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J.-E. WERTH

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## Integrability for representations appearing in geometric pre-quantization

by

J.-E. WERTH

Departamento de Física, Universidade Federal da Paraíba,  
58000 João Pessoa, PB, Brasil

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**ABSTRACT.** — Vectorfield representations  $D_\rho(\theta^n, \theta)$  induced from quasi-complete infinitesimal group actions  $(\theta^n, \theta)$  on quantizing fibre bundles are studied. Examples for non  $\tilde{G}$ -maximal prequantizations  $\theta^n$  with  $G$ -maximal projected symmetry  $\theta$  are given. The connection between geometrical properties of the prequantization procedure and integrability properties of the associated Lie algebra representation is discussed.

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### 1. INTRODUCTION

The geometric quantization theory of Kostant and Souriau ([3] [7]) provides for the construction of skew-adjoint vectorfield representations  $D_\rho(\theta^n, \theta)$  induced from infinitesimal group actions  $(\theta^n, \theta)$  on quantizing bundles  $\eta = (P, \alpha, A, \lambda, M, \omega)$  [8]. Here  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M, \omega)$  denotes a quasi-complete  $\mathfrak{g}$ -action on a symplectic manifold  $(M, \omega)$  which can be lifted to a  $\mathfrak{g}$ -action (*prequantization*)  $\theta^n$  on the total space  $P$ .

A previous paper [2] discussed the integrability properties of vectorfield representations induced from a Mackey-like quantization. It follows from a result in [2] that if  $\rho : A \rightarrow \text{Aut } V$  is faithful and unitary, then  $D_\rho(\theta^n, \theta)$  integrates up to a group representation iff  $\theta^n$  is  $G$ -maximal. Global results for complete  $\mathfrak{g}$ -actions on quantizing bundles are presented in [9].

The central results contained in [2] and [9] will be applicable to pre-

quantizations which are not necessarily complete. Work of Palais [5] shows that if  $\theta$  is complete, then  $G$ -maximality of  $\theta$  implies  $\tilde{G}$ -maximality of  $\theta^n$ , where  $\tilde{G}$  denotes the universal covering group of  $G$ . For non-complete  $\theta$ , however, non  $\tilde{G}$ -maximal prequantizations  $\theta^n$  with  $G$ -maximal projected symmetry  $\theta$  can be obtained. Especially, quasi-complete actions on quantum bundles are considered. We give explicit constructions for the Heisenberg algebra acting on a bundle over  $(\mathbb{R}^2 - (0, 0), dx \wedge dy)$  and for Lie algebra actions on bundles over the momentum phase space. The relationship to integrability conditions for skew-adjoint Lie algebra representations is analysed.

## 2. A GEOMETRIC INTEGRABILITY CRITERION

Let  $\mathcal{H}$  be a separable complex Hilbert space with dense domain  $\mathfrak{D} \subset \mathcal{H}$ .  $S(\mathfrak{D})$  denotes the Lie algebra of skew-symmetric operators in  $\mathcal{H}$  with invariant domain  $\mathfrak{D} \subset \mathcal{H}$  and  $\mathcal{A}(\mathfrak{D})$  is the subset of operators in  $S(\mathfrak{D})$  being essentially skew-adjoint on  $\mathfrak{D}$ . A Lie algebra homomorphism

$$D : \mathfrak{g} \rightarrow S(\mathfrak{D})$$

of a Lie algebra  $\mathfrak{g}$  into  $S(\mathfrak{D})$  is called a skew-adjoint representation of  $\mathfrak{g}$  on  $\mathfrak{D} \subset \mathcal{H}$  if  $\text{Im } D \subset \mathcal{A}(\mathfrak{D})$ .

Let  $G$  be a connected Lie group with Lie algebra of left invariant vectorfields isomorphic to  $\mathfrak{g}$ . Denote by  $\exp$  the exponential map  $\mathfrak{g} \rightarrow G$ .  $\text{Exp} : \mathcal{A}(\mathfrak{D}) \rightarrow \mathcal{U}(\mathcal{H})$  denotes the exponentiation (given by Stone's theorem) from  $\mathcal{A}(\mathfrak{D})$  into the group  $\mathcal{U}(\mathcal{H})$  of unitary operators on  $\mathcal{H}$ .

We say that  $D : \mathfrak{g} \rightarrow \mathcal{A}(\mathfrak{D})$  is  $G$ -integrable [2] if there exists a unitary representation  $U : G \rightarrow \mathcal{U}(\mathcal{H})$  such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{U} & \mathcal{U}(\mathcal{H}) \\ \exp \uparrow & & \uparrow \text{Exp} \\ \mathfrak{g} & \xrightarrow{D} & \mathcal{A}(\mathfrak{D}) \end{array}$$

commutes.

Let  $C(G, e)$  be the set of closed curves in  $G$  starting and ending at  $e$  and take

$$C^q(G, e) = \{ (x_1, \dots, x_k) \mid k \in \mathbb{N} - \{0\}, x_i \in \mathfrak{g}, \exp x_1 \dots \exp x_k = e \}.$$

$C^q(G, e)$  may be regarded as a subset of  $C(G, e)$  via

$$C^q(G, e) \xrightarrow{i} C(G, e)$$

defined by

$$(i(x_1, \dots, x_k))(t) = \exp x_1 \dots \exp x_{n-1} \exp (kt - n + 1)x_n$$

for  $t \in \left[ \frac{n-1}{k}, \frac{n}{k} \right]$ ,  $n = 1, \dots, k$ . Denote by  $P(\mathcal{U}(\mathcal{H}), 1)$  the set of curves in  $\mathcal{U}(\mathcal{H})$  starting at 1. The subset of closed curves in  $P(\mathcal{U}(\mathcal{H}), 1)$  will be denoted by  $C(\mathcal{U}(\mathcal{H}), 1)$ . We now define a map

$$\delta(D, G) : C^g(G, e) \rightarrow P(\mathcal{U}(\mathcal{H}), 1)$$

by putting ( $t \in [0, 1]$ )

$$\delta(D, G)(x_1, \dots, x_k)(t) = \text{Exp } D(x_1) \dots \text{Exp } D(x_{n-1}) \text{Exp } (kt - n + 1)D(x_n)$$

for  $t \in \left[ \frac{n-1}{k}, \frac{n}{k} \right]$ ,  $n = 1, \dots, k$ . Because  $G$  is connected, any  $g \in G$  can be written as

$$g = \exp x_1 \dots \exp x_k$$

for suitable  $x_i \in \mathfrak{g}$ . Using the commutative diagram above, we get the following more geometrical result.

**PROPOSITION 1** [2]. — Let  $D : \mathfrak{g} \rightarrow \mathcal{A}(\mathfrak{g})$  be a skew-adjoint representation. Then the following statements are equivalent:

- i)  $D$  is  $G$ -integrable;
- ii)  $\text{Im } \delta(D, G) \subset C(\mathcal{U}(\mathcal{H}), 1)$ .

### 3. MAXIMALITY AND INTEGRABILITY

Denote by  $\mathcal{M}(M)$  the Lie algebra of smooth vectorfields on  $M$ . For  $\xi \in \mathcal{M}(M)$  let

$$(m, t) \in D(\xi) \subset M \times \mathbb{R} \rightarrow F(\xi)(m, t) = \varphi_t^\xi(m) \in M$$

be the flow of  $\xi$ . Let  $D(\xi, t)$  be the set of points  $m$  of  $M$  such that  $(m, t)$  lies in  $D(\xi)$ .  $D(\xi, t)$  is open for  $t \in \mathbb{R}$  [4].  $\xi$  is complete if  $D(\xi, t) = M$  for  $t \in \mathbb{R}$ .

A Lie algebra action of  $\mathfrak{g}$  on  $M$  (also called *infinitesimal  $G$ -action* on  $M$ ) is a Lie algebra homomorphism

$$\theta : \mathfrak{g} \rightarrow \mathcal{M}(M).$$

$\theta$  is called *complete* if  $\theta(x)$  is complete for  $x \in \mathfrak{g}$ .

$P(M, m)$  and  $C(M, m)$  denote, respectively, the set of curves in  $M$  starting at  $m$  and the subset of closed curves.  $\Omega(M, m)$  will denote the homotopy classes (rel.  $\{0, 1\}$ ) of paths of  $P(M, m)$ .  $\pi_1(M, m) \subset \Omega(M, m)$  denotes the homotopy classes of based paths in  $(M, m)$ . Thus we have the following commutative diagram

$$\begin{array}{ccc} C(M, m) & \xrightarrow{c} & P(M, m) \\ \mu_m \downarrow & & \downarrow v_m \\ \pi_1(M, m) & \xrightarrow{c} & \Omega(M, m) \end{array}$$

where  $v_m$  and  $\mu_m = v_m|C(M, m)$  are the natural projections.

Define  $C(\theta, m; G, e) \subset C^q(G, e) \stackrel{i}{\subset} C(G, e)$  as follows:

$$C(\theta, m; G, e) := \{ (x_1, \dots, x_k) \in C^q(G, e) \mid \varphi_1^{\theta(x_k)} \dots \varphi_1^{\theta(x_1)}(m) \text{ exists} \}.$$

There exist natural maps (compare the definition of  $\delta(D, G)$ )

$$\delta(\theta, G, m) : C(\theta, m; G, e) \rightarrow P(M, m)$$

and

$$\varepsilon(\theta, G, m) : C(\theta, m; G, e) \rightarrow \Omega(M, m)$$

such that the diagram

$$\begin{array}{ccc} & & P(M, m) \\ & \nearrow^{\delta(\theta, G, m)} & \downarrow v_m \\ C(\theta, m; G, e) & & \Omega(M, m) \\ & \searrow_{\varepsilon(\theta, G, m)} & \end{array}$$

commutes.  $\theta$  is called *G-maximal* ([2] [5] [9]) if for  $m \in M$

$$\text{Im } \delta(\theta, G, m) \subset C(M, m)$$

or equivalently

$$\text{Im } \varepsilon(\theta, G, m) \subset \pi_1(M, m).$$

A Lie algebra action  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$  is called *transitive* if for  $m, m' \in M$  there exists a

$$(x_1, \dots, x_k) \in C(\theta, m; G, e)$$

such that  $\delta(\theta, G, m)(x_1, \dots, x_k)(1) = m'$ . The proof of the following result is a straightforward calculation.

**PROPOSITION 2.** — Let  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$  be a transitive Lie algebra action. Then  $\theta$  is *G-maximal* iff there is a  $m \in M$  such that

$$\text{Im } \delta(\theta, G, m) \subset C(M, m).$$

Now consider a covering  $p : M' \rightarrow M$ . Since  $p$  is a local diffeomorphism, there is a natural (injective) Lie algebra homomorphism

$$p' : \mathcal{M}(M) \rightarrow \mathcal{M}(M').$$

In this situation, a Lie algebra action  $\theta' : \mathfrak{g} \rightarrow \mathcal{M}(M')$  is called a *covering* of a Lie algebra action  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$  if

$$\begin{array}{ccc} & & \mathcal{M}(M') \\ & \nearrow^{\theta'} & \uparrow p' \\ \mathfrak{g} & \xrightarrow{\theta} & \mathcal{M}(M) \end{array}$$

commutes. Since a covering has unique path lifting, an application of Proposition 2 gives.

**PROPOSITION 3.** — Let  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$  be a transitive *G-maximal*  $\mathfrak{g}$ -action on  $M$  and let  $p' : M' \rightarrow M$  be a regular covering of  $M$ . Then the

covering  $\theta' : \mathfrak{g} \rightarrow \mathcal{M}(M')$  of  $\theta$  is  $G$ -maximal iff there is a  $m' \in M'$  such that

$$\text{Im } \varepsilon(\theta, G, p(m')) \subset p_*\pi_1(M', m').$$

Any open imbedding  $i : M \subset M^*$  induces a Lie algebra homomorphism

$$i^* : \mathcal{M}(M^*) \rightarrow \mathcal{M}(M)$$

given by  $i^*\xi^* = \xi^*|_M$ ; here  $\xi^*|_M$  denotes the restriction of  $\xi^* \in \mathcal{M}(M^*)$  to  $M$ . Now consider a Lie algebra action  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$  and an open imbedding  $i : M \subset M^*$ . Then  $\theta^* : \mathfrak{g} \rightarrow \mathcal{M}(M^*)$  is called an *extension* of  $\theta$  if

$$\begin{array}{ccc} & & \mathcal{M}(M^*) \\ & \nearrow \theta^* & \downarrow i^* \\ \mathfrak{g} & \xrightarrow{\theta} & \mathcal{M}(M) \end{array}$$

commutes.

**PROPOSITION 4** [5]. — A Lie algebra action  $\theta$  is  $\tilde{G}$ -maximal if and only if there is a complete extension of  $\theta$ .

**COROLLARY.** — Any complete  $\mathfrak{g}$ -action is  $\tilde{G}$ -maximal.

As an example, consider the construction given in [2]: take  $M = \mathbb{R}^2 - (0, 0)$ ,  $\mathfrak{g} = \mathbb{R}^2$  (2-dimensional Abelian Lie algebra) and define  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$  by

$$(*) \quad \theta(a, b) = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$$

for  $(a, b) \in \mathbb{R}^2$ .  $\theta$  is a non-complete  $\tilde{G}$ -maximal action for  $\tilde{G} \cong \mathbb{R}^2$ . A natural complete extension on  $M^* = \mathbb{R}^2$  is given by  $(*)$ , too. Now consider the double covering (Riemannian sheet)  $M_{\mathbb{R}}$  of  $\mathbb{R}^2 - (0, 0)$ . The corresponding covering

$$\theta_{\mathbb{R}} : \mathbb{R}^2 \rightarrow \mathcal{M}(M_{\mathbb{R}})$$

is non  $\tilde{G}$ -maximal. Thus, according to Proposition 4, there is no complete extension of  $\theta_{\mathbb{R}}$ .

A vectorfield  $\xi \in \mathcal{M}(M)$  is called *quasi-complete* if

$$E(\xi, t) := M \setminus D(\xi, t)$$

is a set of measure zero for  $t \in \mathbb{R}$ ; note that  $\xi$  is complete iff  $E(\xi, t) = \emptyset$  for  $t \in \mathbb{R}$ . So a Lie algebra action  $\theta$  is (*quasi-*)complete if  $\theta(x)$  has this property for any  $x \in \mathfrak{g}$ .

Let  $\Omega$  be a volume on  $M$ ;  $\Omega$  is called  $\xi$ -invariant if  $L_{\xi}\Omega = 0$ .  $\mathcal{M}(M, \Omega)$  will denote the Lie algebra of vectorfields  $\xi \in \mathcal{M}(M)$  such that  $L_{\xi}\Omega = 0$ . We say that  $\theta$  acts on  $(M, \Omega)$  if  $\text{Im } \theta \subset \mathcal{M}(M, \Omega)$ .

Denote by  $\mathcal{F}_0(M, \Omega)$  the pre-Hilbert space of compactly supported

complex-valued functions on  $M$ . We denote by  $L^2(M, \Omega)$  the corresponding Hilbert space. A  $\mathfrak{g}$ -action  $\theta$  on  $(M, \Omega)$  induces a representation

$$D(\theta) : \mathfrak{g} \rightarrow \text{End } \mathcal{F}_0(M, \Omega)$$

via  $(f \in \mathcal{F}_0(M, \Omega))$

$$(D(\theta)(x)f)(m) = \left. \frac{d}{dt} \right|_{t=0} f(\varphi_t^{\theta(x)}(m)).$$

**PROPOSITION 5** [2]. — Let  $\theta$  be a quasi-complete  $\mathfrak{g}$ -action on  $(M, \Omega)$ . Then  $D(\theta)$  is a skew-adjoint representation of  $\mathfrak{g}$  on  $\mathcal{F}_0(M, \Omega)$  in  $L^2(M, \Omega)$ . Moreover,  $D(\theta) : \mathfrak{g} \rightarrow \mathcal{A}(\mathcal{F}_0(M, \Omega))$  is  $G$ -integrable if and only if  $\theta$  is  $G$ -maximal.

Hence, in view of the results of Palais, a representation induced from a quasi-complete  $\mathfrak{g}$ -action on  $(M, \Omega)$  is  $G$ -integrable iff the  $\mathfrak{g}$ -action can be regarded as a restriction of a complete one.

#### 4. LIE ALGEBRA ACTIONS ON PRINCIPAL FIBRE BUNDLES

Let  $(P, \pi, M, S)$  denote a principal fibre bundle with projection  $\pi : P \rightarrow M$  and structure group  $S$ . A  $\mathfrak{g}$ -action  $(\mathfrak{g}, \theta)$  on  $(P, \pi, M, S)$  consists of  $\mathfrak{g}$ -actions

$$\mathfrak{g} : \mathfrak{g} \rightarrow \mathcal{M}(P), \quad \theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$$

such that for  $x \in \mathfrak{g}$

- i)  $D(\mathfrak{g}(x)) = (\pi \times id_{\mathbb{R}})^{-1} D(\theta(x))$ ;
- ii)  $\pi(F(\mathfrak{g}(x))(p, t)) = F(\theta(x))(\pi(p), t)$  for  $(p, t) \in D(\mathfrak{g}(x))$ ;
- iii)  $\mathfrak{g}(x)$  is  $S$ -invariant.

Now let  $\theta : \mathfrak{g} \rightarrow \mathcal{M}(M)$  be a  $\mathfrak{g}$ -action and let  $p : M' \rightarrow M$  be a regular covering corresponding to a normal subgroup  $H \subset \pi_1(M, m_0)$ . Denote by

$$\theta' : \mathfrak{g} \rightarrow \mathcal{M}(M')$$

the induced covering action. Then  $(\theta', \theta)$  is a  $\mathfrak{g}$ -action on the principal fibre bundle  $(M', p, M, \pi_1(M, m_0)/H)$ .

For Lie algebra actions on principal fibre bundles with non-discrete structure group see §5.

It is not hard to prove the following result.

**PROPOSITION 6.** — Let  $(\mathfrak{g}, \theta)$  be a  $\mathfrak{g}$ -action on  $(P, \pi, M, S)$ . Suppose that  $\theta$  is transitive and  $G$ -maximal. Then  $\mathfrak{g}$  is  $G$ -maximal iff there is a  $p \in P$  such that

$$\text{Im } \delta(\mathfrak{g}, G, p) \subset C(P, p).$$

### 5. VECTORFIELD REPRESENTATIONS

Let  $\rho : S \rightarrow \text{Aut } V$  be a finite-dimensional representation of  $S$  in a complex vector space  $V$ . Then the  $\rho$ -bundle associated with  $(P, \pi, M, S)$  is the vectorbundle

$$(E_\rho, \pi_\rho, M, V),$$

where  $E_\rho$  is the orbit space of the right  $G$ -action on  $P \times V$  given by letting  $g \in G$  take  $(p, v)$  to  $(pg, \rho^{-1}(g)v)$ . The equivalence class of  $(p, v)$  is denoted by  $[p, v]_\rho$ . We have  $\pi_\rho[p, v]_\rho = \pi(p)$ .  $(E_\rho, \pi_\rho, M, V)$  is sometimes denoted by  $E_\rho(P)$  or simply  $E_\rho$ .

Any  $\mathfrak{g}$ -action  $(\mathfrak{g}, \theta)$  on  $(P, \pi, M, S)$  induces a  $\mathfrak{g}$ -action  $\mathfrak{g}_\rho$  on  $E_\rho$  via

$$F(\mathfrak{g}_\rho(x))( [p, v]_\rho, t) = [F(\mathfrak{g}(x))(p, t), v]_\rho.$$

Observe that this flow is well defined since  $\mathfrak{g}(x)$  is  $S$ -invariant.

Let  $\Gamma_0 E_\rho$  be the space of compactly supported smooth sections in  $(E_\rho, \pi_\rho, M, V)$ . A  $\mathfrak{g}$ -action  $(\mathfrak{g}, \theta)$  on  $(P, \pi, M, S)$  induces a representation—called vectorfield representation—

$$D_\rho(\mathfrak{g}, \theta) : \mathfrak{g} \rightarrow \text{End } \Gamma_0 E_\rho$$

via  $(\sigma \in \Gamma_0 E_\rho)$

$$(D_\rho(\mathfrak{g}, \theta)(x)\sigma)(m) := \left. \frac{d}{dt} \right|_{t=0} (\varphi_t^{\mathfrak{g}_\rho(x)} \circ \sigma \circ \varphi_t^{\theta(x)})(m).$$

For unitary  $\rho$ , we define a pre-Hilbert structure on  $\Gamma_0 E_\rho$  by setting

$$\langle \sigma_1, \sigma_2 \rangle = \int_\Omega \langle \sigma_1(m), \sigma_2(m) \rangle_m.$$

We denote by  $L^2(E_\rho, \Omega)$  the corresponding Hilbert space. The following result generalizes Proposition 5.

**PROPOSITION 7 [2].** — Let  $(\mathfrak{g}, \theta)$  be a  $\mathfrak{g}$ -action on  $(P, \pi, M, S)$  and let  $\rho : S \rightarrow \text{Aut } V$  be a unitary faithful finite-dimensional representation. Suppose that  $\theta$  is quasi-complete on  $(M, \Omega)$ . Then  $D_\rho(\mathfrak{g}, \theta)$  is a skew-adjoint representation of  $\mathfrak{g}$  on  $\Gamma_0 E_\rho \subset L^2(E_\rho, \Omega)$ . Moreover,  $D_\rho(\mathfrak{g}, \theta) : \mathfrak{g} \rightarrow \mathcal{A}(\Gamma_0 E_\rho)$  is  $G$ -integrable if and only if  $\mathfrak{g}$  is  $G$ -maximal.

### 6. APPLICATION TO GEOMETRIC PRE-QUANTIZATION

$(P, A, M)$  will denote a smooth principal fibre bundle with abelian structure group  $A$  over a connected manifold  $M$ .  $\pi$  will denote the projection  $P \rightarrow M$ . Let  $\alpha$  be a connection form on  $P$  and let  $\omega$  be a sym-

plectic structure on  $M$ . Given a linear injective map  $\lambda : \mathbb{R} \rightarrow \alpha$  from the real numbers into the Lie algebra of  $A$ , we say that

$$(P, \alpha, A, \lambda, M, \omega)$$

is a quantizing bundle [8] if

$$d\alpha = \lambda\pi^*\omega.$$

We remark that this definition includes (up to association) Kostant's Hermitian line bundle [3] and Souriau's *espace fibré quantifiant* [7].

Let  $\{, \}$  be the Lie algebra structure on the space  $\mathcal{F}(M, \omega)$  of smooth real-valued functions on  $M$  defined by

$$\{ \varphi, \psi \} = \xi_\varphi\psi = \omega(\xi_\psi, \xi_\varphi)$$

where  $\xi_\varphi$  is the Hamiltonian vectorfield corresponding to  $\varphi \in \mathcal{F}(M, \omega)$ . Suppose we are given a Lie algebra homomorphism

$$\phi : \mathfrak{g} \rightarrow \mathcal{F}(M, \omega).$$

For any quantizing bundle  $\eta = (P, \alpha, A, \lambda, M, \omega)$  over  $(M, \omega)$   $\phi$  induces a  $\mathfrak{g}$ -action  $(\theta_\phi^\eta, \theta_\phi)$  on  $\eta$  as follows:

$$\theta_\phi : \mathfrak{g} \rightarrow \mathcal{M}(M)$$

is given by  $\theta_\phi(x) := \xi_{\phi(x)}$ , and

$$\theta_\phi^\eta : \mathfrak{g} \rightarrow \mathcal{M}(P)$$

is defined via the flows of  $\theta_\phi^\eta(x)$  for  $x \in \mathfrak{g}$ :

$$F(\theta_\phi^\eta(x))(p, t) := F(\theta_\phi^\alpha(x))(p, t) \exp - t\lambda(\phi(x))(\pi(p)).$$

Here  $\theta_\phi^\alpha(x) \in \mathcal{M}(P)$  denotes the horizontal lift of  $\theta_\phi(x) \in \mathcal{M}(M)$  with respect to  $\alpha$ . The Lie algebra action  $(\theta_\phi^\eta, \theta_\phi)$  is called prequantization.

Suppose that  $(M, \omega)$  is  $(A, \lambda)$ -quantizable. Denote by  $\mathbb{Q}(A, \lambda, M, \omega)$  the set of equivalence classes of  $(A, \lambda, M, \omega)$ -bundles. Then there is a free and transitive action

$$\mathbb{Q}(A, \lambda, M, \omega) \times \pi_1^\wedge(M, m_0) \rightarrow \mathbb{Q}(A, \lambda, M, \omega)$$

of the group  $\pi_1^\wedge(M, m_0)$  of group homomorphisms  $\pi_1(M, m_0) \rightarrow A$  on  $\mathbb{Q}(A, \lambda, M, \omega)$  (see e. g. [9]). Denote by  $a_m^\alpha(\gamma) \in A$  the parallel displacement along  $\gamma \in C(M, m)$  with respect to the connection form  $\alpha$  of  $\eta = (P, \alpha, A, \lambda, M, \omega)$ . We know [9] that

$$(*) \quad a_{m_0}^{\alpha\chi}(\gamma) = a_{m_0}^\alpha(\gamma)\chi[\gamma]$$

for  $\gamma \in C(M, m_0)$ ,  $\chi \in \pi_1^\wedge(M, m_0)$ , where  $\alpha\chi$  denotes the connection form of the quantizing bundle  $\eta\chi$ .

We shall now discuss the maximality properties of  $(\theta_\phi^\eta, \theta_\phi)$ .

$$\mu(\phi, G, m) : C(\theta_\phi, m; G, e) - A$$

be the association given by

$$(x_1, \dots, x_k) \rightarrow \exp - \lambda \sum_{i=1}^k (\phi(x_i))(\varphi_1^{\theta_\phi(x_i)} \dots \varphi_1^{\theta_\phi(x_1)}(m)).$$

Suppose that  $\theta_\phi$  is G-maximal. Then

$$\delta(\theta_\phi^\eta, G, p)(\bar{x})(1) = p a_{\pi(p)}^\alpha(\delta(\theta_\phi, G, \pi(p))(\bar{x}))\mu(\phi, G, \pi(p))(\bar{x})$$

for  $\bar{x} \in C(\theta_\phi, m; G, e)$ ,  $p \in P$ . Hence, for G-maximal  $\theta_\phi$ ,  $\theta_\phi^\eta$  is G-maximal iff

$$a_m^\alpha(\delta(\theta_\phi, G, m)(\bar{x})) = \mu^{-1}(\phi, G, m)(\bar{x})$$

for  $\bar{x} \in C(\theta_\phi, m; G, e)$ ,  $m \in M$ . For transitive and G-maximal  $\theta_\phi$  it follows from Proposition 6 that  $\theta_\phi^\eta$  is G-maximal iff

$$a_{m_0}^\alpha(\delta(\theta_\phi, G, m_0)(\bar{x})) = \mu^{-1}(\phi, G, m_0)(\bar{x}).$$

By (\*), we have the following result.

**PROPOSITION 8.** — Suppose that  $\theta_\phi$  is transitive and that  $\theta_\phi^\eta$  is G-maximal. Take  $\chi \in \pi_1^A(M, m_0)$ . Then  $\theta_\phi^{\eta\chi}$  is G-maximal iff

$$\text{Im } \varepsilon(\theta_\phi, G, m_0) \subset \text{Ker } \chi.$$

### 7. AN EXAMPLE FOR THE HEISENBERG ALGEBRA

Let  $H = (\mathbb{R}^3, \cdot)$  be the Heisenberg group with multiplication

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2).$$

Denote by  $\mathfrak{h}$  the Lie algebra of left invariant vectorfields on  $H$ . We shall now construct a Lie algebra homomorphism ( $\dot{\mathbb{R}}^2 = \mathbb{R}^2 - (0, 0)$ )

$$\phi : \mathfrak{h} \rightarrow \mathcal{F}(\dot{\mathbb{R}}^2, dx \wedge dy).$$

Consider the basis

$$\{ \partial/\partial x_1, \partial/\partial x_2 + x_1 \partial/\partial x_3, \partial/\partial x_3 \}$$

of  $\mathfrak{h}$  and define

$$\phi(\partial/\partial x_1) = y, \quad \phi(\partial/\partial x_2 + x_1 \partial/\partial x_3) = x, \quad \phi(\partial/\partial x_3) = 1.$$

It is not hard to conclude that  $\phi$  is a Lie algebra homomorphism and that

$$\theta_\phi : \mathfrak{h} \rightarrow \mathcal{M}(\dot{\mathbb{R}}^2)$$

is given by

$$\theta_\phi(\partial/\partial x_1) = \partial/\partial x, \quad \theta_\phi(\partial/\partial x_2 + x_1 \partial/\partial x_3) = -\partial/\partial y, \quad \theta_\phi(\partial/\partial x_3) = 0.$$

By Proposition 4,  $\theta_\phi$  is H-maximal. Furthermore,

$$\varepsilon(\theta_\phi, H, *) : C(\theta_\phi, *; H, e) \rightarrow \pi_1(\dot{\mathbb{R}}^2, *)$$

is surjective for any  $* \in \dot{\mathbb{R}}^2$ .

Since  $dx \wedge dy$  is exact,  $(\dot{\mathbb{R}}^2, dx \wedge dy)$  is quantizable; denote by

$$\varepsilon = (\dot{\mathbb{R}}^2 \times A, \alpha, A, \lambda, \dot{\mathbb{R}}^2, dx \wedge dy)$$

the trivial  $(A, \lambda)$ -bundle associated with the system  $\{f_{ij}, \alpha_i; i, j \in J\}$  of quantizing functions [8] given by

$$f_{ij} = e, \quad \alpha_i = xdy|U_i.$$

Hence, in view of Proposition 4,  $\theta_\phi^e$  is a H-maximal  $\mathfrak{h}$ -action on  $\dot{\mathbb{R}}^2 \times A$ . The following result now follows from Proposition 8.

PROPOSITION 9. —  $\theta_\phi^{eX}$  is H-maximal if and only if

$$\chi \in \pi_1^A(\dot{\mathbb{R}}^2, *)$$

is trivial.

By using Proposition 7, we can conclude that for unitary and faithful  $\rho$  the skew-adjoint representation  $D_\rho(\theta_\phi^{eX}, \theta_\phi)$  integrates up to a unitary representation of the Heisenberg group if and only if

$$\chi \in \pi_1^A(\dot{\mathbb{R}}^2, *)$$

is trivial.

### 8. ACTIONS ON BUNDLES OVER MOMENTUM PHASE SPACE

We shall now apply the results of § 6 to actions on cotangent bundles. Let  $\delta : \mathfrak{g} \rightarrow \mathcal{M}(X)$  be a  $\mathfrak{g}$ -action on  $X$ . Consider the cotangent bundle  $T^*X$  with projection  $v : T^*X \rightarrow X$ . Then  $\delta$  induces a  $\mathfrak{g}$ -action  $\theta_\delta$  on  $T^*X$  via ( $x \in \mathfrak{g}$ )

$$\varphi_t^{\theta_\delta(x)}(u_q) = (\varphi_{-t}^{\delta(x)})^* u_q,$$

$u_q \in T^*X$ ,  $v(u_q) = q$ . Observe that  $\theta_\delta$  is  $\mathfrak{G}$ -maximal iff  $\delta$  is. Let  $\Omega$  denote the canonical 1-form on  $T^*X$  given by

$$\Omega(\xi_{u_q}) = u_q(v_* \xi_{u_q}).$$

The exterior derivative of  $\Omega$  is the canonical symplectic structure on  $T^*X$ . Since

$$L_{\theta_\delta(x)} \Omega = 0$$

for  $x \in \mathfrak{g}$  [1],

$$x \in \mathfrak{g} \xrightarrow{\phi} -\Omega(\theta_\delta(x)) \in \mathcal{F}(T^*X, d\Omega)$$

defines a Lie algebra homomorphism such that  $\theta_\phi = \theta_\delta$  [6]. The corres-

ponding prequantization action  $\theta_\delta^\varepsilon$  on the trivial quantizing bundle  $\varepsilon = (T^*X \times A, \alpha, A, \lambda, T^*X, d\Omega)$  is the trivial lift of  $\theta_\delta$  [9]. Hence  $\theta_\delta^\varepsilon$  is G-maximal iff  $\delta$  is G-maximal. Observe that  $v : T^*X \rightarrow X$  induces an isomorphism [9]

$$\pi_1(T^*X, u_q) \xrightarrow{\cong} \pi_1(X, q).$$

Thus

$$\pi_1^\wedge(T^*X, u_q) \cong \pi_1^\wedge(X, q)$$

and we have the following result.

**PROPOSITION 10.** — Let  $\delta : \mathfrak{g} \rightarrow \mathcal{M}(X)$  be a quasi-complete transitive G-maximal  $\mathfrak{g}$ -action on X. Denote by  $(\theta_\delta^\varepsilon, \theta_\delta)$  the induced prequantization on  $\varepsilon = (T^*X \times A, \alpha, A, \lambda, T^*X, d\Omega)$ . Take  $\chi \in \pi_1^\wedge(X, q_0)$  and denote by  $\chi^*$  the corresponding element of  $\pi_1^\wedge(T^*X, u_{q_0})$ . Then  $\theta_\delta^{\varepsilon\chi^*}$  is G-maximal if and only if

$$\text{Im } \varepsilon(\delta, G, q_0) \subset \text{Ker } \chi.$$

Hence, in view of Proposition 7, we conclude that for unitary and faithful  $\rho$ ,  $D_\rho(\theta_\delta^{\varepsilon\chi^*}, \theta_\rho)$  is a skew-adjoint representation of  $\mathfrak{g}$  which integrates up to a unitary representation of G if and only if

$$\text{Im } \varepsilon(\delta, G, q_0) \subset \text{Ker } \chi.$$

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