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by

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ABSTRACT. — The main object of the study of the present paper is the pair (L, P) consisting of two nonempty sets: the logic L of experimentally verifiable propositions (also called questions, events, yes-no measurements) and the set P of pure states, and satisfying several physically plausible postulates formulated recently (here, the postulates (A1), (A3) and (PP)). (L, P), endowed with the axioms mentioned above, was recently shown to give a proper non-lattice frame replacing the well-known quantum logic axiomatic scheme. In the present paper we consider in detail two important concepts, the projection postulate and the superposition principle, both within the non-lattice framework of the pair (L, P).

The results we obtained here extend and generalize the results obtained in our earlier papers.

1. BASIC AXIOMS AND NOTATION

We assume, following the quantum logic approach to quantum axiomatics, that with every physical system there is associated a pair (L, S) consisting of two sets L and S, whose elements are called propositions (questions, events, yes-no measurements) and states, respectively, and impose the following axioms (now commonly accepted), which relate L to S:

(A1) L is an orthomodular σ-orthoposet, i. e. a σ-orthocomplete orthocomplemented partially ordered set possessing the following property, called the orthomodularity:

$a \leqslant b$ (where $a, b \in L$) implies $b = a \vee c$ for some $c \in L$, $c \leqslant a'$.

(A2) S is a σ-convex set of probability measures on L.

Remark. — We clearly assume that L $\neq$ { 0, 1 } . By $'$ we denote the ortho-complementation of L, and the symbol $\vee$ ($\wedge$, respectively) stands for the least upper bound (the greatest lower bound, respectively) in L.

If $a \leqslant b'$ (where a, $b \in$ L), then we say that a and b are *orthogonal* and write $a \perp b$. The orthogonality relation is clearly symmetric.

When $a \perp b$, then we write $a + b$ instead of $a \vee b$, and when $a \leqslant b$, we write $b - a$ in place of $(b' + a)' = b \wedge a'$.

Finally, the element c in (A1) is easily seen to be unique, as it is shown to be $c = b - a$ (see Varadarajan [25]).

The set L satisfying the postulate (A1) above is called the *logic of propositions* (briefly, the *logic*). Clearly, the mathematical structure of the logic L established in axiom (A1) is too modest (or, in other words, the axioms (A1) and (A2) are too general) in order to get a significant information about the physical system under study, described by the pair (L, S). So, there usually is assumed a more rich mathematical structure for L, namely the structure of an atomistic orthomodular complete lattice with the covering law holding in it (see, e. g., Varadarajan [26], Piron [22]). However, in view of the conceptual difficulties connected with the physical explanation and justification of some of these axioms (like the complete lattice structure of L, its atomisticity, or the validity of the covering law), the problem arises to find out a system of axioms for quantum theory free of the troubles mentioned above. One of the possible answers to this question has been found in recent papers of the author (Guz [8], [9], [12]). We will here follow the basic idea of these papers, where the main attention has been directed to the structure of the set P of pure states of the physical system, and assume the following:

(A3) There exists a subset P $\subseteq$ S whose members, called pure states, are assumed to satisfy the following requirements:

i) For every nonzero proposition $a \in$ L there exists a pure state $p \in$ P such that $p(a) = 1$.

ii) If for each pure state $p \in$ P satisfying $p(a) = 1$ we always have $p(b) = 1$, where $a, b \in$ L, then $a \leqslant b$.

iii) For any pure state $p \in$ P there is a proposition $a \in$ L such that $p(a) = 1$ and $q(a) < 1$ for all pure states $q \neq p$.

Note that the name « pure state » for a member of the set P, satisfying the conditions *i*), *ii*), *iii*) above, is fully justified, since it can easily be verified (Guz [12]) that every p from P is an extreme point of the σ-convex set of probability measures on L spanned by P.

It should also be emphasized at this moment that the assumptions *i*)-*iii*) are by no means new. For instance, *i*) and *iii*) were assumed as postulates by Mac Laren [16], and *ii*) by Gudder [4]. Their physical significance is

clear; for example, the assumption *iii*) asserts that pure states may be realized in the « laboratory », because *iii*) tell us that there is a measuring device answering the experimental question (described by the proposition *a* in *iii*)) « Is the physical system in the pure state *p* ? ». The interpretation of the other assumptions, *i*) and *ii*), is obvious.

Note also that the assumption *i*) can be obtained as a direct consequence of an obvious physical assumption, the so-called « repeatibility hypothesis », which states that the measurement of a proposition repeated immediately will always give the same result.

Axiom (A3) leads to several important consequences. It has been shown (Guz [8]) that having assumed (A1) and (A3) we are in a position to prove that the propositional logic L is atomistic (i. e. L is atomic and each nonzero $a \in L$ is the least upper bound of the atoms contained in a) and that there is a bijection $s : P \rightarrow A(L)$ of the set P of pure states onto the set A(L) of all atoms of L such that for every $p \in P$

(1) $p(s(p)) = 1$;
(2) $p(a) = 1$, where $a \in L$, implies $a \geqslant s(p)$.

The atomic proposition $s(p)$ is called the *support* or *carrier* of the pure state *p* (Zierler [27], Pool [23]), and it is denoted also by supp *p* or carr *p*.

Now let m_1, m_2 be two arbitrary states from S. We say that m_1 and m_2 are *orthogonal* (Gudder [4]), and write $m_1 \perp m_2$, if there is a proposition $a \in L$ such that $m_1(a) = m_2(a') = 1$. Note that this orthogonality relation is clearly symmetric.

The pair (P, $\perp$) with $\perp$ denoting the orthogonality defined above restricted to P, called the *phase space* of the physical system (see Guz [8]), plays an important role in quantum axiomatics. Before seeing this, however, we must introduce some new definitions.

Let $M \subseteq P$. We define

$$M^{\perp} = \{ p \in P : p \perp q \text{ for all } q \in M \},$$
$$M^{-} = M^{\perp\perp}.$$

Obviously, $M \subseteq M^{-}$, and when $M = M^{-}$, we call the set M *closed*. The family C(P, $\perp$) of all the closed subsets of P is called the *phase geometry* associated with a physical system (Guz [6]). It can easily be verified (see Guz [8]) that under the set-theoretical inclusion C(P, $\perp$) becomes an atomistic complete lattice with the lattice operations given by

$$\bigvee_j M_j = \left(\bigcup_j M_j \right)^{-}, \qquad \bigwedge_j M_j = \bigcap_j M_j$$

(where $\{ M_j \}$ stands for an arbitrary family of closed subsets of P), and that C(P, $\perp$) is orthocomplemented by the correspondence $M \rightarrow M^{\perp}$ ($M \in C(P, \perp)$).

For the empty set $\varnothing$ we put, by definition, $\varnothing^\perp = P$, which immediately leads to $\varnothing, P \in C(P, \perp)$.

The importance of the phase geometry $C(P, \perp)$ consists in the validity of the following embedding theorem (Guz [8]):

For each $a \in L$ the set $a^1 = \{ p \in P : p(a) = 1 \}$ belongs to $C(P, \perp)$, and the correspondence $a \to a^1$ defines an orthoinjection of the propositional logic L into the phase geometry $C(P, \perp)$.

Let m_1, m_2 be two arbitrary states again. The number

$$(m_1 : m_2) = \inf \{ m_1(a) : a \in L, m_2(a) = 1 \}$$

is called the *degree of dependance of m_1 on m_2* (Guz [7]).

It should be noticed at this moment that the number $(m_1 : m_2)$ has independently been introduced several years ago by Mielnik (see [19], [20]) under the name « transition probability between m_1 and m_2 », however, we shall refer to $(m_1 : m_2)$ as to the transition probability only when both m_1 and m_2 are pure states.

It can easily be seen (Guz [12]) that in our axiomatic scheme described by axioms (A1) and (A3) the transition probability between any two pure states $p, q \in P$ is given by

$$(p : q) = p(s(q)).$$

Moreover, one can easily verify the following properties of the transition probability:

i) $0 \leqslant (p : q) \leqslant 1$ for all $p, q \in P$.

ii) $(p : q) = 0$ if and only if $p \perp q$.

iii) $(p : q) = 1$ if and only if $p = q$.

2. COVERING LAW AND PROJECTION POSTULATE

There are several equivalent formulations of the so-called covering law in lattice quantum logics (see Bugajska and Bugajski [1]), among them the most interesting is perhaps the so-called *projection postulate* (later on abbreviated to (PP)), which states the following:

If $p(a) \neq 0$, where $a \in L$ and $p \in P$, then there exists one and only one pure state $q \in P$ such that $q(a) = 1$ and $p(a) = (p : q)$.

The significance of the projection postulate (PP) has been clarified in the papers of Bugajska and Bugajski [1], [2], where were proved several important consequences of this postulate for the theory of non-lattice quantum logics (see [2]). In this section we shall show the equivalence of the projection postulate (PP) with the covering law for the case of a non-lattice quantum logic L.

We shall say that the *covering law* holds in L, or that L possesses the *covering property*, L being an atomic σ-orthoposet, provided

i) for each $a \in$ L and each atom $e \in$ A(L) there exists $a \vee e$ in L;

ii) $a \vee e$ covers a, when $e \not\leqslant a$ (in other words, $a \vee e \geqslant b \geqslant a$ implies either $b = a$ or $b = a \vee e$).

Having assumed the property i) for L, where L is an atomic orthomodular σ-orthoposet, we are in a position to prove that the « covering property » ii) is equivalent to any of several other conditions (see Guz [9]), among them is the well-known Jauch-Piron's condition (abbreviated to (JP)):

(JP) For each $a \in$ L and each $e \in$ A(L), $a \vee e - a$ is either an atom or zero.

Remark. — For lattice quantum logics the equivalence between the covering law and the (JP) condition above has been established by Jauch and Piron [15] long time ago, and their proof of this equivalence is easily seen to hold also for non-lattice quantum logics.

To prove the equivalence of (PP) with the covering law we will follow the arguments of Bugajska and Bugajski [1], which were applied to the case, where L was a lattice. First let us note that owing to one—to—one correspondence $s : p \rightarrow s(p)$ between pure states and atoms of L, one can rewrite the projection postulate (PP) in the following equivalent form:

(PP′) If $p(a) \neq 0$, where $a \in$ L and $p \in$ P, then there is exactly one atomic proposition $e \leqslant a$ such that $p(a) = p(e)$.

We shall now show that (PP′) is equivalent to the pair of assumptions i) and (JP), and hence also to the covering law.

We shall begin by proving the implication from (PP′) to i) + (JP). Since the implication (PP′) $\Rightarrow$ i) is a known fact (see Bugajska and Bugajski [2]) we need only to show the implication (PP′) $\Rightarrow$ (JP).

Let $a \in$ L, $e \in$ A(L), and let $p = s^{-1}(e)$. One can assume without loss of generality that $e \not\leqslant a$ and $e \not\leqslant a'$. Then clearly $p(a) \neq 0$ and $p(a') \neq 0$, so by applying twice the projection postulate (PP′) we obtain

$$p(a) \neq 0 \Rightarrow \exists_{e_1 \in A(L)} \quad e_1 \leqslant a \quad \text{and} \quad p(a) = p(e_1),$$
$$p(a') \neq 0 \Rightarrow \exists_{e_2 \in A(L)} \quad e_2 \leqslant a' \quad \text{and} \quad p(a') = p(e_2).$$

Hence

$$p(e_1 + e_2) = p(e_1) + p(e_2) = 1,$$

so we have

$$e = s(p) \leqslant e_1 + e_2 \leqslant a + e_2,$$

which leads immediately to

$$a \vee e \leqslant a + e_2,$$

so that

$$a \vee e - a \leqslant (a + e_2) - a = e_2,$$

which implies

$$a \vee e - a = e_2 \in A(L).$$

Therefore we have shown that for all $a \in L$ and $e \in A(L)$, $a \vee e - a$ is either zero (when $e \leqslant a$) or an atom (when $e \not\leqslant a$), as claimed.

Now we shall prove the converse implication, i. e. the implication from $i)$ + (JP) to (PP'). So, let us assume that $i)$ and (JP) hold in L, and suppose that $a \in L$ and $p \in P$ are such that $p(a) \neq 0$. Define $e = s(p)$. Since $e \not\leqslant a'$, we find by applying (JP) that $a' \vee e - a'$ is an atom. We have

$$p(a' \vee e - a') = 1 - p(a') = p(a),$$

which concludes the existence part of (PP').

The rest of the theorem, i. e. the uniqueness part of (PP'), follows simply by repeating the arguments of Bugajska and Bugajski [1].

3. CONDITIONING OF PURE STATES

Many attempts have been made to justify the covering property in quantum logics (see, e. g., Pool [23], Jauch and Piron [15], Bugajska and Bugajski [1], [2]), but as long as we are within the conventional quantum logic axiomatic scheme, this property still remains without a satisfactory empirical justification. We are, however, in a position to give a physical justification to the covering law, provided we shall go out this axiomatic framework, and consider the experimental procedures (« filters ») corresponding to propositions from the quantum logic. The covering law can then be obtained as a consequence of physically clear properties of filters (for details, see Guz [9], [10], [12], [13]).

Moreover, the correspondence between propositions and filtering procedures associated with these propositions is sufficient and necessary for the validity of the covering law in a quantum logic. To prove this let us first consider the pair (L, P) satisfying axioms (A1) and (A3). A mapping $E_a(E_a : p \rightarrow p_a)$ of the set P of pure states into itself is said to be a *generalized (pure) filter associated with the proposition* $a \in L$, provided (compare Guz [13]):

$i)$ The domain $D(E_a)$ of E_a consists of those pure states $p \in P$, for which $p(a) > 0$, i. e.

$$D(E_a) = \{ p \in P : p(a) > 0 \}.$$

$ii)$ If $p \in D(E_a)$, then $p(a) = (p : p_a)$ and $p_a(a) = 1$.

Remark. — The number $(E_a p)(b)$ is customarily interpreted as the *conditional probability* that an « event » $b \in L$ will occur, provided the « event » $a \in L$ was found to occur for the system being initially in the pure state p. In other words, $E_a p$ describes the final pure state of the system *conditioned* by the fact of the occurrence of an « event » $a \in L$ for the physical system being initially in the pure state p, and this is the reason why we often call

the map E_a the *conditional probability mapping* associated with the (non-zero) proposition $a \in L$.

It is not difficult to see that every generalized filter E_a has the following property:

$$\forall_{p, q \in D(E_a)} \qquad (p : p_a) \geqslant (p : q_a). \qquad (3.1)$$

Indeed, since $q_a(a) = 1$, we obtain

$$(p : q_a) = \inf \{ p(b) : b \in L, \ q_a(b) = 1 \} \leqslant p(a) = (p : p_a),$$

as desired.

The physical meaning of the inequality (3.1) is obvious: p_a is the final state of the system to which the initial state p goes, after the proposition $a \in L$ has been verified to be true, and this means that the transition probability $(p : q_a)$ has to attain its maximum for $q_a = p_a$. Moreover, it is clear that the inequality (3.1) would be strict, provided $p_a \neq q_a$, and this leads us to the following definition (compare Guz [13]):

A generalized filter E_a will simply be called *filter* if the inequality (3.1) becomes strict, whenever $p_a \neq q_a$.

Suppose at the moment that (L, P), in addition to the axioms (A1) and (A3) satisfies also (PP) (or, equivalently, the covering law). Then it can be shown (Guz [12]) that with every non-zero proposition $a \in L$ there can be associated a filter E_a, namely the one defined by

$$E_a = s^{-1} s_a s, \qquad (3.2)$$

where s stands, as usually, for the support mapping (see Section 1), and s_a is the so-called Sasaki projection restricted to the set $A(L) \cup \{ 0 \}$ ($s_a(e) = a' \lor e - a' =$ an atom or zero; $e \in A(L) \cup \{ 0 \}$).

Conversely, let us assume that (L, P) satisfies axioms (A1) and (A3) only, and let $\{ E_a \}$ be a family of filters associated with the non-zero propositions from L. Then:

(1) The covering law holds in L.

(2) $\{ E_a \}$ is unique, since $\{ E_a \}$ is then induced (via the formula (3.2)) by the Sasaki projections s_a.

Proof. — The validity of the covering law in L, after we assume the properties mentioned above for (L, P), has recently been shown by Guz [12]. So, we only need to prove (2).

Suppose that $p(a) > 0$, where $p \in P$ and $a \in L$, and let $p^a = s^{-1}(a' \lor s(p) - a')$, that is, $s(p^a) = a' \lor s(p) - a'$. In other words, the mapping $a \to p^a$ is the usual pure filter defined by (3.2), so its domain coincides with that of $E_a(E_a : p \to p_a)$.

We have

$$(p_a : p^a) = p_a(s(p^a)) = p_a(a' \lor s(p)) - p_a(a') = p_a(a' \lor s(p)),$$

because by *ii*) we get $p_a(a') = 0$.

Since $p_a(a) = 1$, we obtain $s(p_a) \leqslant a$, and hence $s(p_a) \not\leqslant a'$.
We shall show that $s(p_a) \leqslant a' \vee s(p)$. By $ii)$ we have

$$p(a) = (p : p_a) = p(s(p_a)),$$

but, by orthomodularity,

$$a = (a - s(p_a)) + s(p_a),$$

so that we find

$$p(a - s(p_a)) = 0.$$

Hence

$$s(p) \leqslant (a - s(p_a))' = a' \vee s(p_a), \tag{3.3}$$

and by applying the covering law we obtain

$$s(p_a) \leqslant a' \vee s(p), \tag{3.4}$$

as claimed. Indeed, it follows from (3.3) that $a' \vee s(p) \leqslant a' \vee s(p_a)$; hence $a' \vee s(p) = a' \vee s(p_a)$, since $a' \vee s(p_a)$ covers a'.

But the inequality (3.4) leads immediately to $p_a(a' \vee s(p)) = 1$, so we have $(p_a : p^a) = 1$, and hence $p_a = p^a$.

The equality above proved for all $p \in P$ satisfying $p(a) > 0$, i. e. for all $p \in D(E_a)$ = the domain of the map $p \to p^a$, shows that E_a coincides with the mapping $p \to p^a$, as claimed.

We thus have shown the equivalence of the covering law (or the projection postulate) in (L, P), the latter satisfying axioms (A1) and (A3), with the existence of a (unique) family of pure filters associated with the non-zero propositions from L. Moreover, the latter consists of the filters E_a induced by the Sasaki projections on L according to the formula (3.2).

4. FILTERS ON A TRANSITION PROBABILITY SPACE: AN ALTERNATIVE TO QUANTUM LOGIC

To avoid an obvious inconvenience connected with the domain $D(E_a)$ of a (generalized) filter E_a, which varies when a is changed, it is useful to extend the set P of pure states by adding to it some fictitious « pure » state, called the *zero state* and denoted by 0, which is defined as the zero function on L, i. e. $0(a) = 0$ for all $a \in L$. The extended set of pure states will in the sequel be denoted by P_0, that is $P_0 = P \cup \{0\}$.

It is also convenient to extend the transition probability function (:) and the mapping E_a onto a whole P_0 by setting for an arbitrary $p \in P_0$

$$(0 : p) = (p : 0) = 0,$$

and

$$E_a(p) = \begin{cases} p_a, & \text{if } p(a) > 0 \\ 0, & \text{if } p(a) = 0. \end{cases}$$

The extended (generalized) filter is here denoted by the same letter E_a; also the previous notation will now be retained, that is we shall write p_a instead of $E_a p$.

Remark. — Note that the extended generalized filter is also defined when $a = 0$; then we have, by definition, $E_a p = 0$.

The extended generalized filters possess the following properties (Guz [*11*], [*12*]):

(1) $p(a) = (p : p_a) \geqslant (p : q_a)$ for all $p, q \in P$ and $a \in L$.

(2) $(p : p_a) = 0$ implies $p_a = 0$.

(3) $p_a(a) = 1$, provided $p_a \neq 0$,

or, equivalently:

$$p_a(a') = 0 \quad \text{for all} \quad p \in P_0.$$

As an immediate consequence of the above—mentioned properties we obtain:

(4) Every E_a is an idempotent, i. e. $E_a^2 = E_a$.

The two concepts defined above, i. e. the extended transition probability and the extended (generalized) pure filter, leads us immediately to a general concept of a transition probability space and, similarly, to a general concept of a (generalized) filter, the latter being defined on an abstract transition probability space.

We begin with the definition of a transition probability space (Guz [*11*], [*12*]). A pair $(P_0, (:))$ consisting of a (non-empty) set P_0 together with a real-valued function $(:) : P_0 \times P_0 \rightarrow [0, 1]$, the latter called the *transition probability* in P_0, is said to be a *transition probability space* (later on abbreviated to t. p. s.) if the following conditions are satisfied:

(1) $(p : q) = 1$ implies $p = q$.

(2) There exists an element $p_0 \in P_0$ such that

i) $\forall_{p \in P_0} (p : p_0) = (p_0 : p) = 0$,

ii) $\forall_{p \in P_0, p \neq p_0} (p : p) = 1$.

The elements of the set P_0 are called *pures states*. It can easily be seen, by using (2), that the element p_0 defined above is necessarily unique; we denote it by 0 and call the *improper* or the *zero state*. The set P_0 is therefore of the form $P \cup \{ 0 \}$, where P consists of those pure states, which are different from 0; the latter are called the *proper* pure states.

The concept of the transition probability enables us to define the *orthogonality* in P_0 by setting (here p, q are members of P_0):

$$p \perp q \quad \text{iff} \quad (p : q) = (q : p) = 0.$$

A transition probability space $(P_0, (:))$ is said to be *special* if it satisfies additionally the requirement:

(3) $(p : q) = 0 \Rightarrow (q : p) = 0$.

We are now in a position to give a precise definition of a (generalized) filter, acting in an arbitrary transition probability space.

A mapping $E : P_0 \rightarrow P_0$ is said to be a *generalized filter* on the transition probability space $(P_0, (:))$, provided:

i) E is an indempotent mapping,

ii) $(p : Ep) = 0$ implies $Ep = 0$.

A generalized filter E is simply called a *filter* if it additionally possesses the following property:

iii) $(p : Ep) = (p : Eq) > 0 \Rightarrow Ep = Eq$.

Let E, F be two generalized filters on $(P_0, (:))$. We shall say that E is *stronger* than F, or that E *implies* F, and write $E \leqslant F$, if $FE = E$. We say that E and F are *mutually exclusive* or *orthogonal*, and write $E \perp F$, if $EF = FE = 0$, where 0 denotes the zero mapping of P_0 (defined by $0(p) = 0$ for all $p \in P_0$).

A family $\mathscr{F}$ of generalized filters acting on a transition probability space $(P_0, (:))$ is said to be *full*, provided:

i) $\mathscr{F}$ contains all the mappings $E_p : P_0 \rightarrow P_0$, where p runs over the set P_0, defined by

$$E_p q = \begin{cases} p, & \text{if } q \not\perp p, \\ 0, & \text{if } q \perp p. \end{cases}$$

ii) Every member E of the family $\mathscr{F}$ is uniquely determined by its range $R(E) = \{ Ep : p \in P_0 \}$; that is, if for E, $F \in \mathscr{F}$ we have $R(E) = R(F)$, then $E = F$.

It can be shown that if a transition probability space $(P_0, (:))$ admits a full family of generalized filters acting on it, then $(P_0, (:))$ must necessarily be special.

Indeed, it can easily be seen (by considering two cases: when $p \not\perp q$ and when $p \perp q$) that for all $p, q \in P_0$

$$(p : q) = (p : E_q p),$$

so, by using *ii)* we obtain

$$(p : q) = 0 \Rightarrow E_q p = 0 \Rightarrow p \perp q,$$

where the last implication is a direct consequence of the definition of E_q, and we therefore have shown that $(p : q) = 0$ implies $(q : p) = 0$, as claimed.

A family $\mathscr{L}$ of (generalized) filters on $(P_0, (:))$ is said to be a *logic of (generalized) filters* if it satisfies the following conditions:

a) $\forall_{E \in \mathscr{L}} \exists_{F \in \mathscr{L}} \forall_{p \in P_0 \setminus \{0\}}$ $(p : Ep) + (p : Fp) = 1$.

b) For any sequence $\{ E_i \}_{i=1}^{\infty}$ of pairwise orthogonal (generalized) filters from $\mathscr{L}$ there exists an $E \in \mathscr{L}$ such that

$$(p : Ep) = \sum_{i=1}^{\infty} (p : E_i p).$$

If $\mathscr{L}$ is a full logic of filters, then it can readily be seen that the filters F in a) and E in b) are determined uniquely; we denote them by E' and $\sum_i E_i$, respectively.

Indeed, let us first note that for an arbitrary generalized filter E we have

$$R(E) = \{ \, p \in P_0 : p = Ep \, \} = \{ \, p \in P : p = Ep \, \} \cup \{ \, 0 \, \},$$

and by applying this observation we get $R(F) = R(F_1)$, provided F_1 is another generalized filter satisfying a). Hence $F = F_1$, as desired. The proof of the uniqueness of the filter E in b) is identical.

The physical interpretation of the assumptions a), b) above is standard, and does not differ from the well-known interpretation given in the framework of the quantum logic approach: E' is the filter complementary to E, and $\sum_i E_i$ denotes a single filter, which replaces a sequence of mutually orthogonal filters E_i.

Remark. — Note that if $\mathscr{L}$ is a full logic of generalized filters, then the mappings 0 and I (the zero and the identity map, respectively) belong to $\mathscr{L}$, because $0 = E_0$ and $I = E_0'$.

It is not difficult to check that every full logic of filters $\mathscr{L}$ possesses all the properties (F1)-(F7) postulated in our recent paper (Guz [12]), and therefore the following theorem holds (Guz [12]), which states that the axiomatic scheme based on the pair (L, P) satisfying axioms (A1), (A3) and (PP) can be translated into the corresponding axiomatic framework based on the concept of a transition probability space $(P_0, (\ : \ ))$ and that of a (pure) filter acting on the former:

Let us suppose that for (L, P) the axioms (A1), (A3) and (PP) (or the covering law in place of (PP)) hold; then there exists a transition probability space $(P_0, (\ : \ ))$ such that L is orthoisomorphic to some full logic of filters acting on $(P_0, (\ : \ ))$ and $P = P_0 \setminus \{ \, 0 \, \}$.

Conversely, for an arbitrary full logic $\mathscr{L}$ of filters acting on a transition probability space $(P_0, (\ : \ ))$ there exist a σ-orthoposet L, coinciding actually with the $\mathscr{L}$ itself, and a set P of probability measures on L, whose elements are in one-to-one correspondence with pure states from $P_0 \setminus \{ \, 0 \, \}$, such that (L, P) satisfies axioms (A1), (A3) and (PP).

5. SUPERPOSITION PRINCIPLE

There exist several formulations of the superposition principle in the quantum logic axiomatic framework (cf. Jauch [14], Gudder [4], Emch

and Piron [3], Guz [5], [6], Pulmannova [24]). Here we will follow the formulation of this principle due to Guz [5], [6] (see also Pulmannova [24]), which is the first formulation based upon the concept of the superposition of pure states introduced by Varadarajan [26]. The advantage of this formulation of the superposition principle, besides an extreme clarity of its physical content, is that it enables us to define the « sectors » analogical to those defined within the C*-algebraic approach (see Pulmannova [24]).

Let (L, P) be a pair satisfying axioms (A1), (A3). We say that the *superposition principle holds for* (L, P) (see Guz [5], [6]) if for any pair p, q of distinct pure states there is a third pure state $r \neq p$, q such that $r \in \{ p, q \}^-$.

Here the « closure » operation $^-$ is defined as follows (Varadarajan [26]): For an arbitrary subset $M \subseteq P$ we define M^- to be the set of pure states $p \in P$ satisfying $p(a) = 1$, if the proposition $a \in L$ is such that $q(a) = 1$ for all $q \in M$; that is

$$M^- = \{ p \in P : p(a) = 1, \text{ provided } q(a) = 1 \text{ for all } q \in M \text{ (where } a \in L) \}.$$

The physical interpretation of the members of the set $M^- \setminus M$ is obvious. These are the pure states, which have all the properties possessed by all the elements of M simultaneously, so it is fully justified to call them the *superpositions* of the pure states from M. According to this interpretation, the superposition principle formulated above states that for any two distinct pure states p, q there always exists their superposition r in P.

Remark. — It has been shown (Guz [8]) that $M^- = M^{\perp\perp}$ for an arbitrary $M \subseteq P$.

It can easily be seen (Guz [5]; see also Pulmannova [24]) that if the superposition principle holds for (L, P), where (L, P) satisfies axioms (A1), (A3) and (PP), then the propositional logic L is necessarily irreducible.

This statement can be reversed:

If (L, P) satisfies axioms (A1), (A3) and (PP), and if L is irreducible, then the superposition principle holds for (L, P).

Proof. — First let us note that by replacing pure states by their carriers we can readily rewrite the superposition principle in the following equivalent form (and just this property was named by Emch and Piron [3] the « superposition principle »): (*) For any pair e, f of distinct atoms in L there is an atom $g \in L$, different from those, such that $g \leqslant e \vee f$.

Now we shall show that the irreducibility of L implies the validity of (*), and this will be done in the following way: we will suppose that there exist two distinct atoms e, f such that there is no an atom $g \neq e$, f with $g \leqslant e \vee f$, and prove that L is then reducible.

Since $e \neq f$, we have $e \vee f - e \neq 0$, so by using the Jauch-Piron property (JP) we find that $e \vee f - e$ is an atom contained in $e \vee f$ (and

orthogonal to e), so that, by our hypothesis, $e \vee f - e = f$. Therefore we have $f \perp e$.

We shall now prove that for an arbitrary atom $g \neq e$, f one has $g \perp e$, f. Since, by our assumption, $g \nleq e + f$, we find by applying the Jauch-Piron property that $(e + f) \vee g - (e + f)$ is an atom. Denote the latter by h. We have $h \perp e + f$ and $h \leq (e + f) \vee g$; hence, by applying the covering law we get $g \leq e + f + h$, which leads to $g \vee h - h \leq (e + f + h) - h = e + f$. We shall show that $g = h$. Suppose the contrary, i. e. that $g \neq h$. Then clearly $g \vee h - h \neq 0$, so that $g \vee h - h$ must be an atom by (JP), but since $g \vee h - h \leq e + f$, we must have either $g \vee h - h = e$ or $g \vee h - h = f$. Now let us note that $g \nleq h'$ (indeed, $g \leq h'$ implies $g \vee h - h = g$, so we then have either $g = e$ or $g = f$, which contradicts our assumption that $g \neq e, f$), so $g \vee h' - h'$ is an atom by (JP), and therefore $g \vee h' - h' = h$. Hence $h \leq g \vee h'$, so we have $g < g \vee h \leq g \vee h'$, and by applying the covering law we obtain $g \vee h = g \vee h'$, which leads to

$$g \vee h - h = (g \vee h') \wedge h' = h',$$

so h' is either e or f. But $h' = e$ implies $h = e' \geq f$, so that $h = f$, and similarly $h' = f$ leads readily to $h = e$. In both cases we arrive at a contradiction (since $h \perp e, f$), so the assumption $g \neq h$ is untenable. Therefore we have shown that $g = h$; hence $g \perp e, f$, as claimed.

The last step in the proof of our theorem is the observation that e and f must necessarily belong to the center $C(L)$ of L, where $C(L)$ is by definition the set of all central elements of L, i. e. $C(L) = \{ a \in L : a \leftrightarrow b \text{ for all } b \in L \}$, where $\leftrightarrow$ stands for the well known Mackey's compatibility relation defined by (see [17], [18]): $a \leftrightarrow b$ if and only if there are three pairwise orthogonal propositions a_1, b_1, c such that $a = a_1 + c$ and $b = b_1 + c$.

Indeed, let a be an arbitrary element of the propositional logic L. We shall show that a is compatible with e and f. One can assume with no loss of generality that $e \nleq a$. Owing to the atomisticity of L (see Section 1) we can write $a = \vee e_j$, where e_j are atoms different from e, so we have $a \perp e$, since by the result proved earlier we have $e_j \perp e$ for all j. Thus we have shown that either $e \leq a$ or $e \perp a$, and hence $e \leftrightarrow a$, as claimed. Similarly we prove that $f \leftrightarrow a$, and this concludes the proof that $e, f \in C(L)$; hence $C(L) \neq \{ 0, 1 \}$, and the theorem is therefore proved.

Remark 1. — Note that the superposition principle can in fact be stated in the following form:

For any pair p, q of orthogonal pure states there exists a pure state $r \neq p, q$ such that $r \in \{ p, q \}^{-}$.

Equivalently (when we pass on from pure states to their carriers):

For any pair e, f of orthogonal atomic propositions there is a third atom $g \neq e, f$ such that $g \leq e \vee f$.

It will clearly be sufficient to prove that if $e \not\perp f$ $(e, f \in A(L))$, then there always exists an atom $g \in A(L)$ different from e and f, such that $g \leqslant e \vee f$, and this is in fact a consequence of the orthomodularity and the atomicity of the logic L. Indeed, if e, f are distinct atoms, then $e \vee f - f \neq 0$ (here we have used the orthomodularity of L), so by atomicity of L we find an atom $g \in A(L)$ such that $g \leqslant e \vee f - f$. Clearly, $g \neq f$ (since $g \perp f$), and if $e \not\perp f$, we have also $g \neq e$.

Remark 2. — Note that the implication from the superposition principle to the irreducibility of the logic L can also be proved in a more general context; namely for the pair (L, P) satisfying axioms (A1) and (A3) only.

Indeed, suppose the contrary, i. e. assume that there is a central element $a \in C(L)$ such that $0 < a < 1$, where (L, P) satisfies axioms (A1), (A3) and the superposition principle. Since L is atomic, there exist two atoms $e \leqslant a$ and $f \leqslant a'$. Let $p = s^{-1}(e)$, $q = s^{-1}(f)$, and let $r \in \{ p, q \}^-$. (Note that $p(a) = q(a') = 1$). We shall prove that r must coincide with either p or q. Note first that for an arbitrary pure state $r \in P$ we have either $r(a) = 1$ or $r(a) = 0$, provided a is central, since then we have either $s(r) \leqslant a$ or $s(r) \leqslant a'$, but this is precisely the preceding statement. First let us suppose that $r(a) = 1$; then $r(a') = 0$, so we have $r(b) = r(b \wedge a + b \wedge a') = r(b \wedge a)$ for all $b \in L$. We shall show that in this case $r = p$. Suppose that $p(b) = 0 = p(b \wedge a)$, where $b \in L$; then also $r(b) = r(b \wedge a) = 0$, since $q(b \wedge a) = 0$ and $r \in \{ p, q \}^-$. Thus we have shown that $r \in \{ p \}^-$; hence $r = p$, because $\{ p \}^- = \{ p \}$ (see Guz [8]). Similarly we prove that $r(a) = 0$ implies $r = q$. Therefore we have proved that $\{ p, q \}^- = \{ p, q \}$, which contradicts the assertion of the superposition principle.

Let us finally note that the irreducibility of L (or, equivalently, the validity of the superposition principle in (L, P)) is not a restrictive assumption, because if it does not hold, we can always take into consideration any irreducible part of L instead of a whole L, as it can be proved that this irreducible « sector » of L possesses all the essential properties of the whole (L, P).

We shall demonstrate this for the relatively simple case, where L has a discrete center. We say that the center C(L) of a logic L is *discrete* (see, e. g., Varadarajan [26]) if there is an at most denumerable set $\{ c_i \}_{i \in I}$ of pairwise orthogonal elements of C(L) such that

i) $\displaystyle\bigvee_i c_i = 1$,

ii) C(L) consists precisely of all the lattice sums $\displaystyle\bigvee_{j \in J} c_j$, where J is a subset of I.

The c_i's are called the *atoms* of C(L).

If L is a logic (i. e. an orthomodular σ-orthoposet) with the discrete

center, then every proposition $a \in L$ can be written as $a = a_1 + a_2 + \ldots$, where $a_i \leqslant c_i$, and the a_j are uniquely determined, since $a_j = a \wedge c_j$.

In other words, we have a decomposition of the whole logic L into a direct sum of the irreducible logics $L_i = [0, c_i] = \{ b \in L : b \leqslant c_i \}$, the latter endowed with the partial ordering inherited from L and the ortho-complementation $\perp$ defined by $b^\perp = b' \wedge c_i$ (c_i is clearly the greatest element in L_i).

The result announced above consists of the following:

Let (L, P) be a pair satisfying axioms (A1), (A3) and (PP), and suppose that L has a discrete center C(L). Let $\{ c_i \}_{i \in I}$ be the family of atoms of C(L), and let L_i and P_i be respectively defined by $L_i = [0, c_i]$ and $P_i =$ the set of all probability measures p on L_i such that $\bar{p} \in P$, where $\bar{p}$ is defined by

$$\bar{p}(a) = p(a \wedge c_i), \qquad a \in L.$$

Then (L_i, P_i) satisfies all the axioms (A1), (A3) and (PP).

Proof. — That L_i is again an orthomodular σ-orthoposet satisfying the covering law, it is a well known fact. One therefore needs to prove the validity of (A3) in (L_i, P_i). However, before proving this, we need some elementary facts about the sets P_i and their images $\bar{P}_i$ under the canonical mapping $p \to \bar{p}$ defined above (by definition, $\bar{P}_i = \{ \bar{p} : p \in P_i \}$).

We have:

i) The mapping $—$, when restricted to a single P_i, is one-one, that is $p \neq q$ (where $p, q \in P_i$) implies $\bar{p} \neq \bar{q}$.

ii) $\bar{P}_i \cap \bar{P}_j = \varnothing$, when $i \neq j$.

iii) $P = \bigcup_i \bar{P}_i$.

The property i) is almost obvious, and it needs no proof.

To prove ii), let us suppose the contrary, i. e. assume that there exists some $m \in \bar{P}_i \cap \bar{P}_j$, so that $m = \bar{p} = \bar{q}$, where $p \in P_i$, $q \in P_j$. If $i \neq j$, then $c_i \perp c_j$, and we have $\bar{p}(c_i) = 1$, while $\bar{q}(c_i) = 0$, so the assumption $\bar{p} = \bar{q}$ is untentable. So we have shown that $\bar{P}_i \cap \bar{P}_j \neq \varnothing$ implies $i = j$, which is equivalent to ii).

We shall finally prove the property iii), i. e. that each element of P lies in some $\bar{P}_i$. Suppose that $p \in P$. Let us observe that $p(c_i) = 1$ for some $i \in I$, for if $p(c_j) < 1$ for all j implies $p(c_j) = 0$ for all j (since $c_j \in C(L)$), and hence $p(1) = p\left(\bigvee_j c_j \right) = 0$, so we arrive at a contradiction. Now we shall show that $p = \bar{p}_i$, where p_i stands for the restriction of p to L_i. This shows that $p_i \in P_i$ and, at the same time, that $p \in \bar{P}_i$.

We have for an arbitrary $a \in L$

$$\bar{p}_i(a) = p_i(a \wedge c_i) = p(a \wedge c_i) = p(a \wedge c_i + a \wedge c_i') = p(a),$$

which proves that $p = \bar{p}_i$, as claimed, and the proof of *iii*) is therefore complete.

Now we shall come back to the proof of our theorem and verify the validity of the postulate (A3) in (L_i, P_i).

To prove the validity of (A3*i*) in (L_i, P_i), let us suppose that $a \in L_i$ and $a \neq 0$. Then there exists a pure state $p \in P$ such that $p(a) = 1$. Clearly, $p(c_i) = 1$ and $p(c_j) = 0$ for all $j \neq i$, so that $p = \bar{p}_i$, where p_i is the restriction of p to L_i (see proof of *iii*)), and $p_i(a) = p(a) = 1$. Since $p_i \in P_i$, the proof is complete.

Now we shall show that (A3*ii*) holds for (L_i, P_i), i. e. that $a^1 \cap P_i \subseteq b^1 \cap P_i$ (where $a, b \in L_i$) implies $a \leqslant b$.

Indeed, let $a^1 \cap P_i \subseteq b^1 \cap P_i$, where $a, b \in L_i \setminus \{ 0 \}$, and let p be an arbitrary pure state from P satisfying $p(a) = 1$. We have, as before, $p = \bar{p}_i$ for some i (where $p_i \in P_i$) and $p_i(a) = p(a) = 1$, so by our assumption we obtain also $p(b) = p_i(b) = 1$. Therefore we have shown that $a^1 \subseteq b^1$; hence $a \leqslant b$.

We shall finally prove that (L_i, P_i) satisfies (A3*iii*), i. e. that

$$\forall_{p \in P_i} \exists_{a \in L_i} \quad p(a) = 1 \qquad \text{and} \qquad \forall_{q \in P_i, q \neq p} \quad q(a) < 1 .$$

Note that if $p \in P_i$, then $\bar{p} \in P$, and by (A3) one can find a proposition $a \in L$ such that $\bar{p}(a) = 1$ and $\bar{q}(a) < 1$ for all $q \in P_i$ distinct from p (here we have used the fact that $q \neq p$, where $q, p \in P_i$, implies $\bar{q} \neq \bar{p}$). But, by definition, $\bar{p}(a) = p(a \wedge c_i)$, so we have $p(a \wedge c_i) = 1$, where $a \wedge c_i \in L_i$, and, at the same time, $\bar{q}(a) = q(a \wedge c_i) < 1$, which concludes the proof of (A3*iii*).

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