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Section A :
Physique théorique.

The connection between local and asymptotic structures for isolated gravitating systems with isometries

by

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ABSTRACT. — For asymptotically flat spacetimes containing isometries we establish relations between asymptotically defined quantities and quantities describing the structure of the sources as defined by Dixon. In particular, for stationary spacetimes we derive a mass formula and in axisymmetric spacetimes we prove the equality of asymptotic and local angular momentum.

RÉSUMÉ. — Pour des espaces-temps plats à l'infini on établit des relations entre des quantités asymptotiquement définies et des moments multipolaires selon W. G. Dixon décrivant la structure des sources. On déduit une formule de masse pour des espaces-temps stationnaires et on prouve l'égalité des moments cinétiques asymptotique et local pour des espaces-temps axisymétriques.

§ 1. INTRODUCTION

It has been emphasized in [*1*], [*2*], [*3*] that a more complete description of isolated systems in General Relativity requires a combination of several parts of the theory which have been carried out rigorously but which have not been related to each other so far. Such distinct aspects are the description of the structure and motion of bodies, the asymptotic analysis of asymptotically flat gravitational fields in terms of the structure at null or spacelike infinity, the characteristic and the Cauchy initial value problem.

In this paper, we take the first step towards the combination of the former two aspects. We restrict ourselves to cases where the spacetime modelling an isolated system admits a group of motions and we apply the results of [*4*] (in the sequel referred to as paper I) to spacetimes which are asymptotically flat to obtain formulae relating asymptotic quantities like the Bondi mass and angular momentum to the corresponding local (Dixon-) quantities. In the stationary case (§ 3 *b*) it is shown explicitly how apart from the local (inertial) mass the rotational energy and the gravitational potential energy contribute to the total mass of the system. In the axisymmetric case (§ 3 *c*) we show that the asymptotic and the local angular momentum agree, i. e. that the gravitational field is no source of angular momentum contrary to the case of the masses.

This work is regarded as a prerequisite for the treatment of the analogous but more difficult problem when no symmetries are present. This will be dealt with in a later paper.

§ 2. ASYMPTOTICS

The concept of an isolated system used in this work involves the existence of future null infinity $\mathscr{I}^+$ in the sense of asymptotic simplicity as defined by Penrose (see, e. g. [*5*]). This provides an invariant geometric formulation of asymptotic flatness of a spacetime. Spacetimes being (weakly) asymptotically simple and empty possess the same global asymptotic structure as Minkowski spacetime. In particular, there exists an asymptotic symmetry group which arises as the invariance group of the universal geometrical structure inherent in $\mathscr{I}^+$. This is the BMS-group. Now there are also physical fields defined on the manifold representing future null infinity. Similarly to the standard procedure in special relativistic field theories, one can try to combine these fields with the asymptotic symmetries also defined there to obtain integrated quantities like energy-momentum and angular momentum which are characteristic of the system as a whole, being part of the invariant asymptotic description of the physical behaviour of an isolated source.

There is a compatibility criterion for the construction of such quantities. If the physical spacetime admits a Killing vector field Z^a, one also has available the Komar integral [*6*]

$$C_S(Z) = \frac{1}{4\pi} \oint_S \nabla^a Z^b dS^*_{ab} \tag{2.1}$$

where S is any smooth two-sphere surrounding the matter sources. In fact, by Einstein's vacuum equations, $C_S(Z)$ is independent of the choice of S in the vacuum region and therefore represents a conserved quantity. On the other hand, since every Killing field admits a smooth extension to $\mathscr{I}$, one can evaluate $C_S(Z)$ on a two-sphere cross section of $\mathscr{I}$ in a conformally rescaled spacetime. Thus a quantity of the type referred to above must reduce to the Komar integral if the BMS vector field from which it is constructed arises from the extension of a Killing vector field on spacetime. In this paper, we will therefore adopt the following assumption and definitions.

In addition to the assumptions (A1), (A2) of I we demand

(A3) The spacetime $(\mathscr{M}, g)$ is asymptotically simple and empty in a neighborhood of $\mathscr{I}^+$ and $(\mathscr{M}, g, T)$ is a solution of Einstein's equations.

(2.1) DEFINITION. — *a*) In a stationary spacetime satisfying (A3) with Killing vector field ξ (normalized to unity at infinity), $M := C_S(\xi)$ is called the *total mass of the system* if S is any two-sphere cross section of $\mathscr{I}^+$.

b) In an axisymmetric spacetime satisfying (A3) with Killing vector field η whose orbits are closed curves with parameter length 2π, $J := \frac{1}{2} C_S(\eta)$ is called the *total angular momentum of the system* if S is any two-sphere cross section of $\mathscr{I}^+$.

Remarks. — In a stationary spacetime, M agrees with the Bondi mass (see, e. g. [*7*]). In an axisymmetric spacetime, J agrees with the definition of Tamburino and Winicour [*8*]. If $(\mathscr{M}, g)$ is also asymptotically flat at spatial infinity in the sense of Ashtekar and Hansen [*9*], then M and J also agree with the definitions available there if the corresponding symmetries are present (see [*10*], [*11*]).

§ 3. THE CONNECTION BETWEEN LOCAL AND ASYMPTOTIC STRUCTURE

A relation between the asymptotic and the local conserved quantities will first be established for a general group of motions and hereafter we will specialize to some physically significant cases. All notations follow those of paper I.

a) General group of motions.

We recall Lemma 4.2 (I) and we suppose the assumptions (A1), (A2), (A3) and (S1) to hold. In other words, we are given an isolated body represented by an asymptotically flat spacetime in which a unique center-of-mass line exists and the system is invariant under a group of motions generated by a Killing vector field Z. We have

(3.1) PROPOSITION. — Suppose (A1), (A2), (A3), and (S1). Then

$$C_S(Z) = 2E_\Sigma(Z) - \kappa \int_\Sigma T dS$$

where

$$T := T^{ab} g_{ab}.$$

Proof.

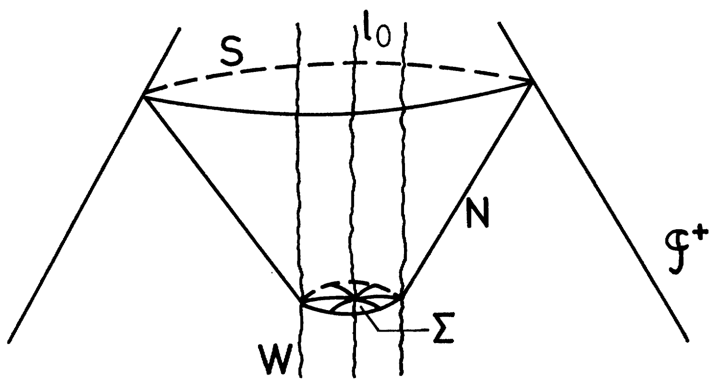

FIG. 3.1. — W = convex hull of supp T^{ab}.

By Stokes' theorem, $C_S(Z)$ may be written as

$$C_S(Z) = \frac{1}{4\pi} \int_{N \cup \Sigma} \nabla_b \nabla^{[a} Z^{b]} dS_a \tag{3.1}$$

since $S = \partial(N \cup \Sigma)$ (see fig. 3.1).

From the Ricci identity

$$\nabla_{[b} \nabla_{a]} Z^b = \frac{1}{2} R_{ab} Z^b$$

one obtains

$$\nabla_b \nabla^{[a} Z^{b]} + \nabla_b \nabla^{(a} Z^{b)} = \nabla_b \nabla^a Z^b = \nabla^a \nabla_b Z^b + R^a{}_b Z^b$$

which, by Killing's equation, reduces to

$$\nabla_b \nabla^{[a} Z^{b]} = R^a{}_b Z^b = 8\pi \left(T^a{}_b - \frac{1}{2} T \delta^a{}_b \right) Z^b. \tag{3.2}$$

The last step made use of the field equation. We insert (3.2) in (3.1), use the defining equations (3.2 (I)) and equation (4.4 (I)). This yields the result. □

The equation obtained in the foregoing proposition may be rewritten as follows.

(3.2) LEMMA. — $C_S(Z)$ may be expressed in terms of the values of Z and ∇Z at the centre of mass (with $Z_k u^k = \kappa$, see (4.3 (I))):

$$C_S(Z) = Z_k u^k\left(2M_D - \int_\Sigma T dS\right) + \nabla_k Z_l S^{kl}.$$

Proof. — This is an immediate consequence of Proposition (3.1), equations (3.3 (I)) and Lemma (4.2 (I)). □

Next we use (3.21 (I)) to express the integral over the trace of T^{ab} in terms of mass, angular momentum and higher order moments.

(3.3) PROPOSITION. — Under the hypotheses of proposition (3.1),

$$C_S(Z) = \kappa\left(M_D - \int_{T_{z(s)}(\mathcal{M})} \widehat{T}^{kl}(G_{kl} + G^m_{kl}X_m)DX - \sum_{v=1}^{3}\left(\frac{d}{ds}\right)^v \langle A^{ab}_{(v)}, g_{ab} \rangle\right) + \nabla_k Z_l S^{kl}.$$

Proof. — We apply equation (3.21 (I)) for $\phi_{ab} = g_{ab}$. The first term yields the Dixon mass M_D while the second term vanishes. Hence all that has to be done is the evaluation of Λ_m. Since $\nabla g = 0$, we see from (3.15 (I)) that λ_a has to be an ordinary Jacobi field, i. e. it is given by (3.1 (I)). Inserting the initial conditions (3.16 (I)), we find

$$\lambda_a = - H_a{}^k \sigma^l g_{kl} = \sigma^{-1}{}_a{}^k \sigma_k = \sigma_a$$

and therefore

$$\Lambda_m(z, X) = H^a{}_m \lambda_a(z, \exp_z X) = - \sigma_m(z, \exp_z X) = X_m.$$

Insertion of the whole expression into the expression given in Lemma (3.2) yields the result. □

This is as far as we can get without further assumptions concerning the nature of the isometry group.

b) Stationary group of motions.

In this subsection, we will obtain a mass formula relating the total mass M to local quantities.

(3.4) THEOREM. — The total mass of a stationary isolated system is given by

$$M = \kappa(M_D - 2\Phi - \Psi - 4E_{rot})$$

where

$$\Phi := \frac{1}{2}\int_{T_{z(s)}(\mathcal{M})} \widehat{T}^{kl} G_{kl} DX$$

$$\Psi := \int_{T_{z(s)}(\mathcal{M})} \widehat{T}^{kl} G^{m}_{kl} X_m DX$$

$$E_{rot} := -\frac{1}{4\kappa} \nabla_k \xi_l S^{kl};$$

κ is the red shift factor, Φ is the gravitational potential energy introduced in [*12*]. E_{rot} is interpreted to be the rotational energy, which is motivated as follows: in a stationary spacetime, the principal directions of the inertia tensor may be chosen in such a way that they are Lie-dragged with respect to the Killing field ξ (if the eigenvalues of the inertia tensor are mutually distinct, no other choice is possible). Then

$$E_{rot} = \frac{1}{4} W_{kl} S^{kl}$$

where W^{kl} is the angular velocity of the principal directions (cf. § 3 (I)). According to proposition (5.8 (I)) and definition (5.6 (I)) we obtain

$$E_{rot} = \frac{1}{4} \Omega_{kl} S^{kl}$$

if the motion is quasi-rigid and the principal moments of inertia are mutually distinct. This expression agrees with Dixon's definition of rotational energy in [*12*].

Proof. — Using definition (2.2 (*a*)), proposition (3.3) and corollary (5.1 (I)), we find

$$M = \kappa(M_D - 2\Phi - \Psi) - 4\kappa E_{rot} - \kappa \sum_{v=1}^{3} \left(\frac{d}{ds}\right)^{v} \langle A^{ab}_{(v)}, g_{ab} \rangle.$$

But the $\langle A^{ab}_{(v)}, g_{ab} \rangle$ are scalar fields along l_0 depending only on the metric g. Therefore

$$\frac{d}{ds} \langle A^{ab}_{(v)}, g_{ab} \rangle = \mathcal{L}_{\xi} \langle A^{ab}_{(v)}, g_{ab} \rangle = 0.$$

This proves the first part of the Theorem.

Using Proposition (5.7 (I)) and the center-of-mass condition yields the remaining statement. □

c) Axisymmetric group of motions.

In this case we can derive a relation between angular momenta:

(3.5) PROPOSITION. — The total angular momentum and the angular momentum at the center-of-mass of an axisymmetric isolated system are equal:

$$J = L.$$

Proof. — Using Definition (2.2 (*b*)), Lemma (3.2) and the fact that [proposition (6.1 (I))]

$$\kappa = 0,$$

we find using (6.5 (I))

$$J = \frac{1}{2} C_S(\eta) = \frac{1}{2} \nabla_k \eta_l S^{kl} = L. \qquad \square$$

This result is surprising, when compared with theorem (3.4). In the stationary case, the total mass M of the system is not equal to the inertial mass M_D of the sources. One may interpret this fact by saying that the extra terms appearing in the mass formula represent the contribution to the mass from the gravitational field. On the other hand, in the axisymmetric case, it is clear that the component of angular momentum in the direction of the Killing field is not radiated to infinity; but this does not yet explain why the corresponding angular momenta are equal. Instead, by this result one is forced to conclude that the field itself is no source of angular momentum whereas it is one for the mass. A possible explanation for this might be the absence of torsion from the affine connection in Einstein's theory of gravitation; only torsion could be a source of angular momentum. However, this hypothesis would have to be explored in more detail.

§ 4. DISCUSSION

In this work we have established relations between source quantities and those asymptotic quantities which are defined by the existence of an isometry in spacetime. Einstein's vacuum equation together with Killing's equation ensure that these relations do not contain terms arising from the integration over hypersurfaces connecting the source region with the asymptotic regime. However, in general this cannot be expected any longer if the quantities to be considered are not linked to isometries but are defined by other means. The next simplest case of this kind would be the relation of angular momenta in a stationary but not necessarily axisymmetric spacetime: the asymptotic angular momenta (in the null as well as the spatial regime) are defined and compared in *[11]*. What is their relation to the sources? The answer to this would also shed light on the conjecture mentioned at the end of § 3 *c* whose verification would require that the intermediate hypersurface terms would vanish also in this case even if by different reasons.

Having learnt about this problem one would proceed to general non-stationary spacetimes in which one definitely has to deal with field contributions from intermediate zones. For the purpose of establishing an energy balance between the radiation field and the sources, these field energy terms have to be estimated against the mechanical energy terms, for instance by defining a near zone. This has still to be achieved but there is some reason to hope that an exact framework as the one set up here offers a possibility to relate the asymptotically observed energy flux to the motion of the sources by making controlled approximations. This must be an aim of general relativistic mechanics.

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