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Section A :
Physique théorique.

Zero mass, 2 dimensional, real time Sine Gordon model without u. v. cut-offs

by

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ABSTRACT. — We prove the existence of a limit for matrix elements of the time evolution operator in the interaction picture, for a zero mass two dimensional Sine Gordon model in a space time box when the ultraviolet cut-off is removed, and for $\alpha^2 < 2\pi$.

RÉSUMÉ. — Nous prouvons l'existence d'une limite pour les éléments de matrice de l'opérateur d'évolution temporel dans le schéma d'interaction, pour le modèle de Sine Gordon de masse nulle à deux dimensions dans une boîte d'espace-temps quand la régularisation ultraviolette est supprimée, et ceci pour $\alpha^2 < 2\pi$.

§ 1. INTRODUCTION

In a previous paper, hereafter refered as [*1*], we have shown that a generalized Poisson process [*2*] can be associated to some relativistic (we mean real time) models. We mentioned that it applies to trigonometric

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interactions [3], and in particular to the Sine-Gordon model [4], which is defined by the interaction (see section 2 for notations):

$$\mathrm{H_I} = \int_\Lambda dx : \cos(\alpha\phi_\varkappa(x)) :_\mu ,$$

$\varkappa$ is an ultraviolet cut-off and Λ a space cut-off.

This model has some broader generality than it could appear since it is equivalent to the massive Thirring model [5], plus Yukawa interaction (see eg. [6] [7]).

Trigonometric interactions have been studied extensively in the Euclidean region (see e. g. [8] to [12]), showing that at least for $\alpha^2 < 4\pi$ the two dimensional Euclidean model exists [13] [14] in the limit $\varkappa \to 0, \Lambda \nearrow \mathrm{R}$.

Then the Osterwalder-Schrader reconstruction theorem (see eg. [15] [17]) allows to assert the existence of a relativistic model.

In this paper, it is this detour that we want to avoid by using Poisson measures instead of Gaussian measures, which can be no longer used since of the characteristics of the relativistic propagators.

Number of estimates as well as the general strategy of [13] and [14] can be adapted in the zero mass case, where we use essentially the existence of the grand partition function of a neutral Coulomb gas in two dimensions [16].

The lack of comparison between the relativistic free propagators for different masses does not allow to treat the $m \neq 0$ model, in contrast to the Euclidean case and also make necessary a further estimate that is not needed for the Euclidean model. This reduces our existence proof to $\alpha^2 < 2\pi$.

In the second section we briefly describe the notations and the main results of [1]. We also mention the definition of the Poisson measure through the characteristic functional of the Poisson process, naturally associated with trigonometric interaction models.

The third section deals with the limit $m = 0$, whereas in the fourth section we derive the main result of this paper, namely the existence of the limit:

$$\lim_{\varkappa\to 0} \left(\psi_{fg} e^{i\mathrm{H}_0 t} e^{-i(\mathrm{H}_0 + \mathrm{H}_\mathrm{I}^{\Lambda\varkappa})t} \psi_{fg}\right)$$

where H_0 is the free relativistic Hamiltonian and ψ_{fg} are coherent states.

§ 2. THE SINE GORDON MODEL WITH CUT-OFF IN TWO DIMENSIONAL SPACE TIME

In this section we define the notations and recall the results of [1] which are needed in the sequel.

ϕ is a scalar neutral relativistic Bose field of mass m in two space time dimensions.

Λ is a finite box in R viz. $\Lambda = \{ x ; | x | < \Lambda/2 \}$.

The sine Gordon model is defined by the interaction:

$$H_I = \lambda \int_\Lambda dx : \cos(\alpha \phi_\varkappa(x)) :_\mu , \tag{2.1}$$

where

$$\phi_\varkappa(x) = \int_R dy \phi(y, 0) \chi_\varkappa(x - y) , \tag{2.2}$$

is the zero time field with ultraviolet cut-off, viz. $\chi_\varkappa$ is the function:

$$\chi_\varkappa(x) = \frac{1}{\varkappa} \chi\left(\frac{1}{\varkappa} x\right), \tag{2.3}$$

where χ is a function of D(R) which is:

$$\text{symmetric positive} \tag{2.4}$$

$$\chi(x) = 0 , \quad \text{if} \quad |x| \geq 1 . \tag{2.5}$$

$$\int_R \chi(x) dx = 1 . \tag{2.6}$$

Moreover, the Wick order in (2.1) is taken with respect to a mass μ, which is strictly positive (see eg. [17]):

$$: \exp(i\alpha\phi_\varkappa(x)) :_\mu = \exp(i\alpha\phi_\varkappa(x)) \exp\left(\frac{\alpha^2}{4} \int_R dk \frac{|\tilde{\chi}_\varkappa(k)|^2}{\sqrt{k^2 + \mu^2}}\right), \tag{2.7}$$

where $\tilde{\chi}_\varkappa$ is the Fourier transform of $\chi_\varkappa$:

$$\tilde{\chi}_\varkappa(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \chi_\varkappa(x) dx .$$

We notice that [17] [7]:

$$\lambda \int_\Lambda dx : \cos(\alpha\phi_\varkappa(x)) :_\mu = \lambda(m, \mu, \varkappa) \int_\Lambda dx : \cos(\alpha\phi_\varkappa(x)) :_m , \tag{2.8}$$

when $m > 0$, with:

$$\lambda(m, \mu, \varkappa) = \lambda \exp\left(\frac{\alpha^2}{4} \int_R dk \, |\chi_\varkappa(k)|^2 \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{\sqrt{k^2 + m^2}}\right)\right). \tag{2.9}$$

Finally H_0 is the relativistic free Hamiltonian corresponding to the mass m.

With these notations one has the following, [2]:

PROPOSITION (2.10). — The application:

$$f \in \mathscr{S}_R(R) \to C(f)$$
$$= \exp\left\{|\lambda| \left(\int_0^t dt \int_\Lambda dx \left\{ \cos\left(\alpha \int_R \int_R d\xi d\zeta \Delta_R^{(m)}(x - \zeta + \xi ; t) \chi_\varkappa(\xi) f(\zeta) \right) - 1 \right\} \right) \right\}$$

is a normalized, continuous, positive type function.

$\Delta_{\mathrm{R}}^{(m)}(x, t) = \dfrac{1}{2\pi} \displaystyle\int \dfrac{dke^{ikx}}{\sqrt{k^2 + m^2}} \Theta(t) \sin (t\sqrt{k^2 + m^2})$ is the usual retarded function.

As a corolary, by Minlos theorem (see eg. [*19*] [*17*]), there exists a positive measure $p_{\Lambda|\lambda|}$ on $\mathscr{S}'_{\mathrm{R}}(\mathrm{R})$ such that:

$$\mathrm{C}(f) = \int_{\mathscr{S}'_{\mathrm{R}}(\mathrm{R})} dp_{\Lambda|\lambda|}(\varphi) \exp (i\varphi(f)) . \tag{2.11}$$

Moreover:

$$\mathrm{C}(f) = \mathrm{E}[\exp (i\Phi_{t=0}(f))] , \tag{2.12}$$

where Φ is a generalized Poisson process.

Actually, in [*1*] we have proved that the generalized Poisson process can be realized as follows: $\Omega = \bigsqcup_n \Omega_n$,

$$\Omega_0 = \{ \omega_0 \} , \tag{2.13}$$

$$\Omega_n = \{ \omega = (n, t_1 \dots t_n, \varepsilon_1 \dots \varepsilon_n, x_1 \dots x_n) \} , \qquad n > 0 , \tag{2.14}$$

where $0 < t_1 < t_2 \dots < t_n < \mathrm{T}$, $\varepsilon_i = \pm 1$ and $x_i \in \Lambda$.

Let $\mathrm{V}^{(n)}_{a_i \eta_i \mathscr{B}_i} = \{ (n, t_i, \varepsilon_i, x_i, t_i \in a_i, \varepsilon_i = \eta_i, x_i \in \mathscr{B}_i \}$, where the a_i's are disjoint ordered Lebesgue measurable subsets of $[0, \mathrm{T}]$, and the $\mathscr{B}_i$'s Lebesgue measurable subsets of Λ. They generate a Borel σ-algebra $\mathscr{F}$. Then:

$$\mathrm{P}_{\Lambda|\lambda|}(\mathrm{V}^{(n)}_{a_i\eta_i\mathscr{B}_i}) = \frac{|\lambda|^n}{2^n} \prod_{i=1}^{n} |a_i| |\mathscr{B}_i| , \tag{2.15}$$

where the $|a_i|$ and $|\mathscr{B}_i|$ are the Lebesgue measure of a_i and $\mathscr{B}_i$ respectively, is a bounded positive measure on Ω, [*19*] [*1*].

The generalized process Φ can be written, [*1*]:

$$\Phi(x, t)(\omega = (n, t_i, \varepsilon_i, x_i)) = \alpha \sum_{i=1}^{n} \varepsilon_i \int_{\mathrm{R}} d\xi \Delta_{\mathrm{R}}^{(m)}(x - x_i - \xi ; t_i - t) \chi_x(\xi) , \tag{2.16}$$

and one has:

$$\mathrm{E}[\exp (i\Phi(f))] = e^{-|\lambda|\Lambda \mathrm{T}} \int_{\Omega} \mathrm{P}_{\Lambda|\lambda|}(d\omega) \{ e^{i\Phi(f)} \} (\omega) . \tag{2.17}$$

In [*1*] it was proved that the expectation value of the operator $\exp (i\mathrm{H}_0\mathrm{T}) \exp (- i(\mathrm{H}_0 + \mathrm{H}_\mathrm{I})\mathrm{T})$ in between coherent states:

$$\Psi_{fg} = \exp (i(\pi(f) - \phi(g)))\Omega_{\mathrm{F}} , \qquad f, g \in \mathscr{S}_{\mathrm{R}}(\mathrm{R}) , \tag{2.18}$$

where π is the conjugate momentum of ϕ and Ω_{F} the vacuum Fock state, is given by:

$$(\Psi_{fg} | \exp (i\mathrm{H}_0\mathrm{T}) \exp (- i(\mathrm{H}_0 + \mathrm{H}_\mathrm{I})\mathrm{T})\Psi_{fg}) = \int_{\Omega} \mathrm{P}_{\Lambda|\lambda|}(d\omega)\mathrm{F}_{xm}(\omega) , \tag{2.19}$$

where $F_{\varkappa m}$ is an integrable function on Ω, with the following structure:

$$F_{\varkappa m}(\omega) = F^{(1)}(\omega)F^{(2)}_{fg}(\omega)F^{(3)}_{\varkappa m}(\omega), \tag{2.20}$$

where:

$$F^{(1)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(\frac{-i\lambda}{|\lambda|}\right)^n, \tag{2.21}$$

$$\begin{aligned} F^{(2)}_{fg}(\omega &= (n, t_i, \varepsilon_i, x_i)) \\ &= \exp\left(i\alpha\sum_{i=1}^{n}\varepsilon_i\int_{\mathrm{R}}\int_{\mathrm{R}} d\xi d\zeta \chi_\varkappa(\xi)\Delta^{(m)}_{\mathrm{R}}(x_i - \xi - \zeta\,; t_i)g(\zeta)\right) \\ &\quad \exp\left(-i\alpha\sum_{i=1}^{n}\varepsilon_i\int_{\mathrm{R}}\int_{\mathrm{R}} d\xi d\zeta \chi_\varkappa(\xi)\partial_0\Delta^{(m)}_{\mathrm{R}}(x_i - \xi - \zeta\,; t_i)f(\zeta)\right), \end{aligned} \tag{2.22}$$

and:

$$\begin{aligned} F^{(3)}_{\varkappa m}(\omega &= (n, t_i, \varepsilon_i, x_i)) \\ &= \exp\left(\frac{n\alpha^2}{4}\int_{\mathrm{R}} |\tilde\chi_\varkappa(k)|^2\left(\frac{1}{\sqrt{k^2+\mu^2}} - \frac{1}{\sqrt{k^2+m^2}}\right)dk\right) \\ &\quad \exp\left(-\frac{\alpha^2}{4}\sum_{i\neq j}\varepsilon_i\varepsilon_j\int_{\mathrm{R}}\int_{\mathrm{R}} d\xi d\zeta \chi_\varkappa(x_i - \xi)\Delta^{(m)}_{\mathrm{F}}(\xi - \zeta, t_i - t_j)\chi_\varkappa(x_j - \zeta)\right), \end{aligned} \tag{2.23}$$

where $\Delta^{(m)}_{\mathrm{F}}$ is the Feynman propagator:

$$\Delta^{(m)}_{\mathrm{F}}(x, t) = \frac{1}{2\pi}\int_{\mathrm{R}} e^{ikx}\frac{e^{-i|t|\sqrt{k^2+m^2}}}{\sqrt{k^2+m^2}}dk. \tag{2.24}$$

$F^{(1)}$ does not depend on the cut-off, and $F^{(2)}$ is of modulus one. $F^{(3)}$ depends explicitly on the cut-off and becomes singular if $\varkappa$ tends to zero. In the next section we shall control its singularity.

§ 3. THE LIMIT $m = 0$

Our goal is to prove that at least for small α, we can deal with the singularity which appears in $\Delta^{(m)}_{\mathrm{F}}$. This will be done only in the case $m = 0$ and in this section we shall prove that the limit $m = 0$ exists in (2.19).

PROPOSITION (3.1). — One has the following limit:

$$\lim_{m\to 0}\int_\Omega P_{\Lambda|\lambda|}(d\omega)F^{(1)}(\omega)F^{(3)}_{\varkappa m}(\omega) = \int_\Omega P_{\Lambda|\lambda'|}(d\omega)F^{(1)}(\omega)F^{(3)}_{\varkappa}(\omega),$$

where:

$$\lambda' = \lambda\left(\frac{2}{\mu}e^{-\gamma}\right)^{\frac{\alpha^2}{4\pi}}, \tag{3.2}$$

γ is the Euler's constant, and:

$$
\begin{aligned}
&F_{\varkappa}^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) \\
&= \exp\left(\frac{n\alpha^2}{4}\int_{\mathbb{R}} dk(|\tilde{\chi}_\varkappa(k)|^2 - \frac{1}{2\pi}\Big)\Big(\frac{1}{\sqrt{k^2+\mu^2}} - \frac{1}{|k|}\right) \\
&\quad \exp\left(\frac{\alpha^2}{4\pi}\sum_{i\neq j}\varepsilon_i\varepsilon_j\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi d\zeta\chi_\varkappa(x_i - \xi)\ln((\xi-\zeta)^2 - (t_i - t_j)^2)^{1/2}\chi_\varkappa(x_j - \zeta)\right), \\
&\qquad \text{if } \sum_{i=1}^{n}\varepsilon_i = 0, \qquad = 0 \quad \text{otherwise} \qquad (3.3)
\end{aligned}
$$

with the usual determination of $\ln(z)$.

Proof. — First we prove the pointwise convergence of $F_{\varkappa m}^{(3)}$ to $F_{\varkappa}^{(3)}$. Indeed if we remark that (see [20] for notations):

$$
\begin{aligned}
\Delta_F^{(m)}(x, t) &= \frac{1}{\pi}K_0(im\sqrt{t^2 - x^2}), \qquad 0 < |x| < |t| \\
&= \frac{1}{\pi}K_0(m\sqrt{x^2 - t^2}), \qquad 0 < |t| < |x|,
\end{aligned} \qquad (3.4)
$$

where:

$$
K_0(z) = -\ln\left(\frac{cz}{2}\right)I_0(z) + 2\sum_{n\geq 1}\frac{1}{n}I_{2n}(z), \qquad (3.5)
$$

where $\ln(c) = \gamma$, the Euler's constant, then:

$$
\begin{aligned}
&F_{\varkappa m}^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(\frac{m}{\mu}\right)^{n\alpha^2/4\pi}\left(\frac{mc}{2}\right)^{\alpha^2/4\pi\left(\left(\sum_i^n\varepsilon_i\right)^2 - n\right)} \\
&\exp\left(\frac{n\alpha^2}{4}\int_{\mathbb{R}} dk(|\tilde{\chi}_\varkappa(k)|^2 - \frac{1}{2\pi}\Big)\Big(\frac{1}{\sqrt{k^2+\mu^2}} - \frac{1}{\sqrt{k^2+m^2}}\right) \\
&\exp\left(\frac{\alpha^2}{4\pi}\ln\left(\frac{mc}{2}\right)\sum_{i\neq j}\varepsilon_i\varepsilon_j\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi d\zeta\chi_\varkappa(x_i - \xi)A_0(m, \xi, \zeta, t_i, t_j)\chi_\varkappa(x_j - \zeta)\right) \qquad (3.6) \\
&\exp\left(-\frac{\alpha^2}{4\pi}\sum_{i\neq j}\varepsilon_i\varepsilon_j\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi d\zeta\chi_\varkappa(x_i - \xi)\sum_{k=1}^{\infty}A_{2k}(m, \xi, \zeta, t_i, t_j)\chi_\varkappa(x_j - \zeta)\right) \\
&\exp\left(\frac{\alpha^2}{4\pi}\sum_{i\neq j}\varepsilon_i\varepsilon_j\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi d\zeta\chi_\varkappa(x_i - \xi)\ln(\sqrt{(\xi-\zeta)^2 - (t_i - t_j)^2})\right. \\
&\qquad\qquad \left.(A_0(m, \xi, \zeta, t_i, t_j) + 1)\chi_\varkappa(x_j - \zeta)\right),
\end{aligned}
$$

where:

$$
A_0(m, \xi, \zeta, t_i, t_j) = I_0(m\sqrt{(\xi-\zeta)^2 - (t_i - t_j)^2}) - 1, \qquad (3.7)
$$

$$
A_{2k}(m, \xi, \zeta, t_i, t_j) = \frac{2}{k}I_{2k}(m\sqrt{(\xi-\zeta)^2 - (t_i - t_j)^2}), \qquad (3.8)
$$

with the usual definition of the square root.

The result follows from the well known behaviour of the functions $I_p(z)$ for small z (see eg. [20]).

Secondly we remark that for any collection $\{ f_i \}_{i=1...n}$ of functions of $D_R(R)$:

$$\sum_{i,j=1}^{n} \int_R \int_R d\xi d\zeta f_i(\xi) \operatorname{Re}(\Delta_F^{(m)}(\xi - \zeta ; t_i - t_j)) f_j(\zeta) \geq 0 . \tag{3.9}$$

Hence we have the inequality:

$$| F^{(3)}_{\varkappa m}(\omega = (n, t_i, \varepsilon_i, x_i)) | \leq \exp\left(\frac{n\alpha^2}{4} \int_R \frac{dk \, | \tilde{\chi}_\varkappa(k) |^2}{\sqrt{k^2 + \mu^2}}\right). \tag{3.10}$$

This last expression defines a function on Ω which is $P_{\Lambda|\lambda'|}$ integrable.

Finally the proposition follows from Lebesgue's dominated convergence theorem applied to the integral with respect to the bounded positive measure $P_{\Lambda|\lambda'|}$.

§ 4. REMOVAL OF ULTRAVIOLET DIVERGENCIES

In this section we remove the ultraviolet cut-off by letting $\varkappa$ going to zero, in the expression we have obtained in the previous section, for $m=0$:

$$(\Omega_F \,|\, \exp(iH_0T) \exp(-i(H_0 + H_I)T)\Omega_F) = \int_\Omega P_{\Lambda|\lambda'|}(d\omega) F^{(1)}(\omega) F^{(3)}_\varkappa(\omega) \tag{4.1}$$

where

$$\lambda' = \lambda \left(\frac{2}{\mu c}\right)^{\alpha^2/4\pi} \tag{4.2}$$

$$F^{(1)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(-\frac{i\lambda'}{|\lambda'|}\right)^n \tag{4.3}$$

$$\begin{aligned} &F^{(3)}_\varkappa(\omega = (n, t_i, \varepsilon_i, x_i)) \\ &= \exp\left(\frac{n\alpha^2}{4} \int_R dk \left(|d\tilde{\chi}_\varkappa(k)|^2 - \frac{1}{2\pi}\right)\left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|}\right)\right) \\ &\quad \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_R \int_R d\xi d\zeta \chi_\varkappa(x_i - \xi) C_F(\xi - \zeta ; t_i - t_j) \chi_\varkappa(x_j - \zeta)\right) \\ &\quad \text{if } \sum_{i=1}^{n} \varepsilon_i = 0 , \qquad = 0 , \quad \text{otherwise} . \end{aligned} \tag{4.4}$$

$$\begin{aligned} C_F(x, t) &= \frac{1}{2} \ln(t^2 - x^2) + \frac{i\pi}{2}, \qquad \text{if} \quad 0 < |x| < |t| , \\ &= \frac{1}{2} \ln(x^2 - t^2) , \qquad \text{if} \quad 0 < |t| < |x| . \end{aligned} \tag{4.5}$$

First we prove that (4.1) is uniformly bounded for $\varkappa \in]0, 1]$, indeed:

$$\left| \int_{\Omega} \mathrm{P}_{\Lambda|\lambda'|}(d\omega)\mathrm{F}^{(1)}(\omega)\mathrm{F}_{\varkappa}^{(3)}(\omega) \right| \leq \int_{\Omega} \mathrm{P}_{\Lambda+2\varkappa|\lambda'|}(d\omega)\mathrm{C}^{(1)}(\omega)\mathrm{I}^{(3)}(\omega), \quad (4.6)$$

where

$$\mathrm{C}^{(1)}(\omega = (n_i, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{n\alpha^2}{4}\int_{\mathrm{R}} dk\left(|\tilde{\chi}(k)|^2 - \frac{1}{2\pi}\right)\left(\frac{1}{\sqrt{k^2+\mu^2}} - \frac{1}{|k|}\right)\right), \quad (4.7)$$

$$\mathrm{I}^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{\alpha^2}{4\pi}\sum_{i \neq j} \varepsilon_i\varepsilon_j \operatorname{Re}\{ \mathrm{C}_{\mathrm{F}}(x_i - x_j; t_i - t_j) \}\right),$$

$$\text{if} \sum_{i=1}^{n} \varepsilon_i = 0, \qquad = 0, \quad \text{otherwise}, \quad (4.8)$$

which is an easy consequence of the Jensen's inequality, [*21*]. However,

$$\operatorname{Re} \mathrm{C}_{\mathrm{F}}(x_i - x_j, t_i - t_j) = \frac{1}{2}\ln(|x_i - x_j - t_i + t_j|) + \frac{1}{2}\ln(|x_i - x_j + t_i - t_j|). \quad (4.9)$$

Hence by the Cauchy-Schwartz inequality:

$$\int_{\Omega'} \mathrm{P}_{\Lambda+2\varkappa|\lambda'|}(d\omega)\mathrm{C}^{(1)}(\omega)\mathrm{I}^{(3)}(\omega) \leq \int_{\Omega'} \mathrm{P}_{\Lambda+2\varkappa|\lambda'|}(d\omega)\mathrm{C}^{(1)}(\omega)\mathrm{J}^{(3)}(\omega), \quad (4.10)$$

where:

$$\mathrm{J}^{(3)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp\left(\frac{\alpha^2}{4\pi}\sum_{i \neq j} \varepsilon_i\varepsilon_j \ln(|x_i - x_j - t_i + t_j|)\right),$$

$$\text{if} \sum_{i=1}^{n} \varepsilon_i = 0, \qquad = 0, \quad \text{otherwise}. \quad (4.11)$$

However, from the results of [*16*] and [*13*], for $\alpha^2 < 2\pi$, this is an integrable function with respect to $\mathrm{P}_{\Lambda+2\varkappa|\lambda'|}$.

Consequently we have:

PROPOSITION (4.12). — For $\alpha^2 < 2\pi$

$$\int_{\Omega} \mathrm{P}_{\Lambda|\lambda'|}(d\omega)\mathrm{F}^{(1)}(\omega)\mathrm{F}_{\varkappa}^{(3)}(\omega) < \mathrm{C}$$

independently of $\varkappa \in]0, 1]$ and:

$$F(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(- \frac{i\lambda'}{|\lambda'|} \right)^n \exp \left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j C_F(x_i - x_j, t_i - t_j) \right)$$

$$\text{if} \sum_{i=1}^{n} \varepsilon_i = 0, \qquad = 0, \quad \text{otherwise} \tag{4.12}$$

is $P_{\Lambda|\lambda'|}$ integrable.

This was a crucial step to remove the ultraviolet cut-off, indeed we can proceed along the same lines as in [*13*]. However, we have to control the factor,

$$\exp \left(\frac{n\alpha^2}{4} \int_R dk \left(|\tilde{\chi}_\varkappa(k)|^2 - \frac{1}{2\pi} \right) \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|} \right) \right) \tag{4.13}$$

This can be done by the following observation: for $\alpha^2 < 2\pi$,

$$\lim_{\varkappa \to 0} \int_\Omega P_{\Lambda|\lambda'|}(d\omega) K^{(1)}(\omega)(K^{(2)}(\omega) - 1) = 0 , \tag{4.14}$$

where

$$K^{(1)}(\omega = (n, t_i, \varepsilon_i, x_i))$$
$$= \exp \left(+ \frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_R \int_R \chi_\varkappa(x_i - \xi) C_F(\xi - \zeta, t_i - t_j) \chi_\varkappa(x_j - \zeta) \right) d\xi d\zeta ,$$

$$\text{if} \sum_{i}^{n} \varepsilon_i = 0, \qquad = 0, \quad \text{otherwise.} \tag{4.15}$$

$$K^{(2)}(\omega = (n, t_i, \varepsilon_i, x_i)) = \exp \left(\frac{n\alpha^2}{4} \int_R dk \left(|\tilde{\chi}_\varkappa(k)|^2 - \frac{1}{2\pi} \right) \left(\frac{1}{\sqrt{k^2 + \mu^2}} - \frac{1}{|k|} \right) \right) \tag{4.16}$$

Indeed, again by Jensen's inequality

$$\left| \int_\Omega P_{\Lambda|\lambda'|}(d\omega) K^{(1)}(\omega)(K^{(2)}(\omega) - 1) \right| \leq \int_\Omega P_{\Lambda + 2\varkappa|\lambda'|}(d\omega) I^{(3)}(\omega)(K^{(2)}(\omega) - 1) \tag{4.17}$$

where $I^{(3)}$ was defined in (4.8). However, the integrand can be easily majorized for $\alpha^2 < 2\pi$ by an integrable function, moreover $K^{(2)} - 1$ tends pointwise to zero for $\varkappa \to 0$.

Now we are able to proceed to prove the convergence of the above matrix element. Namely, one has (see [13]):

$$\int_\Omega \mathrm{P}_{\Lambda|\lambda'|}(d\omega)(\mathrm{F}(\omega) - \mathrm{G}_\varkappa(\omega)) = \frac{\alpha^2}{4\pi} \sum_{n \geq 0} \frac{1}{2n\,!}\left(-\frac{i\lambda'}{2}\right)^{2n} \int_0^{\mathrm{T}} dt_{2n} \dots \int_0^{\mathrm{T}} dt_1 \int_\Lambda dx_{2n}$$

$$\dots \int_\Lambda dx_1 \sum_{\substack{\varepsilon_i = \pm 1 \\ \Sigma\varepsilon_i = 0}} \int_0^1 ds \Big(\sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathrm{R}} \int_{\mathrm{R}} d\xi d\zeta \chi_\varkappa(x_i - \xi)(\mathrm{C_F}(x_i - x_j\,;\, t_i - t_j)$$

$$- \mathrm{C_F}(\xi - \zeta\,;\, t_i - t_j))\chi_\varkappa(x_j - \zeta)\Big) \quad (4.18)$$

$$\exp\left(\frac{\alpha^2}{4\pi} s \sum_{i \neq j} \varepsilon_i \varepsilon_j \mathrm{C_F}(x_i - x_j\,;\, t_i - t_j)\right)$$

$$\exp\left(\frac{\alpha^2}{4\pi}(1-s) \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathrm{R}} \int_{\mathrm{R}} d\xi d\zeta \chi_\varkappa(x_i - \xi) \mathrm{C_F}(\xi - \zeta, t_i - t_j)\chi_\varkappa(x_j - \zeta)\right)$$

where

$$\mathrm{G}_\varkappa(\omega = (n, t_i, \varepsilon_i, x_i))$$

$$= \left(\frac{-i\lambda'}{|\lambda'|}\right)^n \exp\left(\frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathrm{R}} \int_{\mathrm{R}} d\xi d\zeta \chi_\varkappa(x_i - \xi) \mathrm{C_F}(\xi - \zeta, t_i - t_j)\chi_\varkappa(x_j - \zeta)\right),$$

$$\text{if} \sum_{i=1}^n \varepsilon_i = 0, \qquad = 0, \quad \text{otherwise.} \quad (4.19)$$

However, there exists an $s_0 \in [0, 1]$ such that (see [13]):

$$\left| \int_\Omega \mathrm{P}_{\Lambda|\lambda'|}(d\omega)(\mathrm{F}(\omega) - \mathrm{G}_\varkappa(\omega)) \right|$$

$$\leq \int_\Omega \mathrm{P}_{\Lambda|\lambda'|}(d\omega)\, |\, \mathrm{F}_{\alpha\sqrt{s_0}}(\omega)\, |\, |\, \mathrm{G}_{\varkappa,\alpha\sqrt{1-s_0}}(\omega)\, |\, |\, \mathrm{L}_\varkappa(\omega)\, |, \quad (4.20)$$

with the obvious definitions of $\mathrm{F}_{\alpha\sqrt{s_0}}$ and $\mathrm{G}_{\varkappa\alpha\sqrt{1-s_0}}$,
and

$$\mathrm{L}_\varkappa(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(-\frac{i\lambda'}{|\lambda'|}\right)^n \frac{\alpha^2}{4\pi} \sum_{i \neq j} \varepsilon_i \varepsilon_j \int_{\mathrm{R}} \int_{\mathrm{R}} \chi_\varkappa(x_i - \xi)(\mathrm{C_F}(\xi - \zeta, t_i - t_j)$$

$$- \mathrm{C_F}(x_i - x_j\,;\, t_i - t_j))\chi_\varkappa(x_j - \zeta), \quad \text{if} \sum_i^n \varepsilon_i = 0\,, \quad = 0\,, \quad \text{otherwise}. \quad (4.21)$$

Now if we choose $\delta > 0$, such that:

$$\frac{\alpha^2}{2\pi}(1 + \delta) < 1\,, \quad (4.22)$$

by Hölder inequality,

$$\left| \int_\Omega P_{\Lambda|\lambda'|}(d\omega)(F(\omega) - G_\varkappa(\omega)) \right|$$

$$\leq \left(\int_\Omega P_{\Lambda|\lambda'|}(d\omega) \mid F_{\alpha\sqrt{1+\delta}}(\omega) \mid \right)^{\frac{s_0}{1+\delta}} \left(\int_\Omega P_{\Lambda|\lambda'|}(d\omega) \mid G_{\varkappa\alpha\sqrt{1+\delta}}(\omega) \mid \right)^{\frac{1-s_0}{1+\delta}} \tag{4.23}$$

$$\times \left(\int_\Omega P_{\Lambda|\lambda'|}(d\omega) \mid L_\varkappa(\omega) \mid^{1+\frac{1}{\delta}} \right)^{\frac{\delta}{1+\delta}} .$$

By proposition (4.12) the second term is uniformly bounded in $\varkappa$.
By Minkowski inequality, for $N > 1$ (see e. g. [*22*]):

$$\left(\int_\Omega P_{\Lambda|\lambda'|}(d\omega) \mid L_\varkappa(\omega) \mid^N \right)^{1/N}$$

$$\leq \sum_{n\geq 1} \frac{\alpha^2}{4\pi} \frac{\mid \lambda' \mid^{2n/N}}{2n!} \sum_{i \neq j} \left[\int_0^T dt_{2n} \dots \int_0^T dt_1 \int_\Lambda dx_{2n} \dots \int_\Lambda dx_1 \right. \tag{4.24}$$

$$\left. \left| \int_R \int_R d\xi d\zeta \chi_\varkappa(x_i - \xi)(C_F(x_i - x_j ; t_i - t_j) - C_F(\xi - \zeta ; t_i - t_j))\chi_\varkappa(x_j - \zeta) \right|^N \right]^{1/N}$$

$$= \frac{\alpha^2}{2\pi} (A_\varkappa)^{1/N} \sum_{n\geq 1} \frac{\mid \lambda' \mid^{2n/N}}{2n!} n(2n-1) T^{\frac{2(n-1)}{N}} \Lambda^{\frac{2(n-1)}{N}} , \tag{4.25}$$

where:

$$A_\varkappa = \int_0^T dt_1 \int_0^T dt_2 \int_\Lambda dx_1 \int_\Lambda dx_2 \left| \int_R d\xi \int_R d\zeta \right.$$

$$\left. \cdot \chi_\varkappa(x_1 - \xi)(C_F(\xi - \zeta ; t_1 - t_2) - C_F(x_1 - x_2 ; t_1 - t_2))\chi_\varkappa(x_2 - \zeta) \right|^N . \tag{4.26}$$

However, the series in (4.25) is convergent and $A_\varkappa$ tends to zero for $\varkappa$ tending to zero. Then we can state the theorem:

THEOREM (4.27). — Let $\phi_\varkappa$ be the free relativistic Bose field in two space time dimensions of mass m at time zero and with ultra violet cut off.

Let H_0 be the corresponding free relativistic Hamiltonian.

Let H_I be the Sine Gordon interaction viz.

$$H_I = \lambda \int_\Lambda dx : \cos(\alpha\phi_\varkappa(x)) :_\mu , \qquad \alpha \text{ real},$$

with μ a strictly positive constant, and the Wick order being taken w. r. t. $\Delta_F^{(\mu)}$.

Let Ω_F be the Fock vacuum state corresponding to the mass m.

Then for $\alpha^2 < 2\pi$ the following limit exists

$$\lim_{\varkappa\to 0}\lim_{m\to 0}(\Omega_F \mid \exp(-iT(H_0+H_I))\Omega_F) = \int_\Omega P_{\Lambda|\lambda'|}(d\omega).F(\omega)$$

where $P_{\Lambda|\lambda'|}$ is the (unnormalized) Poisson measure with constant $|\lambda'| = |\lambda|\left(\frac{2}{\mu c}\right)^{\alpha^2/4\pi}$ and F is

$$F(\omega = (n, t_i, \varepsilon_i, x_i)) = \left(-\frac{i\lambda'}{|\lambda'|}\right)^n \exp\left(\frac{\alpha^2}{4\pi}\sum_{i\neq j}\varepsilon_i\varepsilon_j C_F(x_i - x_j; t_i - t_j)\right)$$

$$\text{if } \sum_{i=1}^n \varepsilon_i = 0, \qquad = 0, \quad \text{otherwise}.$$

The general matrix elements of $\exp(iH_0T)\exp(-i(H_0 + H_I)T)$ in between coherent states are treated by the same techniques exept that one has to take care of the infrared divergence in the choice of the functions defining the coherent states.

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