ROBERT ALICKI

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<http://www.numdam.org/item?id=AIHPA_1981__35_2_97_0>
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by

Robert Alicki
Institute of Physics, Gdańsk University
PL 80-952 Gdańsk, Poland

Abstract. — We consider the wave operators and scattering matrix for quantum dynamical semigroups. The dynamical semigroups with bounded perturbations are briefly studied using the Cook's method and the simplified model of heavy-ion collision is presented as an example.

1. Introduction

The purpose of this note is to clarify some ideas concerning the phenomenological approach to dissipative scattering. There exists a class of scattering phenomena for example a scattering and capture of a neutron by a nucleus or the heavy-ion collision which can be described in terms of the theory of open systems [1]-[5]. Namely we can eliminate a large number of internal degrees of freedom together with some external fields to obtain the irreversible dynamics for few fixed degrees of freedom (e.g. 3-degrees of freedom of the relative motion of two heavy-ions). Moreover because the interaction of internal degrees of freedom is strong then the relaxation time for them is short and hence one can apply the Markovian approximation [6] [7]. It follows that the dynamics of such open system can be described by quantum dynamical semigroup.

We start by introducing some preliminary mathematical definitions. Let \( \mathcal{H} \) be a Hilbert space associated with the open system with scalar
product $(\cdot, \cdot)$ and norm $\| \cdot \| = \sqrt{(\cdot, \cdot)}$. $L^\infty(\mathcal{H})$ is a real Banach space of hermitian operators with operator norm $\| \cdot \|_\infty$ and $L^1(\mathcal{H})$ is a Banach subspace of $L^\infty(\mathcal{H})$ which contains compact operators.

$L^1(\mathcal{H})$ denote a real Banach space of hermitian trace-class operators with trace norm $\| \cdot \|_1$.

We have also the relations $L^1(\mathcal{H})^* \cong L^1(\mathcal{H})$, $L^1(\mathcal{H})^* = L^\infty(\mathcal{H})$.

Consider a one parameter strongly continuous contracting and positive semigroup $\{ \Lambda_t = e^{tL}, t \geq 0 \}$ on $L^1(\mathcal{H})$.

We call it dynamical semigroup if for all $t \geq 0$ the dual map $\Lambda_t^*$ is completely positive [6]-[8] and conservative dynamical semigroup if moreover $\text{tr}(\Lambda_t \sigma) = \text{tr} \sigma$, for all $\sigma \in L^1(\mathcal{H})$, $t \geq 0$.

REMARKS. — The non conservative semigroups can describe the scattering if some other open channels of reaction are taken into account [3]-[9]. The complete positivity of $\Lambda_t^*$ will be not used manifestly further, but this property is based on strong physical arguments and restricts the class of dynamical semigroups [6]-[8].

2. WAVE OPERATORS AND S-MATRIX

We start by assuming that the free evolution is represented by the dynamical group $\{ U_t ; t \in \mathbb{R}^1 \}$

$$U_t \sigma = e^{-iH_0 t} e^{iH_0} \equiv e^{tH_0} \sigma, \sigma \in L^1(\mathcal{H}),$$

where $H_0$ is a self-adjoint Hamiltonian.

The perturbed dynamics is given by the quantum dynamical semigroup

$$\{ \Lambda_t = e^{tL} ; t \geq 0 \}.$$

As in ordinary scattering theory we define the wave operators $W_1$ and $W_2^*$

I) $W_1 : L^1(\mathcal{H}) \to L^1(\mathcal{H})$

$$W_1 \sigma = \lim_{t \to \infty} \Lambda_t U_{-t} \sigma \quad (2.2)$$

for all $\sigma \in L^1(\mathcal{H})$

II) $\tilde{W}_2 : L^\infty(\mathcal{H}) \to L^\infty(\mathcal{H})$

$$\tilde{W}_2 a = \lim_{t \to \infty} \Lambda_t^* U_t a \quad (2.3)$$

for all $a \in L^\infty(\mathcal{H})$.

We have the dual homomorphism

$$\tilde{W}_2^* : L^\infty(\mathcal{H})^* \to L^\infty(\mathcal{H})^* \cong L^1(\mathcal{H})^*.$$

Because $L^\infty(\mathcal{H})^* \supset L^1(\mathcal{H})$ we can finally define

$$W_2^* = \tilde{W}_2^* \upharpoonright L^1(\mathcal{H}), \quad W_2^* : L^1(\mathcal{H}) \to L^1(\mathcal{H}). \quad (2.4)$$
Therefore we obtain the scattering matrix

\[ S = W_2^* W_1 \]

**Remark.** — The definition of \( W_2^* \) presented here seems to be more appropriate than the strong limit \( W_2^* \sigma = \lim_{t \to -\infty} U_{-t} \Lambda_t \sigma \), \( \sigma \in L^1(\mathcal{H}) \) because for the later and under the assumption that \( \Lambda_t \) is conservative the S-matrix preserves the trace of \( \sigma \) and hence cannot describe for instance the capture of particle by the potential of target which is possible in the case of dissipative scattering.

Moreover the definition (2.3)-(2.4) allows to apply the Cook's criterion. One can easily prove the following properties

i) if \( W_1, W_2^* \) exist then the following probability function

\[ P(\rho_{in} \to \rho_{out}) := \lim_{t \to -\infty} \langle \rho_{out} \rangle \langle e^{-tL_0} e^{2tL_1} e^{-tL_0} \rho_{in} \rangle \]

\[ = \langle \rho_{out} \rangle \langle S \rho_{in} \rangle \rho_{out} \]  

(2.6)

where

\[ \rho_{in} \in L^1(\mathcal{H}), \quad \rho_{in} \geq 0, \quad \text{tr} \rho_{in} = 1, \quad || \rho_{out} ||=1. \]

ii) \( W_1, W_2^* \) are positive contractions on \( L^1(\mathcal{H}) \)

iii) if \( \{ \Lambda_t, t \geq 0 \} \) is conservative then \( W_1 \) is trace preserving

iv) \( e^{\tau L} W_1 = W_1 e^{\tau L_0} \)

\[ W_2^* e^{\tau L_1} = e^{\tau L_0} W_2^*, \quad \tau \geq 0 \]

and hence \( e^{\tau L_0} S = S e^{\tau L_0} \).

One can easily generalize Cook's arguments [11] [12] to prove the existence of \( W_1, W_2^* \) (see also [10]).

**Proposition 1.** — Let \( D \{ D^* \} \) be a dense set in \( L^1(\mathcal{H}) \{ L^\infty(\mathcal{H}) \} \) such that

\[ e^{-tL_0} D \subset \text{dom}(L) \cap \text{dom}(L_0) \{ e^{-tL_0} D^* \subset \text{dom}(L^*) \cap \text{dom}(L_0^*) \} \]

for all \( t \in [s, \infty) \),

and some \( s \geq 0 \).

Assume that the function \( ||(L - L_0 )e^{-tL_0} \sigma ||_1 \{ ||(L^* - L_0^*)e^{-tL_0} \sigma ||_\infty \} \) is integrable on \( [s, \infty) \) for \( \sigma \in D \{ a \in D^* \} \).

Then \( W_1 \{ \hat{W}_2 \) and therefore \( W_2^* \) exists.

### 3. QUANTUM DYNAMICAL SEMIGROUPS WITH BOUNDED PERTURBATIONS

We consider a quantum mechanical Fokker-Planck equation

\[ \frac{d\rho}{dt} = -i[H_0 + U, \rho] + \sum_\alpha V_\alpha \rho V_\alpha^* - \frac{1}{2} \{ B, \rho \} \equiv L_\beta \]  

(3.1)

Here $H_0$ is a self-adjoint operator (free Hamiltonian), $U = U^*$ is bounded (Hamiltonian perturbation) and $\sum_a V_a^* V_a \leq B$, $B$ is also bounded.

By standard theorems [11] and using Lindblad results one can prove that the equation (3.1) generates the quantum dynamical semigroup $\{ \Lambda_t = e^{tA}, t \geq 0 \}$ (conservative if $B = \sum_a V_a^* V_a$). Following Davies [9] the core of $L_0 = -i[H_0, \cdot]$ and hence of $L$ is given by

$$\mathcal{D} = (1 + iH_0)^{-1}L^1(\mathcal{H})(1 + iH_0)^{-1}$$

(3.2)

One can check that for $\rho \in L^1(\mathcal{H})$ and $a \in L^1(\mathcal{H})$

$$\text{tr} (e^{tL}\rho a) = \text{tr} (\rho e^{tL^*}a)$$

(3.3)

where $L_\ast : \text{dom}(L_\ast) \to L^1(\mathcal{H})$, $L_\ast a = i[H_0 + U, a] + \sum_a V_a^* a V_a - \frac{1}{2} \{ B, a \}$. (3.4)

**Remark.** In this paper we denote by $i[H, \cdot]$ the closure of a commutator (in a suitable Banach space of operators) which is a generator of one-parameter group $X \to e^{itH}Xe^{-itH}$.

Using similar arguments one can show that

$$\mathcal{D}_\ast = (1 + iH_0)^{-1}L^1(\mathcal{H})(1 - iH_0)^*$$

(3.5)

is a core for $L_\ast$.

It follows that

$$\tilde{W}_2 a \equiv W_2 a = \lim_{i \to \infty} e^{tL^*} e^{iL} a$$

where $L_\ast \in L^1(\mathcal{H})$ and $\tilde{W}_2^* = W_2^*$ in this case (if $W_2$ exists of course).

Now one can prove the simple form of Cook's criterion valid for the dynamical semigroup governed by (3.1).

**Proposition 2.** Let $\mathcal{H}_0$ be a dense set in $\text{dom}(H_0)$.

Assume that the following functions are integrable on $[s, \infty)$, $s \geq 0$ for all $\psi \in \mathcal{H}_0$.

$$\begin{align*}
a) & \quad \sum_a || V_a e^{-itH_0} \psi ||^2, \quad || Be^{-itH_0} \psi ||, \quad || U e^{-itH_0} \psi ||, \quad (3.6) \\
b) & \quad \sum_a || V_a^* e^{itH_0} \psi ||^2, \quad || Be^{itH_0} \psi ||, \quad || U e^{itH_0} \psi ||, \quad (3.7)
\end{align*}$$

Then $a)$ implies the existence of $W_1$ and $b)$ implies the existence of $W_2^*$.  

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Proof. — Let $D$ be a set of all finite rank hermitian operators whose eigenvectors lie in $\text{dom}(H_0)$.

$D$ is dense in $L^1(\mathcal{H}), L^\infty(\mathcal{H})$ and $D \subset D \cap D^*$. Therefore $D$ can be used as a set $D$ and $D^*$ in Proposition 1.

Taking $\sigma = |\psi\rangle\langle\psi|$, $\psi \in \mathcal{H}_0$ one can easily prove that $a)$ implies integrability of $\| (L - L_0)e^{-tL_0}\sigma \|_1$ and similarly for $a = |\psi\rangle\langle\psi|$ and $\| (L^* - L_0^*)e^{tL_0^*}\sigma \|_\infty$ under the assumption $b)$.

Taking linear combinations we extend the above results to $\sigma, a \in D$ and therefore all assumptions of Proposition 1 are fulfilled.

4. SIMPLE MODEL OF HEAVY-ION COLLISION

In paper 5 one can find the heuristic derivation based on the simple model of heavy-ion collision of the quantum Fokker-Planck equation describing the relative motion of two nuclei. The final result is the following

$$\frac{d\rho}{dt} = -i [H_0 + U, \rho] + \frac{1}{2} \sum_{k=1}^{3} \{ [V_k, \rho V_k^*] + [V_k \rho, V_k^*] \} \equiv L_\delta \quad (4.1)$$

Here $\rho$ is a density matrix on Hilbert space $L^2(\mathbb{R}^3)$ and

$$(H_0\psi)(\vec{x}) = -\frac{1}{2m} \Delta \psi(\vec{x}), \quad \vec{x} = (x_1, x_2, x_3). \quad (4.2)$$

$$(U\psi)(\vec{\xi}) = U(\vec{\xi})\psi(\vec{\xi}) \quad (4.3)$$

$$(V_k\psi)(\vec{\xi}) = W(\vec{\xi}) \left( x_k + \alpha \left. \frac{\partial}{\partial x_k} \right) \psi(\vec{\xi}) , \quad k = 1, 2, 3 \quad (4.4)$$

$$\lim_{|\vec{x}| \to \infty} W(\vec{\xi}) = \lim_{|\vec{x}| \to \infty} U(\vec{\xi}) = 0, \quad \alpha > 0$$

To give the physical motivation of (4.1) we write down the formal Heisenberg evolution equations for position and momentum operators $(\hat{x}_k, \hat{p}_k) k = 1, 2, 3$

$$\frac{d\hat{x}_k}{dt} = \frac{1}{m} \hat{p}_k + \alpha^2 \left. \frac{\partial}{\partial \hat{x}_k} \right) W^2(\hat{\xi}) - \alpha W^2(\hat{\xi}) \hat{x}_k \quad (4.5)$$

$$\frac{d\hat{p}_k}{dt} = -\left. \frac{\partial}{\partial \hat{x}_k} \right) U(\hat{\xi}) - \frac{\alpha}{2} \{ W^2(\hat{\xi}), \hat{p}_k \}, \quad k = 1, 2, 3 \quad (4.6)$$

For large $|\langle \hat{p}_k \rangle|$ or small $\alpha$ (4.5) (4.6) correspond to the classical Newton equation with a friction force $-\alpha W^2(\xi)\hat{p}$ describing the "nuclear friction" in heavy-ion collisions [4].

Under some technical conditions one can construct rigorously the dynamical semigroup generated by (4.1) using the method of minimal solution [9] but unfortunately the domain of obtained generator is not manifestly defined and then we cannot easily adopt the methods presented in Section 3.

However one can introduce the "regularized version" of equation (4.1). Namely we assume that

A) $U(\vec{x}), W(\vec{x}), x_k W(\vec{x})$ are bounded and continuous functions on $\mathbb{R}^3$,
B) operator $V_k$ is replaced by

$$
(V_k^{(a)} \psi)(\vec{x}) = W(\vec{x}) x_k \psi(\vec{x}) + \omega W(\vec{x}) \frac{1}{\varepsilon} [\psi(\vec{x} + \varepsilon \vec{e}_k) - \psi(\vec{x})] \quad (4.7)
$$

The regularized generator $L^{(a)}$ belongs to the class described in Section 4 and $\{ e^{i L^{(a)}}, t \geq 0 \}$ is a conservative dynamical semigroup.

**Proposition 3.** — Assume that A) B) hold and moreover

$$
\int_{\mathbb{R}^3} \{ U^2(\vec{x}) + x^2 W^2(\vec{x}) \} \, d^3 \vec{x} < \infty \quad (4.8)
$$

Then the wave operators $W_1$ and $W_2^*$ exist for the generator $L^{(a)}$.

**Proof.** — Taking into account the structure of $L^{(a)}$, $V_k^{(a)}$ and Proposition 2 it is sufficient to prove that the following functions are integrable on $[s, \infty)$ and $(-\infty, -s]$

$$
\| U \psi_t \|, \quad \| W \psi_t \|, \quad \| W x_k \psi_t \|, \quad k = 1, 2, 3 \quad (4.9)
$$

for $\psi_t = e^{-i t H_0} \psi, \psi \in \mathcal{H}_0 \subset \text{dom}(H_0)$ ($\mathcal{H}_0$ is dense in $L^2(\mathbb{R}^3)$).

Taking $\mathcal{H}_0$ as a linear subspace spanned by all Gaussian functions

$$
\exp \left\{ -\frac{| \vec{x} - \vec{e} |^2}{2a^2} \right\}
$$

we apply the standard method [13] to prove the integrability of (4.9).

**REFERENCES**


(Manuscrit reçu le 11 février 1981)