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Section A :
Physique théorique.

On the acoustic impedence condition for ondulated boundary

by

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ABSTRACT. — This paper deals with the reduced wave equation for non-homogeneous media with an impedence boundary condition. The domain is either bounded or unbounded, but its boundary is an ondulated surface with small spatial period. The asymptotic behaviour of solutions is studied as the period of the ondulations tends to zero. It appears that the limit solution satisfies a different impedence condition. This result is obtained by homogenization techniques, and holds for eigen-solutions as well as scattering problems.

RÉSUMÉ. — On considère l'équation réduite des ondes avec une condition aux limites d'impédance. Le domaine est aussi bien borné que non borné, sa frontière étant une surface ondulée de petite période spatiale. On étudie le comportement asymptotique des solutions lorsque la période des ondulations tend vers zéro. On montre que la solution limite satisfait à une condition d'impédance différente. Ce résultat, obtenu à l'aide de technique d'homogénéisation a lieu pour des fonctions propres et pour des problèmes de diffusion.

0. INTRODUCTION

It is known that, if the heat transfer equation is considered with a boundary condition of the type:

$$\frac{\partial u}{\partial n} + \lambda u = 0 \tag{0.1}$$

(*) The results of this work were obtained whilst the author sojourned at the Laboratoire de Mécanique Théorique, Tour 66, 4, Place Jussieu, 75230 Paris, France.

with positive λ in a region bounded by an oscillating surface, the limit behaviour for small period (of the ondulations) is given by a boundary condition of the type (0.1) with another value of λ, which takes into account the ondulations (see [7], ch. 5, § 7 and 8).

We consider here the generalizations of this result to the wave equation with complex λ. Condition (0.1) then becomes an acoustic impedence. On the other hand, we also consider a non-homogeneous medium with periodic structure: the coefficients of the corresponding equation are periodic functions of the space variables. We then consider the asymptotic behaviour as the period tends to zero, in the framework of the homogenization theory. In this connection, it is to be noticed that we do not consider small wave length problems; the spatial period of the ondulations of the boundary and of the coefficients is the only small parameter in the problem.

The asymptotic processes at the boundary and at the interior of the domain are handled by the homogenization classical techniques (cf. [2] [7]), consequently certain parts of the proofs are analogous to that of other well-known problems and we only give the corresponding reference.

In section 1 we state the problem in bounded domains. The corresponding asymptotic behaviour for solutions and eigen-frequencies is given in section 2. An explicit study of the solutions in the vicinity of the boundary, in a particular case, is given in section 3 by boundary layer techniques; the value of the limit impedence is obtained again as a compatibility condition for the existence of the layer. Section 4 is devoted to the statement of the problem in an unbounded domain and the definition of the scattering frequencies. Sections 5 and 6 contain the corresponding convergence of solutions and scattering frequencies. We note that other scattering properties, in the behaviour of homogenization theory, are studied in [3] and [4].

The author thanks Mr. E. Sanchez-Palencia for his encouragement and helpful advice.

1. PROBLEM IN BOUNDED DOMAINS

We consider a problem depending on two parameters ε, η. The first is related to the homogenization of the coefficients. The domain is made by a non-homogeneous medium with periodic structure and the coefficients a_{ij}^{ε}, b^{ε} of the equation are periodic real functions of period εY:

$$a_{ij}^{\varepsilon}(x) = a_{ij}(x/\varepsilon);$$

$b^{\varepsilon}(x) = b(x/\varepsilon)$ where $a_{ij}(y)$ and $b(y)$ are piecewise constant, Y-periodic functions of period a parallelopiped Y satisfying the ellipticity condition:

$$(1.1) \qquad a_{ij}\xi_i\xi_j \geqslant \gamma \mid \xi \mid^2 \qquad \forall \xi \in \mathbb{R}^3; \quad b(y) > 0.$$

The second parameter η is associated with the form of the domain Ω_η defined as follows: let Ω_0 be a bounded domain with boundary $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2$. In a neighbourhood of Γ_0^1 we consider the local coordinates (s_1, s_2, N) with N the outer unit normal. Let $\mathrm{F}(z_1, z_2) > 0$ be a periodic $\mathscr{C}^\infty$ function with period a rectangle $\mathbb{Z}$; $\mathrm{F}_\eta(s_1, s_2) = \mathrm{F}(s_1/\eta, s_2/\eta)$. Moreover, let $\theta(x, \eta)$ be a smooth function such that: $\theta = 1$ on Γ_0^1 except on a neighbourhood V_η of $\partial\Gamma_0^1$ and mes.$\mathrm{V}_\eta \to 0$, as $\eta \searrow 0$, and $\theta = 0$ in a neighbourhood $\mathrm{V}'_\eta \subset \mathrm{V}_\eta$ of $\partial\Gamma_0^1$. We set:

$$\Gamma_\eta^1 = \{ (s_1, s_2, \mathrm{N}) : \mathrm{N} = \theta\eta\mathrm{F}_\eta(s_1, s_2) \}.$$

We also define the impedence $1/\lambda_\eta$ by taking λ_η as follows:

$$\lambda_\eta |_{\Gamma_\eta^1} = \lambda_1(s_1/\eta, s_2/\eta); \qquad \lambda_\eta |_{\Gamma_0^2} = \lambda_2$$

where $\lambda^1(z_1, z_2)$ is periodic with period the rectangle $\mathbb{Z}$ and λ_2 constant. We take the following hypothesis:

$$\text{(1.2)} \qquad \operatorname{Re} \lambda_\eta > 0 \quad \text{and} \quad \operatorname{Im} \lambda_\eta \leqslant 0$$

Under the preceding hypotheses, we consider the following boundary value problem, where f is a given function of $\mathrm{L}^2(\Omega_0)$ (continued with zero values to Ω_η) and ω is a complex spectral parameter: $u_{\varepsilon\eta} \in \mathrm{H}^1(\Omega_\eta)$

$$\text{(1.3)} \qquad -\frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon \frac{\partial u_{\varepsilon\eta}}{\partial x_j}\right) - \omega^2 b^\varepsilon u_{\varepsilon\eta} = f \qquad \text{in } \Omega_\eta$$

$$\text{(1.4)} \qquad [\![u_{\varepsilon\eta}]\!] = 0 \qquad \left[\!\!\left[a_{ij} \frac{\partial u_{\varepsilon\eta}}{\partial x_j} n_i \right]\!\!\right] = 0$$

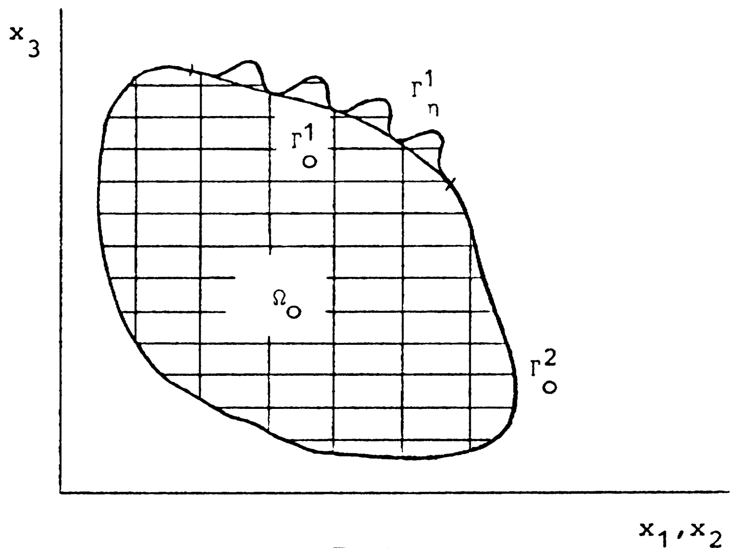

FIG. 1.

on the surfaces of discontinuity of a_{ij}. $[\![\cdot]\!]$ is the symbol for « jump of ».

$$a^{\varepsilon}_{ij}\frac{\partial u_{\varepsilon\eta}}{\partial x_j}n_i + \lambda_\eta u_{\varepsilon\eta} = 0 \qquad \text{on } \partial\Omega_\eta \tag{1.5}$$

If we suppose that $u_{\varepsilon\eta}$ is the pressure, we can interpret the equation (1.3) as the acoustic wave vibration equation. Condition (1.5) means that the velocity of the displacement of the boundaries Γ^1_η and Γ^2_0 is proportional to the pressure with variable coefficient λ_η. It is an impedance boundary condition. (1.4) is the transmission condition.

The operator $A^{\varepsilon\eta}$ corresponding to the problem (1.3)-(1.5) is not self-adjoint, because λ_η is complex. But $(A^{\varepsilon\eta})^{-1}$ is compact and the spectrum of $A^{\varepsilon\eta}$ is formed by isolated points with only infinity as an accumulation point. By (1.2) we see that the spectrum is such that $\operatorname{Re}\omega_k^2 > 0$ and $\operatorname{Im}\omega_k^2 \leqslant 0$. In most of the following we shall consider:

$$\varepsilon = g(\eta)\,, \tag{1.6}$$

g increasing and continuous, $0 = g(0)$ and there will be only a significant parameter.

2. RESULTS OF CONVERGENCE

We multiply (1.3) by $\bar{v} \in H^1(\tilde{\Omega})$, where $\tilde{\Omega} \supset \bigcup_{0<\eta<\eta_0} \Omega_\eta$, η_0 fixed, and integrate by part on Ω_η with $0 < \eta < \eta_0$:

$$\left(a^{\varepsilon}_{ij}\frac{\partial u_{\varepsilon\eta}}{\partial x_j}, \frac{\partial v}{\partial x_i}\right)_{\Omega_\eta} + (\lambda_\eta u_{\varepsilon\eta}, v)_{\Gamma^1_\eta\cup\Gamma^2_0} - \omega^2(b^\varepsilon u_{\varepsilon\eta}, v)_{\Omega_\eta} - (f, v)_{\Omega_\eta} = 0 \tag{2.1}$$

where $(u, w)_\Omega = \displaystyle\int_\Omega u\bar{w}dx$. The problem (1.3)-(1.5) is equivalent to find $u_{\varepsilon\eta} \in H^1(\Omega_\eta)$ satisfying (2.1) $\forall \bar{v} \in H^1(\tilde{\Omega})$. Under the hypothesis (1.6), and

$$\operatorname{Re}\omega^2 \leqslant 0$$

by continuity and the ellipticity property we obtain:

$$\| u_{\varepsilon\eta}|_{\Omega_0} \|_{H^1(\Omega_0)} < k_1\,; \qquad \left\| a^{\varepsilon}_{ij}\frac{\partial u_{\varepsilon\eta}}{\partial x_j}\Big|_{\Omega_0} \right\|_{L^2(\Omega_0)} < k_2 \qquad \forall\varepsilon, \eta$$

Then we consider a subsequence of $u_{\varepsilon\eta}$ such that $u_\varepsilon|_{\Omega_0} \to u^0$ in $H^1(\Omega_0)$ weakly and $L^2(\Omega_0)$ strongly. We remark that mes $(\Omega_\eta \setminus \Omega_0) \to 0$, as $\eta \searrow 0$. By the classical results in the homogenization theory [*2*] the volumic integrals in (2.1) converge to:

$$\left(a^{h}_{ij}\frac{\partial u^0}{\partial x_j}, \frac{\partial v}{\partial x_i}\right)_{\Omega_0} - \omega^2\tilde{b}(u^0, v)_{\Omega_0} - (f, v)_{\Omega_0}\varepsilon,\ \eta \searrow 0$$

where a_{ij}^h and $\tilde{b}$ are the classical homogenized coefficients ([7], ch. 5, § 3). Now we consider the surface integral in (2.1). Using the metric tensor $\underline{g}_\eta$ of Γ_η^1 given by:

$$g_\eta = g_{11}^\eta g_{22}^\eta - (g_{12}^\eta)^2 = \left[1 + \left(\frac{\partial F_\eta}{\partial\left(\frac{s_1}{\eta}\right)}\right)^2\right] \cdot \left[1 + \left(\frac{\partial F_\eta}{\partial\left(\frac{s_2}{\eta}\right)}\right)^2\right] - \left[\frac{\partial F_\eta}{\partial\left(\frac{s_1}{\eta}\right)} \cdot \frac{\partial F_\eta}{\partial\left(\frac{s_2}{\eta}\right)}\right]^2$$

we have

$$(\lambda_\eta u^{\varepsilon\eta}, v)_{\Gamma_\eta^1} = \left(\lambda_1\left(\frac{s_1}{\eta}, \frac{s_2}{\eta}\right) u^{\varepsilon\eta}(s_1, s_2, \theta\eta F_\eta), \bar{v}(s_1, s_2, \theta\eta F_\eta)\sqrt{\underline{g}_\eta}\right)_{\Gamma_0^1}$$

Then by periodicity the weak limit of $\lambda_1(s_1/\eta, s_2/\eta)\sqrt{\underline{g}_\eta}$ is the average Λ_0 in a period ([7], ch. 5, § 4):

$$\Lambda_0 = \frac{1}{|\mathbb{Z}|}\int_{\mathbb{Z}} \lambda_1(z_1, z_2)\sqrt{\underline{g}(z_1, z_2)}\,dz_1 dz_2 \tag{2.2}$$

Moreover, by a result in homogenization of boundary ([7], ch. 5, § 8) we have:

$$u_{\varepsilon\eta}(s_1, s_2, \theta_\eta F_\eta) \to u^0(s_1, s_2, 0)$$

in $L^2(\Gamma_0^1)$ strongly, as ε, $\eta \searrow 0$. Reasoning as before for the integral on Γ_0^2 we have

$$(\lambda_\eta u_{\varepsilon\eta}, v)_{\Gamma_\eta^1 \cup \Gamma_0^2} \to (\Lambda u^0, v)_{\Gamma_0^1 \cup \Gamma_0^2} \varepsilon, \eta \searrow 0$$

where

$$\Lambda = \begin{cases} \Lambda_0 & \text{if} \quad x \in \Gamma_0^1 \\ \lambda_2 & \text{if} \quad x \in \Gamma_0^2 \end{cases}$$

Then we obtain that u^0 is the unique solution of the following problem: $u^0 \in H^1(\Omega_0)$

$$- a_{ij}^h \frac{\partial u^0}{\partial x_i \partial x_j} - \omega^2 \tilde{b} u^0 = f \quad \text{in} \quad \Omega_0 \tag{2.3}$$

$$a_{ij}^h \frac{\partial u^0}{\partial x_j} n_i + \Lambda u^0 = 0 \quad \text{on} \quad \Gamma_0^1 \cup \Gamma_0^2 \tag{2.4}$$

REMARK 2.1. — It is clear that Re $\Lambda > 0$ and Im $\Lambda \leqslant 0$ and then ω^2 is not an eingenvalue of the limit problem. Then u^0 exists and is unique. ■

REMARK 2.2. — We can suppose that $u^{\varepsilon\eta}$, in the interior of Ω_η, for fixed η, has the following formal expansion:

$$u^{\varepsilon\eta}(x) = u_\eta^0(x) + \varepsilon u_\eta^1(x, y) + \varepsilon^2 \dots \qquad y = \frac{x}{\varepsilon}$$

with u_η^l ($l = 1, 2, \dots$) periodic in y with period the parallelopiped Y.

By the classical homogenization theory [2] we have that u_η^0 satisfies the following equation:

$$-\frac{\partial}{\partial x_i}\left(a_{ij}^h\frac{\partial u_\eta^0}{\partial x_j}\right)-\omega^2\tilde{b}u_\eta^0=f \qquad \text{in} \qquad \Omega_n$$

Moreover we can suppose that u_η^0 has the following asymptotic expansion in the parameter η, as x is very near to the boundary Γ_η^1:

$$u_\eta^0(x)=u^{00}(x)+\eta u^{01}(x,z)+\eta^2\dots \qquad z=\frac{s}{\eta}$$

where

$$u^{0l}(x,z) \qquad (l=1,2,\dots)$$

is $\mathbb{Z}$-periodic in z_1, z_2 with period the rectangle $\mathbb{Z}$,

$$a_{ij}^h\frac{\partial u^{0l}}{\partial z_j}\to 0 \qquad \text{as} \qquad z_3\to-\infty \qquad (l=1,2,\dots)$$

As in the homogenization theory for boundary ([7], ch. 5, § 7), we have that u^{00} satisfies:

$$a_{ij}^h\frac{\partial u^{00}}{\partial x_j}n_i+\Lambda u^{00}=0 \qquad \text{on} \qquad \Gamma_0^1\cup\Gamma_0^2. \qquad \blacksquare$$

REMARK 2.3. — We consider the operators $A^{\varepsilon\eta}$ associated to the problem (1.3)-(1.5) and A^0 associated to the limit problem. By hypothesis (1.2) and remark 2.1, the eingenvalues ω^2 of $A^{\varepsilon\eta}$ and A^0 are contained in part of the complex plane such that:

$$\operatorname{Re}\omega^2>0 \qquad \text{and} \qquad \operatorname{Im}\omega^2\leqslant 0. \qquad \blacksquare$$

The convergence of the eingenvalues and the corresponding projectors is studied as in ([7], ch. 11, § 6), and we obtain:

THEOREM 2.1. — If γ is a simple closed curve contained in the resolvent set of A^0, for any $f\in L^2(\Omega_0)$ (we shall continuate f with zero values out of Ω_0, and thus $f\in L^2(\Omega_\eta)$), we have:

$$(P_{\varepsilon\eta}f)\,|_{\Omega_0}\to P_0 f$$

in $L^2(\Omega_0)$ strongly as $\varepsilon,\eta\searrow 0$ where $P_{\varepsilon\eta}$ is the projection

$$P_{\varepsilon\eta}=-\frac{1}{2\pi i}\int_\gamma(A^{\varepsilon\eta}-z)^{-1}dz$$

and P_0 is the analogous projector for A^0. $\blacksquare$

THEOREM 2.2. — If $z^{\varepsilon\eta}$ is a sequence of eigenvalues of $A^{\varepsilon\eta}$ with $\varepsilon,\eta\searrow 0$, such that $z^{\varepsilon\eta}\to z^0$, then z^0 is an eigenvalue of A^0. $\blacksquare$

3. BOUNDARY LAYER

In this section we consider the particular case where the boundary Γ_0^1 is a portion of the plane (x_1, x_2). We suppose that the function F and λ_1 are periodic with period the rectangle $Y_1 \times Y_2 = \mathbb{Z}$. We take $\varepsilon = \eta$;

$$B = \{ Y_1 \times Y_2 \times \mathbb{R} \} \cap \{ -\infty < y_3 < F(y_1, y_2) \}$$

and

$$\Sigma_B = \{ (y_1, y_2, y_3) ; y_3 = F(y_1, y_2) ; (y_1, y_2) \in \mathbb{Z} \} .$$

We suppose $u^\varepsilon(x)$ has the following asymptotic expansion:

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x, y) + \varepsilon^2 \dots ; \qquad y = \frac{x}{\varepsilon}$$

in the interior of the domain, with u^1 Y-periodic in y, and

$$u^\varepsilon(x) = u^0(x) + \varepsilon u^{*1}(x, y) + \varepsilon^2 \dots ; \qquad y = \frac{x}{\varepsilon}$$

for x very near to the boundary Σ_B, with the hypotheses:

(3.1) $\quad u^{*1}$ is $(Y_1 \times Y_2)$-periodic in y_1 and y_2,

(3.2) $$a_{ij} \frac{\partial u^{*1}}{\partial y_j} \to a_{ij} \frac{\partial u^1}{\partial y_j} \qquad \text{as} \quad y_3 \to -\infty$$

We easily see that u^{*1} satisfies:

(3.3) $$-\frac{\partial}{\partial y_i}\left(a_{ij}(y) \frac{\partial u^{*1}}{\partial y_j} \right) = 0 \qquad \text{in} \quad B$$

(3.4) $$a_{ij} \frac{\partial u^{*1}}{\partial y_j} n_i = -\lambda_1 u^0 - a_{ij} \frac{\partial u^0}{\partial x_j} n_i \qquad \text{on} \quad \Sigma_B$$

Moreover u^1 satisfies in B the same equation as u^{*1}. Then we can integrate by parts in B these equations. By (3.1), (3.2) and (3.4) we obtain;

(3.5) $$-\int_B \frac{\partial}{\partial y_i}\left(a_{ij} \frac{\partial (u^{*1} - u^1)}{\partial y_i} \right) dy = \int_{\Sigma_B} \left(-\lambda_1 u^0 - a_{ij} \frac{\partial u^0}{\partial x_j} n_i - a_{ij} \frac{\partial u^1}{\partial y_j} n_i \right) dS_y = 0$$

But, by the flux equation (cf. [7], ch. 5, § 10), (3.5) becomes:

$$\frac{|Y_1| \cdot |Y_2|}{|Y|} \int_Y - a_{ij}\left(\frac{\partial u^0}{\partial x_j} + \frac{\partial u^1}{\partial y_j} \right) dy - u^0 \int_{Y_1 \times Y_2} \lambda_1 \sqrt{g}\, dy_1 dy_2 = 0$$

where $\underline{g}$ is the metric tensor of Σ_B. Then by the classical homogenized coefficients and by (2.2), we have:

$$a^h_{ij} \frac{\partial u^0}{\partial x_j} n_i + \Lambda_0 u^0 = 0 \quad \text{on} \quad \Gamma_0^1 \tag{3.6}$$

The formula (3.6) is the compatibility condition for the existence of the boundary layer, which is obtained by the following theorem:

THEOREM 3.1. — If u^0 is the solution of the problem (2.3)-(2.4) and if u^1 is a solution of the microscopic equation in the interior of the domain (cf. [*7*], ch. 5, § 1) the local problem (3.1), (3.2), (3.3) and (3.4) for u^{*1} has a solution, which is unique up to an additive constant.

Proof. — Let us construct a function $d(y)$ satisfying

$$a_{ij} \frac{\partial d}{\partial y_j} n_i = -\lambda_1 u^0 - a_{ij} \left(\frac{\partial u^0}{\partial x_j} + \frac{\partial u^1}{\partial y_j} \right) n_i \quad \text{on} \quad \Sigma_B$$

$d(y)$ is $(Y_1 \times Y_2)$-periodic; $d(y) = 0$ for sufficiently large $-y_3$

$$[\![d(y)]\!] = 0 ; \qquad \left[\!\!\left[a_{ij} \frac{\partial d}{\partial y_j} n_i \right]\!\!\right] = 0$$

on the surface of discontinuity of a_{ij}.

The function evidently exists and is smooth in any region where a_{ij} are constant. Moreover

$$\int_B \frac{\partial}{\partial y_i} \left(a_{ij} \frac{\partial d}{\partial y_j} \right) dy = 0 \tag{3.7}$$

Now, we take the new unknown $e = u^{*1} - u^1 - d$. The problem for e is:

$$\frac{\partial}{\partial y_i} \left(a_{ij} \frac{\partial e}{\partial y_j} \right) = - \frac{\partial}{\partial y_i} \left(a_{ij} \frac{\partial d}{\partial y_j} \right) \quad \text{in} \quad B \tag{3.8}$$

$$a_{ij} \frac{\partial e}{\partial y_j} n_i = 0 \quad \text{on} \quad \Sigma_B ; \qquad a_{ij} \frac{\partial e}{\partial y_j} \to 0 \quad \text{as} \quad y_3 \to -\infty \tag{3.9}$$

$$e \quad \text{is } (Y_1 \times Y_2)\text{-periodic}. \tag{3.10}$$

Let us define the set V of the functions $w \in H^1_{\text{loc}}(B)$, $(Y_1 \times Y_2)$-periodic and constant for sufficiently large $-y_3$. We introduce the scalar product

$$(\hat{e}, \hat{w})_{\hat{V}} = \int_B a_{ij} \frac{\partial e}{\partial y_i} \frac{\partial \bar{w}}{\partial y_j} dy \tag{3.11}$$

in the space of the equivalence class of V difference of which is a constant. We define $\hat{V}$ as the Hilbert space obtained by completion of the equivalence class space with the norm associated with (3.11). The variational formulation of (3.8)-(3.10) is: find $\hat{e} \in \hat{V}$ such that:

$$(3.12) \qquad (\hat{e}, \hat{w})_{\hat{V}} = \int_B \frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial d}{\partial y_j}\right)\hat{w}dy \qquad \forall \hat{w} \in \hat{V}$$

where the right hand side, by (3.7), is independent of the particular $w \in \hat{w}$ chosen. The existence and uniqueness of $\hat{e}$ will be proved if we show that the right hand side of (3.12) is a bounded functional on $\hat{V}$. To this end, we note that d is zero for sufficiently $-\ y_3$; consequently the domain of integration is in fact a bounded set and then, using the Poincaré's inequality, the proof is achieved. ■

REMARK 3.1. — If we take the impedence $1/\lambda_1$ as the trace of a Y-periodic $\mathscr{C}^0$ function $1/\mu(y_1, y_2, y_3)$, the corresponding limit problem has not uniqueness. For instance it is necessary to impose another hypothesis that no one part of the boundary is included in a hyperplane of rational coefficient ([*2*], ch. 7, § 1). We give two exemples:

1. the boundary Γ_0^1 is the hyperplane $y_3 = 0$ and the period Y is $Y_1 = 3$, $Y_2 = 1$ and $Y_3 = 4$. The function μ is such that $\mu = 1$ in the region $\{0 \geqslant y_3 \geqslant -1\} \cup \{-3 \geqslant y_3 \geqslant -4\}$ and $\mu > 1$ in the region $\{-1 > y_3 > -3\}$. $F(y_1, y_2)$ is such that $y_3 = F(y_1, y_2) < 1$ and then the boundary Σ_B is enclosed in the region where $\mu = 1$. On the boundary the limit problem gives:

$$a_{ij}^h \frac{\partial u^0}{\partial x_j} n_i + \frac{1}{3} | \Sigma_B | u^0 = 0$$

2. The boundary is the hyperplane of the equation: $y_3 = (4/3)y_1$. F and Y are the same as in example 1. The band B′ of periodicity is oblique, it is bounded by the hyperplanes $y_3 = (-3/4)y_1$, $y_3 = (-3/4)(y_1 - 25)$, $y_2 = 0$, $y_2 = 1$ and the normal section is a surface with area equal to 15. We see that there is the region of $\Sigma_{B'}$ such that $\mu > 1$. Then the limit problem on the boundary is:

$$a_{ij}^h \frac{\partial u^0}{\partial x_j} n_i + \Lambda_0 u^0 = 0$$

where

$$\Lambda_0 > \frac{1}{15} | \Sigma_{B'} | = \frac{1}{3} | \Sigma_B | . \qquad ■$$

4. SCATTERING PROBLEM

We consider an exterior domain $\Omega_0 \subset \mathbb{R}^3$, which is the complement of a bounded set E, with smooth boundary. We consider also a bounded set $\Omega_1 \subset \Omega_0$, such that $\Gamma_0^1 = \partial\Omega_0 \cap \overline{\Omega}_1 \neq \varnothing$ and a neighbourhood of Γ_0^1

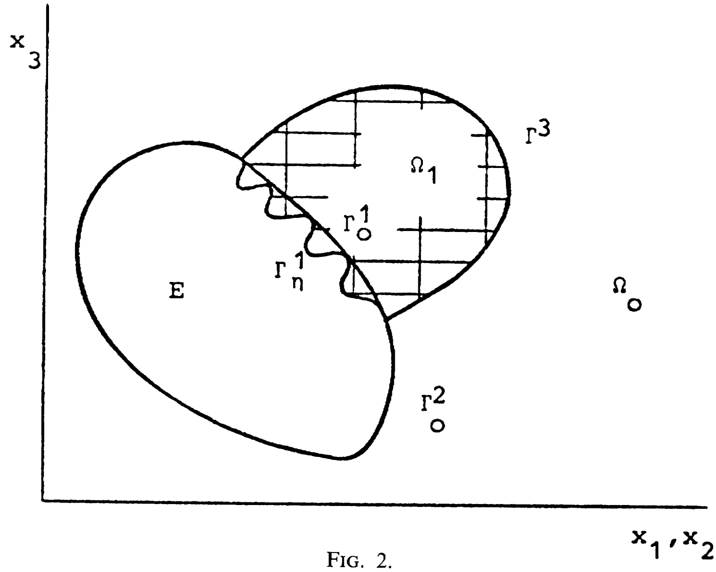

FIG. 2.

have a system of local coordinates. We set $\Gamma_0^2 = \partial\Omega_0 \setminus \Gamma_0^1$. As in section 1, we define the perturbed boundary Γ_η^1, such that the corresponding perturbed domains Ω_η and Ω_1^η containing Ω_0 and Ω_1 respectively. The hypotheses are the same as in section 1, with the following modifications:

$$b^\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega_\eta \setminus \Omega_1^\eta \\ b\left(\dfrac{x}{\varepsilon}\right) & \text{if} \quad x \in \Omega_1^\eta \end{cases}$$

$$a_{ij}^\varepsilon(x) = \begin{cases} \delta_{ij} & \text{if} \quad x \in \Omega_\eta \setminus \Omega_1^\eta \\ a_{ij}\left(\dfrac{x}{\varepsilon}\right) & \text{if} \quad x \in \Omega_1^\eta \end{cases}$$

where $b(y)$ and $a_{ij}(y)$ are almost every where constant and Y-periodic. Under the preceding hypotheses, we consider the following problem $P_{\varepsilon\eta}(\omega, \Omega_\eta)$: let f be a given function of $L^2(\Omega_\eta)$ with $\{\operatorname{supp} f\} \subset \Omega_0$; find $u_{\varepsilon\eta} \in H^1_{loc}(\Omega_\eta)$ such that:

$$(4.1) \qquad -\frac{\partial}{\partial x_i}\left(a_{ij}^\varepsilon \frac{\partial u_{\varepsilon\eta}}{\partial x_j}\right) - \omega^2 b^\varepsilon u_{\varepsilon\eta} = f \qquad \text{in} \quad \Omega_\eta$$

$$a_{ij}^\varepsilon \frac{\partial u_{\varepsilon\eta}}{\partial x_j} n_i + \lambda_\eta u_{\varepsilon\eta} = 0 \qquad \text{on} \quad \Gamma_\eta^1 \cup \Gamma_0^2$$

$$\left[\!\!\left[a_{ij}^\varepsilon \frac{\partial u_{\varepsilon\eta}}{\partial x_j} n_i \right]\!\!\right] = 0\,, \qquad [\![u_{\varepsilon\eta}]\!] = 0$$

on the surface of discontinuity of a_{ij}^ε;

(4.2) $u_{\varepsilon\eta}(x) =$
$$= \frac{1}{4\pi}\int_{|y|=R}\left(-u_{\varepsilon\eta}(y)\frac{\partial}{\partial |y|}\left(\frac{e^{i\omega|x-y|}}{|x-y|}\right) + \frac{\partial u_{\varepsilon\eta}}{\partial |y|}\cdot\frac{e^{i\omega|x-y|}}{|x-y|}\right)dS_y ; \qquad |x| > R$$

The expression (4.2) is equivalent, for ω real, to the outgoing Sommerfeld radiation condition at infinity. We have supposed the time dependence of the form $u_{\varepsilon\eta}(x)e^{-i\omega t}$.

THEOREM 4.1. — If ω is real > 0, the preceding problem $P_{\varepsilon\eta}(\omega, \Omega_\eta)$ has at the most one solution.

Proof. — Using the radiation condition (4.2), by a classical argument ([*9*]) we obtain the conditions to apply the Rellich theorem, which shows that $u_{\varepsilon\eta}$ is zero in a neighbourhood of infinity. The proof is then achieved noting that

$$u_{\varepsilon\eta} = 0 \qquad \text{and} \qquad a_{ij}^{\varepsilon}\frac{\partial u_{\varepsilon\eta}}{\partial x_j}n_i = 0$$

are homogeneous Cauchy conditions at the boundaries between the regions where the coefficients a_{ij}^{ε}, b^{ε} are constant, and that equation (4.1) is piecewise elliptic with constant coefficients. ■

We now deal with the method of reduction to a problem in a bounded domain ([*6*]), which gives the scattering frequencies in a simple way and which also supplies an existence and uniqueness theorem. In this section, ε and η will be considered constants and we shall omit them. Under the preceding hypotheses let ρ be a real number such that the ball $|x| < \rho$ contains $\{E \cup \Omega_1 \cup \operatorname{supp} f\}$. Let g be a function of $L^2(\mathbb{R}^3)$,

$$\operatorname{supp} g \subset \Omega_\eta^\rho = \{\Omega_\eta \cap (|x| < \rho)\}.$$

We consider g known for the time being. We construct the function:

$$w = g * \left(-\frac{1}{4\pi}\frac{e^{i\omega|x-y|}}{|x-y|}\right)$$

which satisfies

$$-\Delta w - \omega^2 w = g \qquad \text{in} \qquad \mathbb{R}^3$$

and the outgoing radiation condition. Also, we construct the function $v \in H^1(\Omega_\eta^{\rho+1})$, where $\Omega_\eta^{\rho+1} = \Omega_\eta \cap \{|x| < \rho + 1\}$, such that:

$$v\big|_{|x|=\rho+1} = w\big|_{|x|=\rho+1} \tag{4.3}$$

$$a_{ij}^{\varepsilon}\frac{\partial v}{\partial x_j}n_i + \lambda_\eta v = 0 \qquad \text{on} \qquad \Gamma_\eta^1 \cup \Gamma_0^2 \tag{4.4}$$

$$\left[\!\left[a_{ij}^{\varepsilon}\frac{\partial v}{\partial x_j}n_i \right]\!\right] = 0 ; \qquad [\![v]\!] = 0 \tag{4.5}$$

on the points of discontinuity of a_{ij}^{ε};

$$-\frac{\partial}{\partial x_i}\left(a_{ij}^{\varepsilon}\frac{\partial v}{\partial x_j}\right) + \lambda_0 v = -\Delta w + \lambda_0 w \tag{4.6}$$

Where λ_0 is chosen such that Im $\lambda_0 > 0$ and then, by (1.2), we have the existence and uniqueness of the solution v of (4.3)-(4.6). Now we consider a function $\gamma(|x|)$ of class $\mathscr{C}^\infty$ which is equal to 1 (resp. 0) for $|x| \leqslant \rho + \frac{1}{3}$ $\left(\text{resp. } |x| > \rho + \frac{2}{3}\right)$, and we construct the function:

$$u = w - \gamma(w - v)$$

u is the solution of the problem $P_{\varepsilon\eta}(\omega, \Omega_\eta)$, if the following relation is satisfied:

$$f = g + T(\omega)g$$

where $T(\omega)$ is a compact operator in $L^2(\Omega_\eta^{\rho+1})$ and holomorphic on the complex plane [*6*]. Moreover using a general theorem on the bounded holomorphic families of compact operators [*8*] we see that $(I + T(\omega))^{-1}$ is a meromorphic function on the complex plane with values in $\mathscr{L}(L^2(\Omega_\eta^{\rho+1}), L^2(\Omega_\eta^{\rho+1}))$.

DEFINITION 4.1. — The poles of the meromorphic function $(I + T(\omega))^{-1}$ are such that there exists a $g \neq 0$ for $f = 0$, and we can construct solutions $\neq 0$ to the problem $P_{\varepsilon\eta}(\omega, \Omega_\eta)$, with $f = 0$, by using w and v. These solutions are *scattering solutions* and the corresponding values of ω are the *scattering frequencies*. ■

5. STUDY OF THE CONVERGENCE

Under the hypotheses of section 4, we consider the limit, as $\eta \searrow 0$ and, by (1.6), $\varepsilon \searrow 0$, for the problem $P_{\varepsilon\eta}(\omega, \Omega_\eta)$ with ω real > 0.

LEMMA 5.1. — The solutions $u^{\varepsilon\eta}$ of the problem $P_{\varepsilon\eta}(\omega, \Omega_\eta)$ are such that:

$$\| u_{\varepsilon\eta} \|_{H^1(\Omega_\eta^{\rho+5})} \leqslant k \qquad \eta, \varepsilon \searrow 0$$

with k independent of η and ε for fixed ρ with

$$\{ E \cup \Omega_1^\eta \cup \text{supp } f \} \subset \{ |x| < \rho \}.$$

Proof. — By contradiction, if the statement is not true, there exists a subsequence of η (and by (1.6) a corresponding subsequence of ε) such that:

$$\| u_{\varepsilon\eta} \|_{H^1(\Omega_\eta^{\rho+5})} \to +\infty \qquad \eta, \varepsilon \searrow 0$$

We define

$$w^{\varepsilon\eta} = \frac{u_{\varepsilon\eta}}{m^{\varepsilon\eta}}; \qquad m^{\varepsilon\eta} = \| u_{\varepsilon\eta} \|_{H^1(\Omega_\eta^{\rho+5})}$$

we have:

$$(5.1) \qquad \| w^{\varepsilon\eta} \|_{H^1(\Omega_\eta^{\rho+5})} = 1 ; \qquad \| w^{\varepsilon\eta} |_{\Omega_0^{\rho+5}} \|_{H^1(\Omega_0^{\rho+5})} \leqslant 1$$

Then for a subsequence, we have:

$$(5.2) \qquad w^{\varepsilon\eta}|_{\Omega_0^{\rho+5}} \to w^0, \qquad \eta, \varepsilon \searrow 0$$

in $H^1(\Omega_0^{\rho+5})$ weakly. We set:

$$\Gamma^3 = \partial\Omega_1^\eta \setminus \Gamma_\eta^1$$

We multiply (4.1) by $v_1/m^{\varepsilon\eta}$ with $v_1 \in H^1(\Omega_1^\eta)$ and $v_1|_{\Gamma^3} = 0$ and integrate by parts on Ω_1^η:

$$(5.3) \quad \left(a_{ij}^\varepsilon \frac{\partial w^{\varepsilon\eta}}{\partial x_j}, \frac{\partial v_1}{\partial x_i}\right)_{\Omega_1^\eta} + (\lambda_\eta w^{\varepsilon\eta}, v_1)_{\Gamma_\eta^1} - \omega^2(b^\varepsilon w^{\varepsilon\eta}, v_1)_{\Omega_1^\eta} = \left(\frac{f}{m^{\varepsilon\eta}}, v_1\right)_{\Omega_1^\eta}$$

$\forall v_1 \in H^1(\Omega_1^\eta)$, $v_1|_{\Gamma^3} = 0$. Now we multiply (4.1) by $v_2/m^{\varepsilon\eta}$ with

$$v_2 \in H^1(\Omega_\eta^{\rho+5} \setminus \Omega_1^\eta) \qquad \text{and} \qquad v_2|_{\Gamma^3 \cup \{|x| = \rho+5\}} = 0$$

and integrate by parts on $\Omega_\eta^{\rho+5} \setminus \Omega_1^\eta$:

$$(5.4) \quad (\nabla w^{\varepsilon\eta}, \nabla v^2)_{\Omega_\eta^{\rho+5}\setminus\Omega_1^\eta} + (\lambda_\eta w^{\varepsilon\eta}, v_2)_{\Gamma_0^2} - \omega^2(w^{\varepsilon\eta}, v_2)_{\Omega_\eta^{\rho+5}\setminus\Omega_1^\eta} = \left(\frac{f}{m^{\varepsilon\eta}}, v_2\right)_{\Omega_\eta^{\rho+5}\setminus\Omega_1^\eta}$$

$\forall v_2 \in H^1(\Omega_\eta^{\rho+5} \setminus \Omega_1^\eta)$, $v_2|_{\Gamma^3 \cup \{|x| = \rho+5\}} = 0$. Next we multiply (4.2) by $1/m^{\varepsilon\eta}$ and in (4.2) we take $R = \rho + 2$:

$$(5.5) \quad w^{\varepsilon\eta}(x) = \frac{1}{4\pi}\int_{|y|=\rho+2}\left[- w^{\varepsilon\eta} \cdot \frac{\partial}{\partial|y|}\frac{e^{i\omega|x-y|}}{|x-y|} + \frac{\partial w^{\varepsilon\eta}}{\partial|y|}\frac{e^{i\omega|x-y|}}{|x-y|}\right]dS_y; \qquad |x| > \rho+2$$

We proceed by steps:

1st step. — We want to study the properties of w^0 in the region $\{\rho < |x| < \rho + 5\}$. In this region $w^{\varepsilon\eta}$ satisfies an elliptic equation with constant coefficients. Moreover, by interior regularity of solutions of the elliptic equations, we obtain that the norm of $w^{\varepsilon\eta}$ in $H^2(\rho+1<|x|<\rho+4)$ is bounded. By the fact that $\Delta w^{\varepsilon\eta} \to \Delta w^0$ in $L^2(\rho < |x| < \rho + 5)$ strongly and by the trace theorem we can take the limit of (5.5) for $\eta, \varepsilon \searrow 0$ and we obtain the uniform convergence of the function and derivatives, in $\{\rho + 3 < |x| < \rho + 5\}$, to:

$$(5.6) \quad w^0(x) = \frac{1}{4\pi}\int_{|y|=\rho+2}\left[- w^0(y)\frac{\partial}{\partial|y|}\frac{e^{i\omega|x-y|}}{|x-y|} + \frac{\partial w^0}{\partial|y|}\frac{e^{i\omega|x-y|}}{|x-y|}\right]dS_y$$

We consider the analytic continuation of w^0 for $|x| > \rho + 2$ given by the same expression (5.6). We consequently consider (5.6) for $|x| > \rho+2$, which implies that $w^0(x)$ satisfies the outgoing radiation condition.

2nd step. — Ne now study the properties of w^0 at finite distance i. e. for $|x| < \rho + 5$. As in section 2, we take the limit of (5.3) and (5.4) as $\eta, \varepsilon \searrow 0$.

We obtain:

$$(5.7)\qquad -a^h_{ij}\frac{\partial^2 w^0}{\partial x_i \partial x_j} - \omega^2 \tilde{b} w^0 = 0 \quad \text{in} \quad \Omega^1$$

$$(5.8)\qquad a^h_{ij}\frac{\partial w^0}{\partial x_j} n_i + \Lambda_0 w^0 = 0 \quad \text{on} \quad \Gamma^1_0$$

$$(5.9)\qquad -\Delta w^0 - \omega^2 w^0 = 0 \quad \text{in} \quad \Omega_0^{\rho+5} \setminus \Omega_1$$

$$(5.10)\qquad \frac{\partial w^0}{\partial n} + \lambda_2 w^0 = 0 \quad \text{on} \quad \Gamma^2$$

The equations (5.7) and (5.9) are elliptic with constant coefficients and then w^0 is holomorphic in Ω_1 and in $\{ \Omega_0^{\rho+5} \setminus \Omega_1 \}$ and can have discontinuities on Γ^3. By virtue of (5.2) we have $w^0 \in H^1(\Omega_0^{\rho+5})$. Then by a standard development and an integration by parts, we obtain the transmission conditions on Γ^3. Then we can write (5.7) and (5.9) in the following form:

$$(5.11)\qquad -\frac{\partial}{\partial x_i}\left(a^H_{ij}\frac{\partial w^0}{\partial x_j}\right) - \omega^2 b^H w^0 = 0 \quad \text{in} \quad \Omega_0^{\rho+5}$$

in the sense of distributions, where:

$$a^H_{ij} = \begin{cases} \delta_{ij} & \text{if} \quad x \in \Omega_0 \setminus \Omega_1 \\ a^h_{ij} & \text{if} \quad x \in \Omega_1 \end{cases} ; \qquad b^H = \begin{cases} 1 & \text{if} \quad x \in \Omega_0 \setminus \Omega_1 \\ \tilde{b} & \text{if} \quad x \in \Omega_1 \end{cases}$$

3rd step. — By (5.6), (5.8), (5.10) and (5.11) and by the uniqueness theorem (see section 4), we have:

$$(5.12)\qquad w^0 = 0$$

We are going to verify that (5.2) is also true in $H^1(\Omega_0^{\rho+5})$ strongly. We multiply the equation (4.1) by $\overline{w^{\varepsilon\eta}}$ and integrate by parts on $\Omega_\eta^{\rho+2}$. Using the elliptic condition (1.1) and the trace theorem on $|x| = \rho + 2$ we obtain

$$(5.13)\qquad \| \nabla w^{\varepsilon\eta} \|_{H^1(\Omega_\eta^{\rho+2})} \to 0$$

But using the uniform convergence of the function and derivatives on $\{ \rho + 1 < |x| < s \}$, with $s > \rho + 5$, we have that (5.13) is true in $H^1(\Omega_\eta^{\rho+5})$ and this is a contradiction with (5.1). ■

THEOREM 5.1. — Let $u_{\varepsilon\eta}$ be the solution of the problem $P_{\varepsilon\eta}(\omega, \Omega_\eta)$, where ω is real. Then, as $\eta, \varepsilon \searrow 0$, one has:

$$u_{\varepsilon\eta}|_{\Omega_0} \to u^0 \quad \text{in} \quad H^1_{loc}(\Omega_0)$$

weakly where u^0 is the solution of the problem $P_H(\omega, \Omega_0)$, which is given by taking u^0, a^H_{ij}, b^H, Λ, in place of $u_{\varepsilon\eta}$, a^ε_{ij}, b^ε, λ_η in the problem $P_{\varepsilon\eta}(\omega, \Omega_{\varepsilon\eta})$.

Proof. — By lemma 5.1 we can extract a subsequence, still denoted by $u_{\varepsilon\eta}$, such that: $u_{\varepsilon\eta}|_{\Omega_0^{\rho+5}} \to u^0$ in $H^1(\Omega_0^{\rho+5})$ weakly. Afterwords we reason as in the first and second steps of the proof of the preceding lemma. ■

6. CONVERGENCE OF SCATTERING FREQUENCIES AND SOLUTIONS

In section 4 we have studied the scattering frequencies which are the poles of $(I + T(\omega))^{-1}$, and the associated solutions. We now consider a point ω of the complex plane which is an accumulation point of scattering frequencies of the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$ (see section 4), as $\eta, \varepsilon \searrow 0$. There exists a subsequence of η and ε, such that $\omega_{\varepsilon\eta} \to \omega$. We consider, for any $\omega_{\varepsilon\eta}$, a corresponding scattering solution $u_{\varepsilon\eta} \neq 0$. We suppose the $u_{\varepsilon\eta}$ are normalized:

$$\| u_{\varepsilon\eta} \|_{L^2(\Omega_\eta^{\rho+5})} = 1 \tag{6.1}$$

Each $u_{\varepsilon\eta}$ satisfies the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$ with $f = 0$.

LEMMA 6.1. — Under the preceding hypotheses the scattering solutions $u_{\varepsilon\eta}$ are such that:

$$\| u_{\varepsilon\eta} \|_{H^1(\Omega_\eta^{\rho+3})} \leqslant k \qquad \eta, \varepsilon \searrow 0 \tag{6.2}$$

The proof is analogous to that of formula (5.13). ■

LEMMA 6.2. — Under the preceding hypotheses, if $\eta, \varepsilon \searrow 0$ (and then $\omega_{\varepsilon\eta} \to \omega$), we have:

$$u_{\varepsilon\eta}|_{\Omega_0^{\rho+3}} \to u^0 \qquad \text{in} \qquad H^1(\Omega_0^{\rho+3}) \tag{6.3}$$

weakly where u^0 is the solution of the homogenized problem $P_H(\omega, \Omega_0)$ (see section 5) with $f = 0$.

Proof. — By (6.2) we can extract from $u_{\varepsilon\eta}$ a subsequence, still denoted $u_{\varepsilon\eta}$, such that (6.3) is verified. Moreover as in the first step of the proof of lemma 5.1, with $u_{\varepsilon\eta}$ in the place of $w^{\varepsilon\eta}$, we show that $u_{\varepsilon\eta}$ satisfies the outgoing radiation condition. Next with the same reasoning as in the second stage of lemma 5.1 we achieve the proof. ■

THEOREM 6.1. — If $\omega \in \mathbb{C}$ is an accumulation point of scattering frequencies $\omega_{\varepsilon\eta}$ of the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$ as $\eta, \varepsilon \searrow 0$, then ω is a scattering frequency of the homogenized problem $P_H(\omega, \Omega_0)$.

Proof. — By the lemmas 6.1 and 6.2 we have that u^0 satisfies the problem $P_H(\omega, \Omega_0)$. Then we must show that

$$u^0 \neq 0 \tag{6.4}$$

Thus we note that, taking the limit, as $\eta, \varepsilon \searrow 0$, in the outgoing radiation condition, the convergence is uniform in every annulus $\{ \rho + 2 < |x| < s \}$ for each $s > \rho + 2$, then $u_{\varepsilon\eta} \to u^0$ in $L^2(\rho + 2 < |x| < s)$ strongly. Then with (6.3), we deduce that $u_{\varepsilon\eta} \to u^0$ in $L^2(\Omega_0^{\rho+5})$ strongly, $\eta, \varepsilon \searrow 0$. But, by the hypothesis (6.1), we have (6.4). ∎

THEOREM 6.2. — Let $u_{\varepsilon\eta} \neq 0$ be a scattering solution, associated with the scattering frequency $\omega_{\varepsilon\eta}$ of the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$, with $f = 0$. If $u^0 \neq 0$ is the limit of $u_{\varepsilon\eta}|_{\Omega_0}$, $\eta, \varepsilon \searrow 0$, in $H^1_{loc}(\Omega_0)$ weakly, then u^0 is a scattering solution of the homogenized problem $P_H(\omega, \Omega_0)$.

Proof. — We reason as in ([*4*] proof of theorem 5.2).

Now we consider a scattering frequency $\omega \in \mathbb{C}$ of the homogenized problem $P_H(\omega, \Omega_0)$ and a circle D centered at ω. We suppose that the boundary ∂D of D is such that no scattering frequency of the problem $P_H(\omega, \Omega_0)$ belongs to ∂D. If we also suppose that no scattering frequency of the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$ $\forall \eta, \varepsilon$, belongs to ∂D, we have the existence of a unique solution of the problem $P_{\varepsilon\eta}(\sigma_{\varepsilon\eta}, \Omega_\eta)$ with fixed f and $\sigma_{\varepsilon\eta} \in \partial D$.

LEMMA 6.3. — The unique solutions of the problems $P_{\varepsilon\eta}(\sigma_{\varepsilon\eta}, \Omega_\eta)$, $\sigma_{\varepsilon\eta} \in \partial D$, are bounded in $L^2(\Omega_\eta^{\rho+5})$, $\eta, \varepsilon \searrow 0$, by a constant independent of η and ε.

Proof. — By contradiction, if the solutions $u_{\varepsilon\eta}$ are not bounded, there is a subsequence of $u_{\varepsilon\eta}$ such that $u_{\varepsilon\eta} = u_{\varepsilon\eta}(\sigma_{\varepsilon\eta})$ with $\sigma_{\varepsilon\eta} \in \partial D$ and:

$$\| u_{\varepsilon\eta} \|_{L^2(\Omega_\eta^{\rho+5})} = m^{\varepsilon\eta} \to +\infty \qquad \eta, \varepsilon \searrow 0$$

The $\sigma_{\varepsilon\eta} \in \partial D$ are bounded and we can assume that they converge, as $\eta, \varepsilon \searrow 0$, to $\sigma \in \partial D$. We normalize $u_{\varepsilon\eta} : m^{\varepsilon\eta} = \| u_{\varepsilon\eta} \|_{L^2(\Omega_\eta^{\rho+5})}$,

$$w^{\varepsilon\eta} = \frac{u_{\varepsilon\eta}}{m^{\varepsilon\eta}}, \qquad \| w^{\varepsilon\eta} \|_{L^2(\Omega_\eta^{\rho+5})} = 1$$

As in the proof of lemma 6.1, we obtain:

$$w^{\varepsilon\eta}|_{\Omega_0^{\rho+3}} \to w^0 \qquad \text{in} \qquad H^1(\Omega_0^{\rho+3})$$

weakly.

Moreover reasoning as in lemma 5.1, we have that $w^0 \neq 0$ satisfies the problem $P_H(\sigma, \Omega_0)$ with $f = 0$ and then w^0 and σ are a scattering solution and frequency of the homogenized problem. This leads to a contradiction because $\sigma \in \partial D$. ∎

THEOREM 6.3. — Let ω be a scattering frequency of the homogenized problem $P_H(\omega, \Omega_0)$. Then, as $\eta, \varepsilon \searrow 0$, there exists at least a scattering frequency $\omega_{\varepsilon\eta}$ of the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$ such that $\omega_{\varepsilon\eta} \to \omega$.

Proof. — Let f be an arbitrary element of $L^2(\Omega_\eta^\rho)$; f will be fixed throughout. The proof uses the fact that, if σ is not a scattering frequency of the problem $P_H(\sigma, \Omega_0)$, there exists a unique outgoing solution $u(\sigma) \in H^1_{loc}(\Omega_0)$. We know also (see section 4) that $u(\sigma)$ has an isolated singularity (pole) for $\sigma = \omega$, i. e. the scattering frequency ω is isolated. Let D be a circle centered at ω and ∂D its boundary; assume ∂D is sufficiently small so that no further scattering frequencies, except ω, belong to $\overline{D}$. If the statement is not true, for any η and ε the corresponding problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$ has no scattering frequencies on $\overline{D}$. Then we can consider the unique outgoing solution $u_{\varepsilon\eta}(\sigma_{\varepsilon\eta})$, with $\sigma_{\varepsilon\eta} \in \partial D$ of the problem $P_{\varepsilon\eta}(\sigma_{\varepsilon\eta}, \Omega_\eta)$. Then by lemma 6.3 the norm of $u_{\varepsilon\eta}(\sigma_{\varepsilon\eta})$ is bounded in $L^2(\Omega_\eta^{\rho+5})$, $\eta, \varepsilon \searrow 0$, and by the same reasoning of lemma 6.1 it is bounded in $H^1(\Omega_\eta^{\rho+3})$.

Then, for any fixed $\sigma \in \partial D$, we take the limit of homogenization as in lemma 6.2:

$$(6.5) \qquad u_{\varepsilon\eta}(\sigma) \to u(\sigma) \quad \text{in} \quad L^2(\Omega_0^{\rho+3})$$

strongly, $\eta, \varepsilon \searrow 0$ where $u(\sigma)$ is the solution of the problem $P_H(\sigma, \Omega_0)$. Moreover by the fact that $u(\sigma)$ has a pole in ω, we can take the Laurent's series of $u(\sigma)$ and there is an entire $m > 0$ such that $(\sigma - \omega)^m u(\sigma)$ has a residue $A \neq 0$ on ∂D. We can calculate:

$$(6.6) \qquad \int_{\partial D} (\sigma - \omega)^m u_{\varepsilon\eta}(\sigma) d\sigma = A^{\varepsilon\eta}$$

and by the hypothesis that $\sigma \in \partial D$ and $u_{\varepsilon\eta}(\sigma)$ have no singularities on $\overline{D}$, $\forall \eta, \varepsilon$, we have:

$$(6.7) \qquad A^{\varepsilon\eta} = 0 \qquad \forall \eta, \varepsilon$$

Moreover, by (6.5) and by the Lebesgue dominated convergence theorem, we can take the limit in (6.6):

$$\int_{\partial D} (\sigma - \omega)^m u_{\varepsilon\eta}(\sigma) d\sigma \to \int_{\partial D} (\sigma - \omega)^m u(\sigma) d\sigma \qquad \eta, \varepsilon \searrow 0$$

then $A^{\varepsilon\eta} \to A \neq 0$ and we have contradiction with (6.7).

Theorem 6.3 is proved. ■

THEOREM 6.4. — Let $u \neq 0$ be a scattering solution, associated to the scattering frequency ω, of the problem $P_H(\omega, \Omega_0)$. There is a sequence $u_{\varepsilon\eta} \neq 0$ of scattering solutions of the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$ which converges to u in $H^1_{loc}(\Omega_0)$ weakly.

Proof. — We suppose u normalized and, proceeding as in the proof of theorem 6.3, we obtain a sequence of scattering frequencies $\omega_{\varepsilon\eta}$ of the problem $P_{\varepsilon\eta}(\omega_{\varepsilon\eta}, \Omega_\eta)$, such that $\omega_{\varepsilon\eta} \to \omega$ for $\eta, \varepsilon \searrow 0$. We normalize the scattering solution $u_{\varepsilon\eta} \neq 0$ corresponding to $\omega_{\varepsilon\eta}$. Proceeding just as in the proof of lemma 6.1 and 6.2 we show that $u_{\varepsilon\eta} \to u$ in $H^1_{loc}(\Omega_0)$ weakly. ■

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