Lambda_{(1)} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_{(2)} & \Lambda_{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{(1)} + \Lambda_{(1)}a_{(2)} & \Lambda_{(1)}\Lambda_{(2)} \end{pmatrix} = B_{12}\,. \tag{3.2}$$

The inverse of the matrix (3.1) is given by

$$[B^{-1}]^{A}_{B} = [B^{-1}(\Lambda, a)]^{A}_{B} = \begin{pmatrix} 1 & 0 \\ -\,[\Lambda^{-1}]^i{}_j a^j & [\Lambda^{-1}]^i{}_k \end{pmatrix}. \tag{3.3}$$

An affine frame in Minkowski space is determined by a four-vector $\tilde{x}$ and a set of four independent base vectors $\tilde{e}_j$; $j = 0, 1, 2, 3$, i. e.:

$$E_A = (\tilde{x}, \tilde{e}_j)\,. \tag{3.4}$$

$\tilde{x}$ denotes the origin of the frame $\tilde{e}_j$ (say with respect to a set of orthonormal axes denoted by e_i), and for the $\tilde{e}_j$ we take also a set of orthonormal frame vectors. An affine frame E_A can be obtained from the particular frame $\overset{\circ}{\mathrm{E}}_{\mathrm{A}} = (0, e_i)$ by a boost with an element of ISO(3, 1) i. e. with the matrix $\mathrm{B}^{-1}(\overline{\Lambda}, \overline{a})$ according to

$$\mathrm{E_A} = \overset{\circ}{\mathrm{E}}_{\mathrm{B}}[\mathrm{B}^{-1}(\overline{\Lambda}, \overline{a})]^{\mathrm{B}}_{\mathrm{A}}, \tag{3.5}$$

where

$$\tilde{x} = -\,[\overline{\Lambda}^{-1}]^i{}_k \overline{a}^k e_i = \tilde{x}^i e_i, \tag{3.6}$$

and

$$\tilde{e}_j = [\overline{\Lambda}^{-1}]^i{}_j e_i. \tag{3.7}$$

Two frames $\mathrm{E_A} = (\tilde{x}, \tilde{e}_j)$ and $\hat{\mathrm{E}}_{\mathrm{A}} = (\hat{\tilde{x}}, \hat{\tilde{e}}_j)$ obtained from the same fixed frame $\overset{\circ}{\mathrm{E}}_{\mathrm{A}}$ by a boost with $\mathrm{B}^{-1}(\overline{\Lambda}, \overline{x})$ and $\mathrm{B}^{-1}(\hat{\overline{\Lambda}}, \hat{\overline{x}})$, respectively, are related by a Poincaré transformation $\mathrm{B}(\Lambda, a)$ according to

$$\hat{\mathrm{E}}_{\mathrm{A}} = \mathrm{E_B}[\mathrm{B}^{-1}(\Lambda, a)]^{\mathrm{B}}_{\mathrm{A}}, \tag{3.8}$$

with

$$\Lambda^i{}_k = [\hat{\overline{\Lambda}}]^i{}_j [\overline{\Lambda}^{-1}]^j{}_k, \tag{3.9}$$

and

$$\hat{\tilde{x}}^i = [\Lambda]^i{}_k \tilde{x}^\kappa + a^i, \tag{3.10}$$

and

$$\hat{\tilde{e}}_i = [\Lambda^{-1}]^k{}_i \tilde{e}_k. \tag{3.11}$$

Let us now consider space-time dependent Poincaré transformations (3.1) with x-dependent translations and Lorentz transformations $a^i(x)$ and $[\Lambda(x)]^i{}_k$. They are realized as gauge transformations on the affine tangent bundle over space-time. The affine tangent bundle

$$\mathrm{T_A(U_4) = E(U_4, F \cong ISO(3,1)/SO(3,1)\,;\ G = ISO(3,1))} \tag{3.12}$$

is a soldered [*9*] [*10*] [*8*] fiber bundle over space-time possessing as fiber the homogeneous space ISO(3, 1)/SO(3, 1) isomorphic to Minkowski space $\mathrm{M_4}$ (considered as an affine space) and having the structural or gauge group G = ISO(3, 1). Moreover, the set of all affine frames in the local tangent spaces at all points $x \in \mathrm{U_4}$ can be given the structure of a principal fiber bundle over $\mathrm{U_4}$ with fiber and structural group given by the Poincaré group. Let us call this bundle with ten dimensional fiber $\mathrm{L_A(U_4)}$, the affine frame bundle over space-time. A connection on $\mathrm{L_A(U_4)}$ — called an *affine connection* associated with the linear connection discussed in Sect. II — is given by the following matrix of one-forms ([6])

$$\mathrm{W^A_B} = \begin{pmatrix} 0 & 0 \\ -\,\tilde{\theta}^i & \omega_k{}^i \end{pmatrix}. \tag{3.13}$$

Here $\omega_k{}^i = -\,\omega^i{}_k$ are the connection forms on the Lorentz frame bundle discussed in Sect. II, and the $\tilde{\theta}^i$ are the soldering forms [*10*] being related to the fundamental one-forms θ^i on $\mathrm{U_4}$ by the equations

$$\tilde{\theta}^i = \theta^i + \nabla \tilde{x}^i \tag{3.14}$$

([6]) Compare Chapter 45 of the book by Lichnerowicz [*6*]. In the terminology of ref. [*7*] the forms (3.13) define a « generalized affine connection ».

with $\nabla\tilde{x}^i$ denoting the covariant derivative of $\tilde{x}^i$ with respect to ω_{ki}, i. e.

$$\nabla\tilde{x}^i = d\tilde{x}^i + \omega_k{}^i\tilde{x}^k . \tag{3.15}$$

The expression (3.13) represents the connection on $L_A(U_4)$ and, correspondingly, on the associated bundle $T_A(U_4)$ in a particular gauge (cross section dependent form of the connection). By changing the gauge the matrix W undergoes an inhomogeneous B-transformation the effect of which is that the transformed matrix $\hat{W}$ has the entries

$$\hat{\tilde{\theta}}^i = [\Lambda(x)]^i{}_j\tilde{\theta}^j + \hat{\nabla}a^i(x) , \tag{3.16}$$

and

$$\hat{\omega}_k{}^i = [\Lambda(x)]^i{}_j[\Lambda^{-1}(x)]^l{}_k\omega_l{}^j + [\Lambda(x)]^i{}_jd[\Lambda^{-1}(x)]^j{}_k \tag{3.17}$$

In (3.16) the covariant derivative with respect to the transformed connection $\hat{\omega}_k{}^i$ appears. Eq. (3.17) is identical with eq. (γ) of the footnote (3), as it should be. It is a simple matter to specialize the above equations to pure gauge translations $B = \begin{pmatrix} 1 & 0 \\ a^i & 1 \end{pmatrix}$. In this case eq. (3.16) assumes the form

$$\hat{\tilde{\theta}}^i = \theta^i + \nabla\hat{\tilde{x}}^i = \tilde{\theta}^i + \nabla a^i(x) = \theta^i + \nabla(\tilde{x}^i + a^i(x)) \tag{3.16'}$$

with $\hat{\tilde{x}}^i = \tilde{x}^i + a^i(x)$ (see eq. (3.10)) implying that the origin of the frame in a certain gauge suffers an x-dependent translation on $T_A(U_4)$ in changing the gauge (with the direction of the axes remaining unchanged, $\hat{\omega}_k{}^i = \omega_k{}^i$).

Let us now go back to eq. (3.14) and expand it in a natural basis of one forms. Calling the coefficients of the soldering forms $v^i_\mu(x)$, i. e. writing $\tilde{\theta}^i = v^i_\mu dx^\mu$, one obtains (compare eq. (2.6))

$$v^i_\mu = \lambda^i_\mu + \nabla_\mu\tilde{x}^i . \tag{3.18}$$

Locally, at a given point x_0 on the base space, one can always transform $\nabla_\mu\tilde{x}^i$ into the form $-\delta^i_\mu$ (7) so that eq. (3.18) assumes the form

$$\lambda^i_\mu = \delta^i_\mu + v^i_\mu \qquad \text{at } x = x_0 . \tag{3.19}$$

Hence the v^i_μ – the coefficients of the soldering forms – appear in this case as the « nontrivial part of the vierbein fields » to use the terminology of Cho [*5*]. However, the eqs. (3.19) are in this form not Lorentz gauge independent. The covariant form of eq. (3.19) is eq. (3.18) where the translational gauge variables $\tilde{x}^i$ appear in Lorentz covariant and general covariant form. While the coefficients $\Gamma_{\mu k}{}^i$ of $\omega_k{}^i$ play the role of *rotational* gauge potentials the v^i_μ play the role of *translational* gauge potentials. Geometrically speaking the $\Gamma_{\mu k}{}^i$ and v^i_μ are the coefficients of an affine connection on $L_A(U_4)$, which is a soldered principal fiber bundle over

(7) Compare ref. [*6*].

space-time, with the v^i_μ representing the coefficients of the soldering forms.

Let us now turn to the structural equations satisfied by the affine connection $(\omega_{ik}, \tilde{\theta}^i)$. Eq. (2.1) is now replaced by

$$\tau^i = \nabla\tilde{\theta}^i = \nabla\theta^i + \nabla\nabla\tilde{x}^i = \Omega^i + \Omega_k{}^i\tilde{x}^k \tag{3.20}$$

with $\tau^i = \frac{1}{2}\theta^k \wedge \theta^l \tau_{kl}{}^i$ playing the role of the torsion two-forms in the affine case, while eq. (2.2) remains unchanged. The Bianchi identities (2.27) now take the form

$$\nabla\tau^i = \Omega_k{}^i \wedge \tilde{\theta}^k . \tag{3.21}$$

Under Poincaré gauge transformations $(\Lambda(x), a(x))$ the Ω_{kl} transform homogeneously as

$$\hat{\Omega}_{ij} = [\Lambda^{-1}(x)]^k{}_i[\Lambda^{-1}(x)]^l{}_j\Omega_{kl}, \tag{3.22}$$

while the τ^i obey the transformation rule

$$\hat{\tau}^i = [\Lambda(x)]^i{}_j\tau^j + \hat{\Omega}_k{}^i a^k(x), \tag{3.23}$$

i. e.

$$\hat{\tau}^i = \hat{\Omega}^i + \hat{\nabla}\hat{\nabla}\hat{\tilde{x}}^i, \tag{3.24}$$

with $\hat{\tilde{x}}^i = [\Lambda(x)]^i{}_j\tilde{x}^j + a^i(x)$ (compare also eqs. (α) and (β) of the footnote (3) in this context).

Having defined the affine frame bundle and the affine tangent bundle over space-time both having the structural group $G = ISO(3, 1)$ we now turn to the definition of a whole sequence of soldered fiber bundles over space-time ranging between $T_A(U_A)$ and $L_A(U_A)$. We call these bundles collectively the soldered H-bundles over space-time. They are of the type

$$H_{G'}(U_4) = E(U_4, F \cong H = G/G' \,;\, G = ISO(3, 1)) \tag{3.25}$$

having base space U_4 and possessing as fiber a homogeneous space H of the Poincaré group which is of the form $H = M_4 \otimes S$ i. e. contains flat Minkowski space-time. The soldering to the base is made through this subspace of H, i. e. by indentifying the local tangent space of U_4 at x with the Minkowski subspace of H through an isomorphism (compare ref. [*9*]).

The classification of homogeneous spaces H is obtained by utilizing a corresponding classification of stability subgroups G′ of the Poincaré group. In fact, it is shown in ref. [*13*] that G′ must be a subgroup of the Lorentz group if $H = M_4 \otimes S$ is required to hold true. The dimension of H is given by $N = \dim H = 10 - \dim G'$. N ranges from 4 ($G' = SO(3, 1)$ to 10 ($G' = 1$). In these two extreme cases the H-bundles are identical with $T_A(U_4)$ and $L_A(U_4)$, respectively, as mentioned before. We shall not specify in this paper the particular H-bundle to be used in physics nor do we fix the dimension N of H. However, we do assume in the following that the space H possesses a Poincaré invariant measure. This will turn out to be necessary for a definition of a Poincaré gauge invariant current to be possible (see Sect. IV below). Moreover, it is probably necessary from the physical point of view that the homogeneous space H can carry half-integer spin fields. This requires N to be bigger then seven [12] [13].

It was shown in ref. [13] that the representation of the generators of the Poincaré group in the form of differential operators for scalar functions defined on the homogeneous space H is given by

$$\tilde{\mathrm{P}}_i = i\tilde{\partial}_i , \tag{3.26a}$$

and

$$\tilde{\mathrm{M}}_{ik} = \tilde{\mathrm{L}}_{ik} + \tilde{\mathrm{S}}_{ik} = i(\tilde{x}_i\tilde{\partial}_k - \tilde{x}_k\tilde{\partial}_i) + \tilde{\mathrm{S}}_{ik} . \tag{3.26b}$$

$\tilde{\mathrm{P}}_i$ and $\tilde{\mathrm{L}}_{ik}$ form a representation of the Lie algebra of ISO(3, 1) in Minkowski space, while the $\tilde{\mathrm{S}}_{ik}$ are differential operators in the additional variables needed to describe the part of the space H which we called S. The dimension of S ranges from zero to six. The coordinates of a point in H we shall denote in the following by $\tilde{\mathrm{X}} = (\tilde{x}, \tilde{y})$. Explicit forms of the operators $\tilde{\mathrm{S}}_{ik}$ in the variables $\tilde{y}$ were given by Bacry and Kihlberg [13]. All we need for the subsequent discussion in this paper is their algebraic properties: The $\tilde{\mathrm{S}}_{ik}$ behave like spin operators obeying the commutation relations

$$i[\tilde{\mathrm{S}}_{ij}, \tilde{\mathrm{S}}_{kl}] = \eta_{ik}\tilde{\mathrm{S}}_{jl} + \eta_{jl}\tilde{\mathrm{S}}_{ik} - \eta_{il}\tilde{\mathrm{S}}_{jk} - \eta_{jk}\tilde{\mathrm{S}}_{il} , \tag{3.27}$$

with $[\tilde{\mathrm{L}}_{ij}, \tilde{\mathrm{S}}_{kl}] = 0$, and $[\tilde{\mathrm{P}}_i, \tilde{\mathrm{S}}_{kl}] = 0$.

Clearly the operators $\tilde{\mathrm{P}}_i$, $\tilde{\mathrm{M}}_{ik}$ satisfy the commutation relations of the Poincaré group, i. e.

$$i[\tilde{\mathrm{M}}_{ij}, \tilde{\mathrm{M}}_{kl}] = \eta_{ik}\tilde{\mathrm{M}}_{jl} + \eta_{jl}\tilde{\mathrm{M}}_{ik} - \eta_{il}\tilde{\mathrm{M}}_{jk} - \eta_{jk}\tilde{\mathrm{M}}_{il} , \tag{3.28a}$$

$$i[\tilde{\mathrm{P}}_i, \tilde{\mathrm{M}}_{jk}] = \eta_{ik}\tilde{\mathrm{P}}_j - \eta_{ij}\tilde{\mathrm{P}}_k , \tag{3.28b}$$

$$[\tilde{\mathrm{P}}_i, \tilde{\mathrm{P}}_j] = 0 . \tag{3.28c}$$

It is now a straightforward matter to define a scalar wave function $\phi(x; \tilde{\mathrm{X}}) = \phi(x; \tilde{x}, \tilde{y})$ on the H-bundle (3.25) and introduce a Poincaré gauge covariant derivative for it. Let us, however, first add a comment concerning the use of the function $\phi(x; \tilde{\mathrm{X}})$ in the context of this paper. The suggestion of considering a field $\phi(x; \tilde{\mathrm{X}})$ for a description of matter in connection with a Poincaré gauge formulation of gravity is a gauge extension of Lurçat's proposal which was based on the idea to allow the spin degrees of freedom to play a dynamical role in particle physics. This idea is carried over here into the framework of a gauge theory for scalar matter wave functions defined on a bundle with ISO(3, 1) as structural group. We further remark in passing that in order to fix the mass and the spin described by the field $\phi(x; \tilde{\mathrm{X}})$ one could demand that the Casimir operators of the Poincaré group take definite values where applied to $\phi(x; \tilde{\mathrm{X}})$ i. e.

$$\tilde{\mathrm{P}}_i\tilde{\mathrm{P}}^i\phi(x; \tilde{\mathrm{X}}) = m^2\phi(x; \tilde{\mathrm{X}}) \tag{3.29}$$

$$\tilde{\mathrm{W}}_i\tilde{\mathrm{W}}^i\phi(x; \tilde{\mathrm{X}}) = - m^2 s(s + 1)\phi(x; \tilde{\mathrm{X}}) \tag{3.30}$$

with
$$\tilde{W}^i = \frac{1}{2}\varepsilon^{ijkl}\tilde{M}_{jk}\tilde{P}_l = \frac{1}{2}\varepsilon^{ijkl}\tilde{S}_{jk}\tilde{P}_l \tag{3.31}$$

being the Pauli-Lubanski operator associated with the space H. However, we shall not project out a definite mass and spin value by demanding eqs. (3.29) and (3.30) to be fulfilled – at least not at this level of the discussion. We shall first try to answer the question of what ISO(3, 1) gauge invariant equation $\phi(x;\tilde{X})$ has to obey on the bundle space $H_{G'}(U_4)$.

The internal dynamics described by the $\tilde{X}$-dependance of the ϕ-field is a gauge dynamics, i. e. the $\tilde{X}$-distribution in a whole neighbourhood of a point $x \in U_4$ can be changed by a Poincaré gauge transformation. However, the internal Poincaré gauge degrees of freedom will have to be characterized and, in fact, determined by the requirement that $\phi(x;\tilde{X})$ *be a solution of a particular gauge invariant field equation on the* H-*bundle* (3.25). This question will be addressed in the next section after having first established there what further relations – beyond those known from general relativity – can be introduced between matter quantities and geometrical quantities in a theory based on a Riemann-Cartan space-time.

We conclude the discussion in this section by giving the expression for the Poincaré gauge covariant derivative of $\phi(x;\tilde{X})$ which we denote by the symbol D_k (or $D_\mu = \lambda^k_\mu D_k$), i. e.

$$D_k\phi(x;\tilde{X}) = \left[\partial_k + \frac{i}{2}\Gamma_{kij}\tilde{M}^{ij} - iv^i_k\tilde{P}_i\right]\phi(x;\tilde{X}). \tag{3.32}$$

This equation can compactly be written as

$$D\phi(x;\tilde{X}) = (d + i\Gamma)\phi(x;\tilde{X}), \tag{3.33}$$

with

$$\Gamma = \frac{1}{2}\omega_{ik}\tilde{M}^{ik} - \tilde{\theta}^i\tilde{P}_i \tag{3.34}$$

denoting the Lie algebra valued one-form associated with the Poincaré group defining a connection in the soldered H-bundle (3.25) constructed over a Riemann-Cartan space-time.

IV. FIELD EQUATIONS

a) Field equations in a U_4.

In the formulation of the gravitational interaction including also a description of matter in quantum mechanical i. e. in wave function form [8]

[8] We first consider here a one-particle Schrödinger wave function type description of matter. A many-particle formalism will have to be developped at a later stage.

we shall specify here *two* source equations describing the interrelation between the distribution of matter and the properties of the underlying Riemann-Cartan space-time geometry. The first set of equations are essentially Einstein's equations of general relativity coupling the Riemannian part of the contracted curvature tensor to the classical energy-momentum tensor $\overline{T}_{\mu\nu}$, while the second set of equations essentially relates a completely antisymmetrical source current $J_{\nu\lambda\varkappa}(\phi)$ associated with a quantum mechanical description of matter to the torsion tensor $K_{\mu\nu\rho}$. This tensor is moreover assumed here to be totally antisymmetric in all its indices. We thus propose two ways in which matter can influence the underlying geometry: On the one hand the classical energy-momentum distribution of matter induces a Riemannian curvature in the geometry yielding the Einsteinian metrical description of gravity as a classical macroscopic field; on the other hand – provided an additional quantum mechanical current distribution is present describing the non-classical wave function properties of matter – this current is assumed to induce a nonzero torsion in the Riemann-Cartan geometry. We, therefore, do not consider it necessary in this approach to regard the source term in Einstein's equations as an expectation value of a quantized expression. We rather leave this part classical as it is and relate it in the usual way to the classical long range gravitational field. In addition, however, we represent the capacity of matter which is described in a quantum mechanical manner and which is embedded in a metrical field generated by other distant masses distributed in a classical way, to act on the underlying geometry through a set of new equations, called the current-torsion-equations (see eqs. (4.2), (4.22) and (4.27) below). They have the same form as the source equations in a Yang-Mills-type theory. The new interaction introduced in this way – although obtained here through a sequence of geometrical arguments connected with general relativity – need not necessarily be a gravitational interaction. We, therefore, introduce a new coupling constant for it. Let us now first state the field equations in a U_4 and then reduce them to simpler and more transparent forms in the spirit of the remarks just made:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = \varkappa T_{\mu\nu}, \tag{4.1}$$

$$\nabla^{\lambda} R_{\mu\nu\varkappa\lambda} = \bar{\varkappa} J_{\varkappa\nu\mu}. \tag{4.2}$$

All geometrical quantities appearing on the l.-h. s. of these equations are supposed to refer to a Riemann-Cartan U_4 characterized by a completely antisymmetric torsion tensor $K_{\mu\nu\rho}$ (compare Sect. II and in particular the equations following eq. (2.49)).

Splitting now $T_{\mu\nu}$ in eq. (4.1) into a classical symmetrical part, $\overline{T}_{\mu\nu} = \overline{T}_{\nu\mu}$, and into a part $T_{\mu\nu}(\phi)$ associated with that form of matter to be described in a quantum mechanical way, i. e.

$$T_{\mu\nu} = \overline{T}_{\mu\nu} + T_{\mu\nu}(\phi), \tag{4.3}$$

eq. (4.1) is then seen to separate into the following three equations the last two of which are obtained by regarding in turn the symmetrical and antisymmetrical part of $T_{\mu\nu}(\phi)$ (compare eqs. (2.52) and (2.55)):

$$\overline{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\overline{R} = \varkappa\overline{T}_{\mu\nu}, \tag{4.4}$$

$$P_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu}P = \varkappa T_{(\mu\nu)}(\phi), \tag{4.5}$$

$$P_{[\mu\nu]} = -\overline{\nabla}^{\rho}K_{\mu\nu\rho} = \varkappa T_{[\mu\nu]}(\phi). \tag{4.6}$$

Eqs. (4.4) are Einstein's field equations for the metrical classical gravitational field, $\varkappa$ is Einstein's gravitational constant, and eqs. (4.5) and (4.6) are equations to be obeyed by the quantities $T_{\mu\nu}(\phi)$. However, we shall not specify here such a tensor $T_{\mu\nu}(\phi)$ in terms of fields ϕ. We rather *define* $T_{\mu\nu}(\phi)$ by the l.-h. s. of eqs. (4.5) and (4.6). The effects which matter, treated à la Schrödinger in wave function form, has on the geometry will be determined by our second set of equations, i. e. these effects will be induced in the geometry through a current which we shall later express explicitely in a bilinear fashion in terms of the ϕ-fields.

We now separate also the source term in eq. (4.2) — which, by definition, is antisymmetric in the last two indices — into a classical and a quantum part according to

$$\tilde{\varkappa}J_{\varkappa\nu\mu} = \varkappa\overline{J}_{\varkappa\nu\mu} + \tilde{\varkappa}J_{\varkappa\nu\mu}(\phi) \tag{4.7}$$

with each term on the r.-h. s. of this equation having the length dimension $[l^{-3}]$ as required by the l.-h. s. of eq. (4.2). Let us observe, that the components of the classical current $\overline{J}_{\varkappa\nu\mu}$ introduced here need not be specified in Einstein's theory since they are derived quanties: The contracted Bianchi identities in a V_4 yield, together with the field equations (4.4),

$$\begin{aligned}\overline{\nabla}^{\lambda}\overline{R}_{\mu\nu\varkappa\lambda} &= -\overline{\nabla}^{\lambda}\overline{R}_{\lambda\varkappa\mu\nu} = \overline{\nabla}_{\nu}\overline{R}_{\mu\varkappa} - \overline{\nabla}_{\mu}\overline{R}_{\nu\varkappa} \\ &= \varkappa\left(\overline{\nabla}_{\nu}\overline{T}_{\mu\varkappa} - \overline{\nabla}_{\mu}\overline{T}_{\nu\varkappa} - \frac{1}{2}\partial_{\nu}\overline{T}\cdot g_{\mu\varkappa} + \frac{1}{2}\partial_{\mu}\overline{T}\cdot g_{\nu\varkappa}\right) = \varkappa\overline{J}_{\varkappa\nu\mu}.\end{aligned} \tag{4.8}$$

The last equality is our definition of $\overline{J}_{\varkappa\nu\mu}$. Moreover, $\overline{T}$, the trace of $\overline{T}_{\mu\nu}$, is given by

$$\varkappa\overline{T} = \varkappa\overline{T}_{\mu}{}^{\mu} = -\overline{R}. \tag{4.9}$$

With eqs. (4.7) and (4.8) eqs. (4.2) now assume the form

$$\overline{\nabla}^{\lambda}P_{\mu\nu\varkappa\lambda} - K^{\sigma}{}_{\mu}{}^{\rho}R_{\rho\nu\varkappa\sigma} - K^{\sigma}{}_{\nu}{}^{\rho}R_{\mu\rho\varkappa\sigma} - K^{\sigma}{}_{\varkappa}{}^{\rho}R_{\mu\nu\rho\sigma} = \tilde{\varkappa}J_{\varkappa\nu\mu}(\phi). \tag{4.10}$$

Notice that except for the first term on the l.-h. s. the full U_4 curvature tensor $R_{\mu\nu\varkappa\lambda} = \overline{R}_{\mu\nu\varkappa\lambda} + P_{\mu\nu\varkappa\lambda}$ appears in these equations. To throw out the Riemannian part we take the cyclic sum of the indices $\mu\nu\varkappa$ on both sides of these equations remembering that the source current was supposed to be totally antisymmetric in its indices i. e. obeys

$$J_{\{\varkappa\nu\mu\}}(\phi) = 3J_{\varkappa\nu\mu}(\phi). \tag{4.11}$$

Then, because of (2.37′), eqs. (4.10) read

$$\overline{\nabla}^\lambda P_{\{\mu\nu\varkappa\}\lambda} - K_\mu{}^{\rho\sigma}P_{\{\nu\varkappa\rho\}\sigma} - K_\nu{}^{\rho\sigma}P_{\{\varkappa\mu\rho\}\sigma} - K_\varkappa{}^{\rho\sigma}P_{\{\mu\nu\rho\}\sigma} = \nabla^\lambda P_{\{\mu\nu\varkappa\}\lambda} = 3\overline{\varkappa}J_{\varkappa\nu\mu}(\phi). \tag{4.12}$$

Using now the cyclic identites (2.38′) for the $P_{\mu\nu\varkappa\rho}$, remembering moreover eq. (2.51), i. e. using the relations

$$P_{\{\mu\nu\varkappa\}} = 2\nabla_{\{\mu}K_{\nu\varkappa\}\lambda} = 2\overline{\nabla}_{\{\mu}K_{\nu\varkappa\}\lambda}, \tag{4.13}$$

eqs. (4.2) take finally the following form:

$$\overline{\nabla}^\lambda\overline{\nabla}_{\{\mu}K_{\nu\varkappa\}\lambda} - K_{\{\mu}{}^{\rho\sigma}\overline{\nabla}_{\{\nu}K_{\varkappa\}\rho\}\sigma} = -\frac{3}{2}\overline{\varkappa}J_{\mu\nu\varkappa}(\phi). \tag{4.14}$$

This is a set of relativistically covariant second order nonlinear differential equations coupling a totally antisymmetrical current $J_{\mu\nu\varkappa}(\phi)$ to a totally antisymmetrical torsion tensor in a Riemann-Cartan space-time. We shall below investigate these equations in local geodesic coordinates (with respect to the background metric generated by $\overline{T}_{\mu\nu}$), or, what amounts to the same, in a situation where the Riemannian part of the curvature tensor is identically zero.

Let us, however, first mention in passing that the classical part of the current defined by eqs. (4.8) clearly obeys the identities $\overline{J}_{\{\varkappa\nu\mu\}} = 0$ and would, therefore, no longer appear in eq. (4.2) after having taken the cyclic sum $\{\ \mu\nu\varkappa\ \}$ on both sides of these equations yielding thus immediately eq. (4.12). We, furthermore, mention that the field equations (4.1) and (4.2) together with (4.3), (4.7), (4.8) and the contracted eqs. (2.39′) imply the relations

$$\overline{J}_\rho{}^\rho{}_\mu = \frac{1}{2}\partial_\mu\overline{T} \tag{4.15}$$

as well as, using (4.17) below,

$$\nabla^\rho\left[T_{\mu\rho}(\phi) - \frac{1}{2}g_{\mu\rho}T(\phi)\right] = 0, \tag{4.16}$$

where $T(\phi)$ is the trace of $T_{\mu\nu}(\phi)$. We shall come back to these equations later when we discuss the conservation laws satisfied in this theory based on eqs. (4.1) and (4.2).

Contracting, moreover, eq. (4.10) with $g^{\varkappa\nu}$ one obtains for a totally antisymmetric $J_{\varkappa\nu\mu}(\phi)$ the result:

$$\nabla^{\lambda}P_{\mu\lambda} = \overline{\nabla}^{\lambda}P_{\mu\lambda} - K_{\mu}{}^{\sigma\rho}P_{[\sigma\rho]} = 0 . \tag{4.17}$$

This equation is seen to be equivalent to eq. (4.16) as the consequence of eqs. (4.5) and (4.6).

We return now to eqs. (4.14) and rewrite these equations by introducing the dual curvature tensor and the dual to the current and torsion tensor, the latter quantities being both axial vectors, i. e. [9]

$$*P^{sk}{}_{jl} = \frac{1}{2}\,\varepsilon^{skpq}P_{pqjl} , \tag{4.18}$$

$$*P^{s}{}_{j} = -\frac{1}{2}\,\varepsilon^{spqk}P_{pqkj} = -\,\varepsilon^{skpq}\overline{\nabla}_{k}K_{pqj} , \tag{4.19}$$

$$*J^{s}(\phi) = -\frac{1}{6}\,\varepsilon^{sijk}J_{ijk}(\phi) , \tag{4.20}$$

$$*K^{s} = -\frac{1}{6}\,\varepsilon^{sijk}K_{ijk} . \tag{4.21}$$

In (4.19) also eqs. (4.13) have been used. Moreover, for totally antisymmetric K_{ijk} one has $\nabla_{s}K_{ijk} = \overline{\nabla}_{s}K_{ijk}$. Now eq. (4.14) can be rewritten as

$$\nabla^{\lambda}*P_{\mu\lambda} = \overline{\nabla}^{\lambda}*P_{\mu\lambda} - K_{\mu}{}^{\sigma\rho}*P_{[\sigma\rho]} = -\,3\overline{\varkappa}*J_{\mu}(\phi) , \tag{4.22}$$

where $*P_{\mu\lambda}$ is expressed in terms of $*K_{\mu}$ by

$$*P_{\mu\lambda} = 2(\overline{\nabla}_{\lambda}*K_{\mu} - g_{\mu\lambda}\overline{\nabla}^{\rho}*K_{\rho}) . \tag{4.23}$$

One could replace $*P_{\mu\lambda}$ in these equations and in eqs. (4.22), by $*R_{\mu\lambda}$ since the dual of the Ricci tensor, $*\overline{R}_{\mu\lambda}$, vanishes identically because of the cyclic identities (2.37').

Using, finally, Minkowskian coordinates by specializing to a situation where the Riemannian curvature is zero eqs. (4.17) and (4.22) take the form, with $\overline{\nabla}_{\rho} = \partial_{\rho}$ and writing for convenience again latin indices,

$$\overset{(K)}{\nabla}{}^{l}P_{il} = 0 , \tag{4.24}$$

and

$$\overset{(K)}{\nabla}{}^{l}*P_{il} = -\,3\overline{\varkappa}*J_{i}(\phi) . \tag{4.25}$$

[9] ε_{ijkl} is the Levi-Civita tensor with $\varepsilon_{0123} = +1 = -\varepsilon^{0123}$. Conversion to greek indices results in the occurence of the tensor $\eta_{\mu\nu\varkappa\lambda} = \varepsilon_{ijkl}\lambda^{i}_{\mu}\lambda^{j}_{\nu}\lambda^{k}_{\varkappa}\lambda^{l}_{\lambda} = \sqrt{-g}\,\varepsilon_{\mu\nu\varkappa\lambda}$, with $g = |g_{\mu\nu}|$ and $\sqrt{-g} = |\lambda^{i}_{\mu}|$.

where the symbol $\overset{(K)}{\nabla}$ denotes here the covariant divergence with respect to K_{ijk} i. e., since $K_j{}^j{}_s = 0$,

$$\begin{aligned}\overset{(K)}{\nabla}{}^l P_{il} &= \partial^l P_{il} - K_i{}^{rs} P_{rs} \\ &= \partial^l P_{il} - K_i{}^{rs} P_{[rs]} .\end{aligned} \tag{4.26}$$

Let us give to eqs. (4.14) (or rather (4.22)) one further form in Minkowski coordinates:

$$\square {}^*K_s - \partial_s(\partial^j {}^*K_j) - \varepsilon_s{}^{ijk} {}^*K_i \partial_j {}^*K_k = -\frac{3}{2} \varkappa {}^*J_s(\phi) , \tag{4.27}$$

with $\square = \eta^{ik}\partial_i\partial_k = \partial_0^2 - \Delta$ being the d'Alembert operator. Eq. (4.27) would clearly be the relevant equation even for a larger domain in a curved space-time if the metrical background field, for other reasons, could be assumed to be nearly Minkowskian ($g_{\mu\nu} \approx \eta_{\mu\nu}$). It would, on the other hand, be an exact equation – as would be eqs. (4.24) and (4.25) – in the limit of a vanishing Riemannian curvature in the U_4, i. e. for the Riemann-Cartan space degenerating to a so-called Weitzenböck space T_4 ([10]) [*4*].

${}^*K_s$ and ${}^*J_s(\phi)$ are, as mentioned, axial vectors behaving under parity transformations as

$${}^*K^0 \xrightarrow{P} - {}^*K^0 ; \qquad {}^*\vec{K} \xrightarrow{P} {}^*\vec{K} \tag{4.28}$$

and similarly for ${}^*J_s(\phi)$. Moreover, eqs. (4.27) imply the following divergence relation for the axial current ${}^*J_s(\phi)$ which is valid in a space-time T_4 or approximately valid in a U_4 with a weak metrical gravitational background field:

$$\partial^s {}^*J_s(\phi) = \frac{2}{3}\frac{1}{\varkappa} \varepsilon^{sijk}(\partial_s {}^*K_i)(\partial_j {}^*K_k) . \tag{4.29}$$

Eq. (4.29) shows that the axial vector current ${}^*J_s(\phi)$ is not conserved. It obeys a divergence equation similar in nature to the PCAC anomaly relation in spinor electrodynamics [*21*] or their extension to general

([10]) For brevity we call here a Weitzenböck space T_4 a space-time with zero *Riemannian* curvature. Thus, in this case, the $\overline{\omega}_{ik}$ can globally be transformed to zero (compare eqs. (2.11) and (2.32) in this context). This space is *not* the four-dimensional space (called A_4 in [*4*]) possessing a teleparallelism obtained by putting $\omega_{ik} = 0$ (see eq. (2.21) with $\overline{\omega}_{ik} = -\tau_{ik} \neq 0$ in eq. (2.11)) in which case the structural equations (2.1) and (2.2) assume in this local Lorentz gauge the form $d\theta^i = \Omega^i$, $\Omega_{ij} = 0$. The latter equation implies, in our notation, $\overline{R}_{klij} = - P_{klij}$ yielding $P_{\{kli\}j} = 0$. The Bianchi identities in the teleparallel case read $d\Omega^i = 0$, i. e. (compare eqs. (2.38′).

$\partial_{\{i} S_{jk\}}{}^l - S_{\{ij}{}^t S_{k\}t}{}^l = 0$, which is identically satisfied by $S_{jk}{}^l = 2\Omega_{jk}{}^l$ with $\Omega_{jk}{}^l$ as given by eq. (2.15). In fact, each term on the left-hand side of these equations is separately zero since $\Omega_{\{ij}{}^t\Omega_{k\}t}{}^l = 0$ as a consequence of the Jacobi identities,

$$[[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] = 0 ,$$

with $[e_i, e_j] = - 2\Omega_{ij}{}^s e_s$ (see eqs. (2.6′)).

relativity as discussed by Kimura [*22*]. In the present case the r.-h. s. of the divergence equation for the current $^*J_s(\phi)$ is given by a pseudoscalar term constructed from the torsion of the underlying space-time geometry.

We shall refer to eqs. (4.22) and to their local (or T_4) versions (4.25) or (4.27) as to the current-torsion-equations and now proceed to study further the conservation laws implied by the field equations and the Bianchi identities.

It is well known that eqs. (4.4) imply the conservation laws

$$\overline{\nabla}^{\mu}\overline{T}_{\mu\nu} = 0\,, \tag{4.30}$$

which can, because of the symmetry of $\overline{T}_{\mu\nu}$ in μ, ν, also be written as

$$\nabla^{\mu}\overline{T}_{\mu\nu} = 0\,. \tag{4.31}$$

Using eq. (2.52) and the familiar formula for the commutator of two covariant derivatives valid in Riemannian geometry

$$[\overline{\nabla}_{\lambda}, \overline{\nabla}_{\sigma}]K_{\mu\nu\rho} = -\,\overline{R}_{\lambda\sigma\mu}{}^{\varkappa}K_{\varkappa\nu\rho} - \overline{R}_{\lambda\rho\nu}{}^{\varkappa}K_{\mu\varkappa\rho} - \overline{R}_{\lambda\sigma\rho}{}^{\varkappa}K_{\mu\nu\varkappa}\,, \tag{4.32}$$

it is easy to derive with the help of eqs. (2.57) the equations

$$\overline{\nabla}^{\lambda}P_{[\mu\lambda]} = 0\,. \tag{4.33}$$

Combining this with eqs. (4.17) one obtains furthermore

$$\overline{\nabla}^{\lambda}P_{(\mu\nu)} - K_{\mu}{}^{\sigma\rho}P_{[\sigma\rho]} = 0\,. \tag{4.34}$$

Eqs. (4.33) imply

$$\overline{\nabla}^{\lambda}T_{[\mu\lambda]}(\phi) = 0\,, \tag{4.35}$$

(compare eqs. (4.6)), while eqs. (4.34) and (4.5) combine to yield, again using (4.6),

$$\begin{aligned} -\frac{1}{2}\partial_{\mu}P &= \varkappa(\overline{\nabla}^{\lambda}T_{(\mu\lambda)}(\phi) - K_{\mu}{}^{\rho\sigma}T_{[\rho\sigma]}(\phi)) \\ &= \varkappa\nabla^{\lambda}T_{\mu\lambda}(\phi) \end{aligned} \tag{4.36}$$

which is seen to be identical with eqs. (4.16) using $-\,P = \varkappa T(\phi)$, a relation following from (4.5).

Turning now to the currents it is easy to show from eq. (4.8) that the classical part in conserved according to the equations

$$\overline{\nabla}^{\rho}\overline{J}_{\rho\mu\nu} = 0\,. \tag{4.37}$$

The divergence relation for $J_{\rho\mu\nu}(\phi)$ is most conveniently expressed in terms of the axial current $^*J_{\rho}$ and is given by eq. (4.29) in the T_4 case. For a general U_4 one obtains

$$\nabla^{\mu *}J_{\mu}(\phi) = \overline{\nabla}^{\mu *}J_{\mu}(\phi) = \frac{2}{3\varkappa}\left\{\, \eta^{\mu\nu\rho\varkappa}(\overline{\nabla}_{\mu}{}^*K_{\nu})(\overline{\nabla}_{\rho}{}^*K_{\varkappa}) + \overline{\nabla}_{\mu}(\overline{R}^{\mu\nu *}K_{\nu}) \,\right\}. \tag{4.38}$$

In deriving this result use has been made of the formula (4.32) applied here to $^*K_{\rho}$ as well as of the cyclic identities (2.37′), the field equations (4.4), and the conservation laws (4.30). Notice the interesting coupling between Riemannian and torsion quantities appearing in this equation in the form $\overline{\nabla}_\mu(\overline{R}^{\mu\nu}{}^*K_\nu)$. With the help of eqs. (4.23) as well as the contracted quantities $^*P$ obtained from them, i. e.

$$^*P = {}^*P_\mu{}^\mu = -\,6\overline{\nabla}^\rho{}^*K_\rho\,, \tag{4.39}$$

one derives for this last term on the r.-h. s. of (4.38) the expression

$$2\overline{\nabla}_\mu(\overline{R}^{\mu\nu}{}^*K_\nu) = \varkappa\left[\,\overline{T}^{\mu\nu}{}^*P_{\mu\nu} - \frac{1}{6}\,\overline{T}{}^*P - (\partial^\mu\overline{T}){}^*K_\mu\,\right]. \tag{4.40}$$

If there is no matter present in the geometry which is distributed in classical form one falls back to the PCAC-type anomaly equation (4.29).

b) Field equations on the H-bundle over U_4.

Let us finally define in the second part of this section a totally antisymmetric current $J_{ijk}(\phi)$ by a bilinear expression in the scalar field $\phi(x;\tilde{X})$ which was introduced in Sect. III. We end this section by establishing a Poincaré gauge invariant wave equation on the soldered H-bundle over U_4 which is to be satisfied by $\phi(x;\tilde{X})$.

A totally antisymmetric bilinear source current for the torsion tensor on U_4 is obtained by integrating over the local fiber $F_x = H_x$ an antisymmetric density constructed in terms of the generators of the Poincaré group associated with the homogeneous space H (compare eqs. (3.26) ([11])).

$$J_{ijk}(\phi) = \frac{1}{R_0^N}\int_{H_x}\phi^*(x;\tilde{X})\tilde{M}_{\{ij}\tilde{P}_{k\}}\phi(x;\tilde{X})d\mu(\tilde{X})\,. \tag{4.41}$$

For dimensional reasons we have introduced here a factor R_0^{-N} with R_0 being an elementary length parameter. Choosing for the ϕ-field the canonical length dimension ([12]) $[l^{-1}]$ the current $J_{ijk}(\phi)$ defined by (4.41) possesses the length dimension $[l^{-3}]$ as required by the field equations (4.2) if $\varkappa$ is regarded as a dimensionless coupling constant.

Let us immediately add concerning the definition (4.41) that the $\tilde{M}_{ij}$ can be replaced by the $\tilde{S}_{ij}$ since

$$\tilde{L}_{\{ij}\tilde{P}_{k\}} \equiv 0\,. \tag{4.42}$$

The internal spin-type degrees of freedom described by the coordinates $\tilde{y}$ are thus seen to be essential. This has, for example, the consequence that

([11]) Remember that we demanded a Poincaré invariant measure on H to exist. We denote it by $d\mu(\tilde{X})$. $N = \dim H$.

([12]) This is justified by the procedure adopted below of taking a flat-space-limit of the formalism presented in this paper. ϕ then reduces to a Klein-Gordon field.

basing this theory on the affine tangent bundle, $T_A(U_4)$, would not be possible since because of eqs. (4.42) no completely antisymmetric source current can be constructed in this case. It is, therefore, necessary to go to higher dimensional homogeneous spaces of the Poincaré group with $N > 4$ in order to bring the generators $\tilde{S}_{ij}$ into the game. They are required for the construction of a fully antisymmetric source current. This is in parallel with Lurçat's conjecture that spin should play a dynamical role. Eq. (4.41) thus defines what one could call a *spin-translational gauge current*. Going now over to the dual current (4.20) one sees at once that $^*J^s(\phi)$ has the form

$$^*J^s(\phi) = -\frac{1}{R_0^N}\int_{H_x} \phi^*(x;\tilde{X})\tilde{W}^s\phi(x;\tilde{X})d\mu(\tilde{X}, \tag{4.43}$$

where $\tilde{W}^s$ is the Pauli-Lubanski operator of the Poincaré group associated with the homogeneous space H.

Performing a Poincaré gauge transformation on the soldered H-bundle the distribution in the variables $\tilde{X} = \tilde{x}, \tilde{y}$ described by the ϕ-field is transformed according to

$$\hat{\phi}(x;\hat{\tilde{X}}) = \phi(x;\tilde{X})\,|_{\tilde{X}=g^{-1}\hat{\tilde{X}}}, \tag{4.44}$$

where $\tilde{X} = g^{-1}\hat{\tilde{X}}$, with $g \in ISO(3,1)$, is a short-hand notation for $\tilde{x} = \Lambda^{-1}(x)(\hat{\tilde{x}} - a(x))$, being the transformation in the M_4 part, and $\tilde{y} = y(\hat{\tilde{y}})$, being the transformation in the S part of H. Using eq. (4.44) and the Poincaré invariance of the measure the current $J_{ijk}(\phi)$ is seen to transform as a covariant local Lorentz tensor of third rank under Poincaré gauge transformations, i. e.

$$\hat{J}_{ijk}(\hat{\phi}) = [\Lambda^{-1}(x)]^p{}_i[\Lambda^{-1}(x)]^q{}_j[\Lambda^{-1}(x)]^s{}_k J_{pqs}(\phi)\,. \tag{4.45}$$

Correspondingly, $^*J^s(\phi)$ transforms as a local (latin indexed) axial vector under Poincaré gauge transformations. This is in accord with the required transformation character of the source terms in the corresponding field equations (compare the latin indexed forms of eqs. (4.14) or (4.22)).

The most difficult question of our analysis is the choice of a Poincaré gauge invariant field equation for $\phi(x;\tilde{X})$. Here a certain amount of guessing and arguing in analogies is unavoidable. We shall postulate an equation for $\phi(x;\tilde{X})$ which goes over into the Klein-Gordon equation in a flat-space-limit, and which goes over into an equation proposed by Penrose [*23*] and Chernikov and Tagirov [*24*] [*25*] in the V_4 limit ([13]).

We start by considering a generally covariant and Poincaré gauge covariant set of equations of second order of the form ([14]).

$$\overline{D}_\mu D_\nu\phi(x;\tilde{X}) = (D_\mu D_\nu - \Gamma_{\mu\nu}{}^\rho D_\rho)\phi(x;\tilde{X}) = M_{\mu\nu}\phi(x;\tilde{X})\,. \tag{4.46}$$

([13]) Compare also ref. [*26*] in this context.

([14]) In subsequent formulae we shall suppress the arguments of ϕ.

The bar on the first differentiation symbol is meant to indicate that the generally covariant differential of the space-time vector quantity $D_\nu\phi$ is to be taken in computing the second Poincaré gauge covariant derivative. $D_\nu\phi$ was defined in eq. (3.32). On the r.-h. s. of (4.46) an arbitrary second rank tensor $M_{\mu\nu}$ with dimension $[l^{-2}]$ defined on U_4 should appear (not one defined on the H-bundle over U_4). The only candidates at our disposal are the tensors $g_{\mu\nu}$ and $R_{\mu\nu} = \bar{R}_{\mu\nu} + P_{\mu\nu}$. Splitting eqs. (4.46) into a symmetrical and an antisymmetrical part in μ, ν one, therefore, has the equations

$$\frac{1}{2}(\bar{D}_\mu D_\nu + \bar{D}_\nu D_\mu)\phi = \alpha g_{\mu\nu}\phi + \beta R_{(\mu\nu)}\phi\,, \tag{4.47}$$

$$\frac{1}{2}(\bar{D}_\mu D_\nu - \bar{D}_\nu D_\mu)\phi = \beta P_{[\mu\nu]}\phi\,, \tag{4.48}$$

with α being a constant of dimension $[l^{-2}]$ and β a number. Contraction of eqs. (4.47) with $g^{\mu\nu}$ yields

$$\bar{D}^\mu D_\mu\phi = (g^{\mu\nu}D_\mu D_\nu - g^{\mu\nu}\bar{\Gamma}_{\mu\nu}{}^\rho D_\rho)\phi = 4\alpha\phi + \beta R\phi\,. \tag{4.49}$$

If this equation is required to reduce in a flat-space-limit to the Klein-Gordon equation α is seen to be given by $\alpha = -m^2c^2/4\hbar^2$ with m denoting the mass associated with the ϕ-field. Let us, moreover, require that in a V_4 limit (i. e. disregarding the Poincaré gauge degrees of freedom and putting the torsion tensor to zero) eq. (4.49) is identical with the scalar wave equation in a Riemannian space-time discussed in refs. [*23*]-[*25*] which has the property of being conformally invariant for $m = 0$. Making these assumptions β must be equal to $-1/6$, and (4.49) takes the final form (using again units in which $c = 1$ and $\hbar = 1$).

$$\left(\tilde{\Box} + \frac{1}{6}R + m^2\right)\phi = 0\,, \tag{4.50}$$

where

$$\tilde{\Box} = D^\mu D_\mu - g^{\mu\nu}\bar{\Gamma}_{\mu\nu}{}^\rho D_\rho = \frac{1}{\sqrt{-g}}\, D_\mu g^{\mu\nu}\sqrt{-g}D_\nu \tag{4.51}$$

is the Poincaré gauge invariant d'Alembertian on the H-bundle over space-time, and $R = \bar{R} + P$ is the curvature scalar in the U_4 with $P = -6{}^*K_s{}^*K^s$. Thus the term $\frac{1}{6}R + m^2$ in eq. (4.50) acts like an effective mass squared being associated with regions where the geometry is different from a flat Minkowski space-time.

We now turn to the equation antisymmetric in μ, ν. Using the formula for the commutator of two Poincaré gauge covariant derivatives obtainable from eqs. (3.32) and (3.28) (15), i. e.

$$[D_i, D_k]\phi = \left\{ \frac{i}{2} R_{ikjl}\tilde{M}^{jl} - i\tau_{ik}{}^{j}\tilde{P}_j - 2\Omega_{ik}{}^{s}D_s \right\} \phi , \tag{4.52}$$

eqs. (4.48) read when written in latin indexed form using, moreover, eqs. (2.1) (16):

$$\left\{ \frac{i}{2} R_{ikjl}\tilde{M}^{jl} - i\tau_{ik}{}^{j}\tilde{P}_j - S_{ik}{}^{s}D_s \right\} \phi = -\frac{1}{3} P_{[ik]}\phi . \tag{4.53}$$

Introducing the two-form

$$\Pi = \frac{1}{2}\theta^i \wedge \theta^k P_{[ik]} = -\frac{1}{2}\theta^i \wedge \theta^k \overline{\nabla}^j K_{ikj} , \tag{4.54}$$

eqs. (4.53) can now compactly be written as

$$i\left[\frac{1}{2}\Omega_{jl}\tilde{M}^{jl} - \tau^j\tilde{P}_j\right]\phi = \Omega^j D_j\phi - \frac{1}{3}\Pi\phi . \tag{4.55}$$

The l.-h. s. of this equation is determined by the Lie algebra valued curvature two-form associated with the Poincaré group (17)

$$\Sigma = d\Gamma + \frac{i}{2}[\Gamma, \Gamma] = \frac{1}{2}\Omega_{jl}\tilde{M}^{jl} - \tau^j\tilde{P}_j \tag{4.56}$$

with Γ being the ISO(3, 1) Lie algebra valued connection form on the H-bundle defined in eq. (3.34). Thus eq. (4.55) could be written in terms of Lie algebra valued forms operating on ϕ as

$$\Sigma\phi = \frac{i}{3}\Pi\phi - i\Omega^j D_j\phi . \tag{4.57}$$

The first term on the r.-h. s. of this equation is associated with the divergence of the torsion field, the second term is associated with the torsion field itself. Eq. (4.57) ties together the internal motion (motion in the fiber) and the translational motion on the base space of the H-bundle over U_4. It should be regarded as an identity constraining the field ϕ which is required to obey the second order field equation (4.50).

(15) Compare also eqs. (2.15) and (3.20).

(16) Written in components Eq. (2.1) reads $\frac{1}{2} S_{ik}{}^{s} = \Omega_{ik}{}^{s} + \Gamma_{[ik]}{}^{s}$.

(17) If $\Gamma = \Gamma_A T^A$ is a Lie algebra valued form with Γ_A being a set of forms and T^A a basis for the Lie algebra then $[\Gamma, \Gamma]$ is defined by $[\Gamma, \Gamma] = \Gamma_A \wedge \Gamma_B(T^A T^B - T^B T^A)$. For the Lie algebra valued one-form (3.34) one has $\frac{1}{2}[\Gamma, \Gamma] = \Gamma \wedge \Gamma$.

V. DISCUSSION

In the framework of a gauge description of gravity based on the Poincaré group we introduced in this paper an additional set of field equations connecting, besides Einstein's equations, matter and geometric quantities. These equations relate the divergence of the contracted dual curvature tensor, $^*R_{\mu\nu} = {}^*P_{\mu\nu}$, in a Riemann-Cartan space-time to the dual current vector, $^*J_\mu$. They are called the current-torsion-equations since the $^*R_{\mu\nu}$ are expressible in terms of the dual of the torsion tensor. Their form is similar to the corresponding equations in a Yang-Mills theory (compare eqs. (4.2) or (4.22)). Let us point out that the study of these nonlinear differential equations, with a source current specified, is interesting in itself even without connecting them to the dynamics of an underlying ϕ-field as was done in Sect. IV (subsection (*b*)) of this paper. Such an investigation will have to be carried out in another context.

The short distance quantum aspect of matter was supposed to be represented by a scalar Poincaré gauge field $\phi(x; \tilde{X})$ defined on a soldered fiber bundle over a Riemann-Cartan space-time possessing a homogeneous space of the Poincaré group as fiber. For short this bundle was called the soldered H-bundle over space-time. Without choosing a particular homogeneous space H from the classification given by Finkelstein [*12*] and by Bacry and Kihlberg [*13*] we defined a completely antisymmetric source current in terms of a bilinear integral expression in the ϕ-fields. If one requires the space H to be able to carry integer as well as half-integer spin representations and to possess a Poincaré invariant measure the dimension N of H should at least be 8. There are various possibilities open now for H having dimension 8, 9 or 10, yielding the corresponding soldered H-bundles over space-time (the last one in the sequence being the affine frame bundle, $L_A(U_4)$, over U_4). It is probably not an easy matter to choose, on physical grounds, a particular one among these bundles. The solution to this problem has also to do with the nature of the interaction described by the current-torsion-equations, in particular, if $^*J_\mu(\phi)$ is related to the dynamics of a Poincaré gauge field defined on the H-bundle as is proposed in this paper. The definition (4.43) of the current contained an elementary length parameter R_0 of unknown size. We imagine that R_0 can be determined with the dimensional coupling constant $\bar{\varkappa}$ put equal to one, i. e. with R_0 measuring the strength of the coupling. Let us, moreover, regard R_0 for a moment to have a particular fixed small value. Since the strength of the source current is scaled by a power of R_0 the strength of $^*J_\mu(\phi)$ is enhanced in going to a bundle with higher dimensional fiber providing the arena for a more complicated internal dynamics associated with the operators $\tilde{S}_{ik}$. (Remember that the soldering to the base space through the M_4 part of the fiber is always the same). Hence the feed-back between the current and the geometry i. e. the strength of the interaction described by the current-torsion-equations is expected to grow in going over to higher dimensional H-bundles. It, therefore, seems possible – if not likely – that the current-torsion-equations obtained here by a chain of arguments aimed at giving an extended geometric description of gravitation modified by the presence of torsion describes, in fact, a nongravitational interaction albeit in a geometrical manner.

Let us conclude by pointing out that one can study various special cases of the U_4 theory presented in this paper. Considering an underlying metrical space-time, V_4, and disregarding the internal Poincaré gauge degrees of freedom described by $\phi(x; \tilde{X})$ one recovers Einstein's theory for the classical gravitational field. Specializing, on the other hand, to a metrically flat space-time possessing, however, a nonzero torsion — i. e. considering a Weitzenböck space-time T_4 — one can study the current-torsion-equations in their simplest form (4.27) unconnected with general relativistic phenomena. An interesting side aspect is the axial vector anomaly equation (4.29) following from eqs. (4.27). Finally, one can specialize the geometry to a M_4 i. e. obtain the residual dynamics — or rather kinematics — of special relativity.

Regarding the representation of matter in this theory we here introduced an energy momentum tensor $T_{\mu\nu}$ *and* a current tensor $J_{\mu\nu\rho}$. Both were split in the course of the derivations of Sect. IV into a classical part referring to matter distributed in classical form, and into a term related to the Poincaré gauge field $\phi(x; \tilde{X})$ representing the quantum mechanical or wave function aspect of matter in this framework. While the classical part of the current tensor is a derived quantity in Einstein's theory the quantum mechanical part of the energy momentum tensor is considered here as a derived quantity expressed in terms of the torsion quantities. The classical parts $\overline{T}_{\mu\nu}$ and $\overline{J}_{\mu\nu\rho}$ are characterized by their symmetry properties $\overline{T}_{[\mu\nu]} = 0$ and $\overline{J}_{\{\mu\nu\rho\}} = 0$, while for the quantum mechanical or ϕ-parts of the energy momentum and current tensors the antisymmetry of their components plays the crucial role such that the formulation of the equations in this case can best be given in terms of the dual quantities.

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Note added in proof

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