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Positive maps of the CCR algebra with a finite number of non-zero truncated functions

by

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ABSTRACT. — We prove that if for a positive linear map on the CCR algebra only the first N (N $<\infty$) truncated functions are not zero then N cannot exceed two. We study also generators of completely positive semigroups on the CCR àlgebra and show that here as well the order of these generators cannot be greater than two.

RÉSUMÉ. — Pour une forme linéaire positive sur l'algèbre de relations de commutation dont seulement les N (N $<\infty$) premières fonctions tronquées ne sont pas nulles nous démontrons que N est borné par 2. Nous étudions aussi les générateurs de semi-groupes complètement positifs sur l'algèbre de relations de commutation, et dans ce cas-ci aussi nous démontrons que l'ordre de ces générateurs est borné par 2.

§ 1. INTRODUCTION

By analogy with the states on the CCR-algebra one can define under certain regularity conditions the truncated correlation functions corresponding to a positive map from this algebra into a Von Neumann algebra.

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We know that if for a certain state only a finite number of these n-point functions are non-vanishing then at most only the first two are not zero. This was proved in [1] by using a theorem of Marcinkiewicz [2] for characteristic functions. Here we use the sub-harmonicity of the spectral radius of analytic functions of bounded operators to generalize this theorem to operator valued characteristic functions and use it to show that the same result holds for positive maps on the CCR algebra. This is done in § 3.

In § 4 we deal with a related problem for generators of completely positive semigroups on the CCR-algebra. Kolmogorov [3] proved that the only differential operators which are generators of classical Markovian semigroups are of second order and elliptic (cf. also [4]). Here we prove an analogue of this for the generators of completely positive semigroups. The relation between this and § 3 is that the objects which replace the coefficients in the non-commutative case are the derivatives with respect to the parameter of the semigroup at zero, of the truncated correlation functions. The quasi-free completely positive semigroups studied in [5], [6], [7] may be considered as the analogues of classical diffusion Markovian processes with generators in which the coefficients are constants while the ones we study here correspond to generators with non-constant coefficients.

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§ 2. PRELIMINARIES

Let X be a set and $\mathscr{H}$ a Hilbert space. A map $K : X \times X \to \mathscr{L}(\mathscr{H})$, the bounded linear operators on $\mathscr{H}$, is said to be a positive-definite kernel if for each positive integer n and each choice of vectors $\phi_1, \ldots, \phi_n$ in $\mathscr{H}$ and elements $x_1, \ldots, x_n$ in X, the inequality

$$\sum_{i,j} (\phi_j, K(x_j, x_i)\phi_i) \geqslant 0$$

holds. $K : X \times X \to \mathscr{L}(\mathscr{H})$ is positive-definite if and only if it has a Kolmogorov decomposition i. e. if and only if there is a Hilbert space $\mathscr{M}$ and a map $V : X \to \mathscr{L}(\mathscr{H}, \mathscr{M})$ with $K(x, y) = V(x)^*V(y)$ for all x, y in X. If K is positive definite then it satisfies the Schwartz inequality [8]

$$K(x, x)\,\| K(y, y) \| \geqslant K(x, y)K(y, x) \qquad \text{for} \quad x, y \in X \tag{1}$$

If X is a group, T a map of X into $\mathscr{L}(\mathscr{H})$ then the kernel $K : X \times X \to \mathscr{L}(\mathscr{H})$ where $K(x, y) = T(xy^{-1})$ is positive definite if and only if there is a Hilbert space $\mathscr{M}$ and a unitary representation $\pi : X \to \mathscr{L}(\mathscr{M})$ and V in $\mathscr{L}(\mathscr{H}, \mathscr{M})$ such that $T(x) = V^*\pi(x)V$ [8].

A function $f : \mathbb{R} \to \mathscr{L}(\mathscr{H})$ is a characteristic function if, i) the kernel

$K : \mathbb{R} \times \mathbb{R} \to \mathcal{L}(\mathcal{H})$ defined by $K(t, s) = f(t - s)$ is, *i)* positive-definite, *ii)* $f(0) = I$ and *iii)* f is continuous. As above there is a Hilbert space $\mathcal{M}$ a unitary representation $\pi : \mathbb{R} \to \mathcal{L}(\mathcal{M})$ and V in $\mathcal{L}(\mathcal{H}, \mathcal{M})$ such that $f(t) = V^*\pi(t)V$ for all $t \in \mathbb{R}$. From the normalization $V^*V = I$ we see that VV^* is an orthogonal projection and that $\| V \| = \| V^* \| = 1$.

Let H be a Hilbert space and W a representation of the CCR over H. We denote the C*-algebra generated by $\{ W(h) ; h \in H \}$ by W(H). A linear map $T : W(H) \to \mathcal{L}(\mathcal{H})$ is completely positive if and only if the kernel $K : H \times H \to \mathcal{L}(\mathcal{H})$, where

$$K(h, k) = T(W(k - h)) \exp \frac{i}{2} \operatorname{Im} (h, k),$$

is positive definite [8]. If T is completely positive then there is a representation $\tilde{W}$ of the CCR over H in $\mathcal{L}(\mathcal{M})$ and $V \in \mathcal{L}(\mathcal{H}, \mathcal{M})$ such that $TW(h) = V^*\tilde{W}(h)V$ [8].

§ 3. POSITIVE MAPS WITH A FINITE NUMBER OF NON-VANISHING TRUNCATED FUNCTIONS

If $f : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ is a characteristic function the continuity of f implies that $t \mapsto \pi(t)$ is strongly continuous and therefore by Stone's theorem there is a self-adjoint operator R such that $f(t) = V^*e^{itR}V$. If f has an analytic continuation then this representation can be extended to the region of analyticity:

LEMMA 1. — Let $S_\rho = \{ z : z \in \mathbb{C}, | \operatorname{Im} z | < \rho \}$ and suppose $f : S_\rho \to \mathcal{L}(\mathcal{H})$ is analytic and the restriction of f to $\mathbb{R}$ is a characteristic function then if $z \in S_\rho$, $V\mathcal{H} \subseteq \mathcal{D}(e^{\frac{1}{2}izR})$ and $(\phi, f(z)\psi) = (V\phi, e^{izR}V\psi)$ for all $\phi, \psi \in \mathcal{H}$.

Proof. — The function $z \to (\phi, f(z)\phi)$, is analytic in the strip S_ρ and its restriction to $\mathbb{R}$ is a characteristic function. Therefore in S_ρ it can be represented by its Fourier integral (Th. 7.11 [2]) which in this case means that for $z \in S_\rho$

$$(\phi, f(z)\phi) = \int e^{iz\lambda} d(V\phi, E_\lambda V\phi).$$

This implies that for each $z \in S_\rho$, $V\mathcal{H} \subseteq \mathcal{D}\left(\exp \frac{1}{2} izR\right)$ and

$$(\phi, f(z)\phi) = (V\phi, e^{izR}V\phi).$$

By polarization

$$(\phi, f(z)\psi) = (V\phi, e^{izR}V\psi). \qquad \blacksquare$$

It follows immediately from this lemma that if $f : \mathbb{C} \to \mathcal{L}(\mathcal{H})$ is entire analytic and its restriction to $\mathbb{R}$ is a characteristic function then the kernel

$K : \mathbb{C} \times \mathbb{C} \to \mathscr{L}(\mathscr{H})$ defined by $K(z, z') = f(z - \bar{z}')$ is positive definite since

$$\sum_{i,j} (\phi_i, f(z_i - \bar{z}_j)\phi_j) = \left\| \sum_i e^{iz_i R} V\phi_i \right\|^2 \geqslant 0 .$$

Thus (1) implies that for $z, z' \in C$,

$$f(z - \bar{z}) \| f(z' - \bar{z}') \| \geqslant f(z - \bar{z}') f(z' - \bar{z}) . \tag{2}$$

LEMMA 2. — If $f : \mathbb{C} \to \mathscr{L}(\mathscr{H})$ is of the form $f(z) = \exp(izA_1 - z^2 B(iz))$ where A_1, $B(z)$ are in $\mathscr{L}(\mathscr{H})$, $z \mapsto B(z)$ is entire analytic and if its restriction to $\mathbb{R}$ is a characteristic function then for each $y \in \mathbb{R}$ $B(-y) \geqslant 0$.

Proof. — With $z = \frac{1}{2} iy$, $z' = 0$, (2) becomes

$$f(iy) \geqslant \left[f\left(\frac{1}{2} iy\right)\right]^2 .$$

Since the logarithmic function is convex

$$- yA_1 + y^2 B(-y) \geqslant - yA_1 + \frac{1}{2} y^2 B\left(-\frac{1}{2} y\right)$$

or

$$B(-y) \geqslant \frac{1}{2} B\left(-\frac{1}{2} y\right)$$

By repeating the argument we obtain

$$B(-y) \geqslant \frac{1}{2^n} B\left(-\frac{1}{2^n} y\right) .$$

Letting $n \to \infty$ this gives

$$B(-y) \geqslant 0 . \qquad \blacksquare$$

From this we deduce immediately that if B is a polynomial then the degree of the polynomial must be even. Suppose

$$B(z) = A_2 + zA_3 + \dots + z^{2k-2} A_{2k} \qquad (k < \infty)$$

then it follows also that $A_{2k} \geqslant 0$.

Before proving the generalization of the theorem of Marcinkiewicz we need the following lemma about subharmonic functions.

LEMMA 3. — If $\phi : \mathbb{C} \to \mathbb{C}$ is subharmonic at the origin and $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ then there is a sequence (ζ_n) in $\mathbb{C}$ such that $\zeta_n \to 0$, $\arg \zeta_n \in (\theta_1, \theta_2)$ and $\phi(\zeta_n) \to \phi(0)$.

Proof. — Since ϕ is subharmonic

$$\overline{\lim_{\substack{\zeta \to 0 \\ \zeta \neq 0}}} \phi(\zeta) = \phi(0)$$

and

$$2\pi\phi(0) < \int_0^{2\pi} \phi(re^{i\theta})d\theta \tag{3}$$

for r small enough. If $\kappa = \{4\pi/(\theta_2 - \theta_1)\} - 1$ and $n \in \mathbb{N}$ we can find r_0 such that if $r < r_0$, $\phi(re^{i\theta}) < \phi(0) + \frac{1}{n\kappa}$ for $\theta \in [0, 2\pi)$ and (3) is satisfied. For each $r < r_0$ let $\mathscr{A}_r = \left\{ \theta \in [0, 2\pi), \phi(re^{i\theta}) \leqslant \phi(0) - \frac{1}{n} \right\}$. Then if $r < r_0$ and $\mu(\mathscr{A}_r)$ is the Lebesgue measure of $\mathscr{A}_r$

$$\begin{aligned}\int_0^{2\pi} \phi(re^{i\theta})d\theta < \mu(\mathscr{A}_r)\left(\phi(0) - \frac{1}{n}\right) + (2\pi - \mu(\mathscr{A}_r))\left(\phi(0) + \frac{1}{n\kappa}\right) \\ = 2\pi\phi(0) + \frac{2\pi}{n\kappa}\left(1 - \frac{\mu(\mathscr{A}_r)}{2\pi}(\kappa + 1)\right) \\ = 2\pi\phi(0) + \frac{2\pi}{n\kappa}\left(1 - \frac{2\mu(\mathscr{A}_r)}{\theta_2 - \theta_1}\right).\end{aligned}$$

Therefore not to contradict (3) we must have $\mu(\mathscr{A}_r) < \frac{\theta_2 - \theta_1}{2}$ and so the interval (θ_1, θ_2) cannot be a subset of $\mathscr{A}_r$. Thus we can find $\zeta_n \in \mathbb{C}$ such that $|\zeta_n| < \frac{1}{n}$, $\arg \zeta_n \in (\theta_1, \theta_2)$ and $|\phi(\zeta_n) - \phi(0)| < \frac{1}{n}$. Since we can repeat the argument for every n the lemma follows. ■

THEOREM 4. — Suppose that $f : \mathbb{C} \to \mathscr{L}(\mathscr{H})$ is of the form

$$f(z) = \exp \sum_{n=1}^{N} (iz)^n A_n (N < \infty), \qquad A_N \neq 0, \tag{4}$$

and that the restriction of f to $\mathbb{R}$ is a characteristic function, then $N \leqslant 2$.

Proof. — Suppose that $N > 2$, then N has to be even. With $z = \frac{1}{2}iy$, $z' = x + \frac{1}{2}iy$ (2) becomes

$$f(x + iy)^* f(x + iy) \leqslant f(iy) \| f(iy) \| .$$

Therefore f satisfies the « Wedge Property »

$$\| f(x + iy) \| \leqslant \| f(iy) \| \tag{5}$$

Let $G(\zeta) = (\zeta)^{N-1}A_1 + (\zeta)^{N-2}A_2 + \ldots + \rho A_{N-1} + A_N$ so that

$$f(z) = \exp\, (iz)^N G_N\left(\frac{1}{iz}\right)$$

and let $\phi(\zeta)$ denote the spectral radius of $G_N(\zeta)$.

Now

$$\| f(iy) \| = \left\| \exp y^{\mathrm{N}}\mathrm{G}_{\mathrm{N}}\left(-\frac{1}{y}\right) \right\| \leqslant \exp (y^{\mathrm{N}} \| \mathrm{A}_{\mathrm{N}} \| + \mathrm{C} \mid y \mid^{\mathrm{N}-1}) \quad (6)$$

for some constant C and if $r(f(z))$ is the spectral radius of $f(z)$, $z = x + iy$, from (6) we obtain

$$r(f(z)) \leqslant \| f(z) \| \leqslant \exp (y^{\mathrm{N}} \| \mathrm{A}_{\mathrm{N}} \| + \mathrm{C} \mid y \mid^{\mathrm{N}-1})$$

Thus $\log r(f(z)) \leqslant y^{\mathrm{N}} \| \mathrm{A}_{\mathrm{N}} \| + \mid y \mid^{\mathrm{N}-1}\mathrm{C}$.

But

$$r(f(z)) = \sup \left\{ \mid e^{(iz)^{\mathrm{N}}\lambda} \mid : \lambda \in \sigma\left(\mathrm{G}_{\mathrm{N}}\left(\frac{1}{iz}\right)\right) \right\}$$

$$= \sup \left\{ \exp \mathscr{R}(iz)^{\mathrm{N}}\lambda : \lambda \in \sigma\left(\mathrm{G}_{\mathrm{N}}\left(\frac{1}{iz}\right)\right) \right\}$$

So that

$$\log r(f(z)) = \sup \left\{ \mathscr{R}((iz)^{\mathrm{N}}\lambda) : \lambda \in \sigma\left(\mathrm{G}_{\mathrm{N}}\left(\frac{1}{iz}\right)\right) \right\}$$

$$= \frac{1}{r^{\mathrm{N}}} \sup \{ \xi \cos \mathrm{N}\theta + \eta \sin \mathrm{N}\theta : \xi + i\eta \in \sigma(\mathrm{G}_{\mathrm{N}}(re^{i\theta})) \}$$

$$= \frac{\cos \mathrm{N}\theta}{r^{\mathrm{N}}} \sup \{ \xi + \eta \tan \mathrm{N}\theta : \xi + i\eta \in \sigma(\mathrm{G}_{\mathrm{N}}(re^{i\theta})) \}$$

where $iz = \frac{1}{r}e^{-i\theta}$.

Thus

$$\sup \{ \xi + \eta \tan \mathrm{N}\theta : \xi + i\eta \in \sigma(\mathrm{G}_{\mathrm{N}}(re^{i\theta})) \} \leqslant \frac{\| \mathrm{A}_1 \| \cos^{\mathrm{N}} \theta}{\cos \mathrm{N}\theta} + \frac{\mathrm{C}r \mid \cos \theta \mid^{\mathrm{N}-1}}{\cos \mathrm{N}\theta} \quad (7)$$

If $\mathrm{N} > 2$ and N is even we can always find real numbers θ_1, θ_2, α such that $0 < \theta_1 < \theta_2 \leqslant 2\pi$ and $\alpha < 1$ such that the set $\{ \tan \mathrm{N}\theta : \theta \in (\theta_1, \theta_2) \}$ is bounded and $\frac{\cos^{\mathrm{N}}\theta}{\cos \mathrm{N}\theta} < \alpha$ for $\theta \in (\theta_1, \theta_2)$.

Now $\zeta \mapsto r(\mathrm{G}_{\mathrm{N}}(\zeta)) = \phi(\zeta)$ is subharmonic [*9*] [*10*] [*11*] and therefore by lemma 3 we can find a sequence (ζ_n) such that $\zeta_n \to 0$,

$$\theta_n = \arg \zeta_n \in (\theta_1, \theta_2) \qquad \text{and} \qquad \phi(\zeta_n) \to \phi(0) = \| \mathrm{A}_{\mathrm{N}} \| .$$

Also from the upper semi-continuity of the spectrum since

$$\sigma(\mathrm{G}_{\mathrm{N}}(0)) = \sigma(\mathrm{A}_{\mathrm{N}}) \subseteq [0, \| \mathrm{A}_{\mathrm{N}} \|],$$

$$\operatorname{dist} (\sigma(\mathrm{G}_{\mathrm{N}}(\zeta_n)), [0, \| \mathrm{A}_{\mathrm{N}} \|]) \to 0 \qquad \text{as} \quad n \to \infty .$$

Therefore $\sup \{ \xi + \eta \tan \mathrm{N}\theta_n : \xi + i\eta \in \sigma(\mathrm{G}_{\mathrm{N}}(\zeta_n)) \} \to \| \mathrm{A}_{\mathrm{N}} \|$ as $n \to \infty$.

Putting $iz = \frac{1}{\zeta_n} = \frac{1}{r_n} e^{-i\theta_n}$ in (7) we obtain

$$\sup \{ \xi + \eta \tan N\theta_n : \xi + i\eta \in \sigma(G_n(\zeta_n)) \} \leqslant \frac{\| A_1 \| \cos^N \theta_n}{\cos N\theta_n} + \frac{C r_n | \cos \theta_n |^{N-1}}{\cos N\theta_n}$$

Letting $n \to \infty$ we obtain

$$\| A_N \| \leqslant \| A_N \| \alpha .$$

Therefore $\| A_N \| = 0$. ■

THEOREM 5. — Let H and $\mathscr{H}$ be Hilbert spaces and let W be a representation of the canonical commutation relations over H. Let $T : W(H) \to \mathscr{L}(\mathscr{H})$ be positive and suppose that

$$T(W(h)) = \exp \sum_{n=1}^{N} \frac{i^n}{n!} R_n(h, \ldots, h) \qquad N < \infty$$

where R_n is a map from the n-fold Cartesian product $H \times \ldots \times H$ to $\mathscr{L}(\mathscr{H})$ which is real linear in each variable, then $R_n = 0$ for $n > 2$.

Proof. — The proof follows immediately from Theorem 4. The restriction of T to the commutative C*-algebra $\{ W(th) : t \in \mathbb{R} \}$, $h \in H$ is positive and therefore completely positive since the algebra is commutative [12]. But this implies that for each $h \in H$, $f : \mathbb{R} \to \mathscr{L}(\mathscr{H})$, where $f(t) = T(W(th))$ is a characteristic function, so that $R_n(h, \ldots, h) = 0$ for $n > 2$ for each h and therefore by polarisation $R_n = 0$ for $n > 2$. ■

From Theorem 6 we see that the only possible characteristic function of the type in equation (4) is $f(t) = \exp(itA_1 - t^2 A_2)$, A_1, A_2 self-adjoint. A necessary condition is that $A_2 \geqslant 0$. We have not been able to find necessary and sufficient conditions for functions of this type to be characteristic functions. However if $A_1 = 0$ (or if $[A_1, A_2] = 0$) the positivity of A_2 is sufficient. The next proposition is a straightforward generalization of this fact.

Note that the R_n's are formally the truncated functions defined recursively by

$$R_1(h) = R(h)$$

$$n \geqslant 2 \qquad R_n(h, \ldots, h) = T(R(h)^n) - \sum \frac{n!}{k!} \frac{1}{n_1! n_2! \ldots n_k!} R_{n_1}(h, \ldots, h) \ldots R_{n_k}(h, \ldots, h)$$

where the summation is over sets of integers $\{ n_1, n_2, \ldots, n_k \}$ with $n_1 \geqslant 1$, $n_2 \geqslant 1, \ldots, n_k \geqslant 1$, $n_1 + n_2 + \ldots + n_k = n$ and where $R(h)$ is the self-adjoint field operator given by $W(h) = \exp iR(h)$.

PROPOSITION 6. — Let H, $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $K : H \times H \to \mathcal{L}(\mathcal{H})$ a positive definite kernel and A a self-adjoint element of $\mathcal{L}(\mathcal{K})$. Then the kernel $K_A : H \times H \to \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ defined by

$$(h' \otimes k', K_A(x, y)h \otimes k) = \int (h', K(\lambda x, \lambda y)h)d(k', E_\lambda k),$$

where $\{ E_\lambda \}$ are the spectral projections of A, is positive definite.

Proof. — Let $K(x, y) = V^*(x)V(y)$, $V(x) \in \mathcal{L}(\mathcal{H}, \mathcal{M})$ be the Kolmogorov decomposition of K and define $V_A(x) : \mathcal{H} \otimes \mathcal{K} \to \mathcal{M} \otimes \mathcal{K}$ by

$$(m' \otimes k', V_A(y)h \otimes k) = \int (m', V(\lambda y)h)d(k', E_\lambda k).$$

It is straightforward to check that $K_A(x, y) = V_A(x)V_A(y)$ and therefore K_A is positive definite. ■

It follows immediately from this that if

$$\mu : H \to \mathbb{C}, \qquad \mu(h) = \exp\left(-\frac{1}{4} q(h, h)\right)$$

is a generating functional for a representation of the CCR, W is any representation of the CCR over H and $T : W(H) \to \mathcal{L}(\mathcal{H})$ is defined by $T(W(h)) = \exp\left(-\frac{1}{4} q(h, h)B\right)$ where $B \in \mathcal{L}(\mathcal{H})$ and $B \geqslant 1$ then T is completely positive.

Again we have not been able to find necessary and sufficient conditions for maps of the type $T(W(h)) = \exp\left(iR_1(h) - \frac{1}{2} R_2(h, h)\right)$ to be completely positive.

§ 4. FINITE ORDER GENERATORS OF COMPLETELY POSITIVE SEMIGROUPS OF THE CCR

Suppose that a classical Markovian semigroup is given by the kernel $P_t(x, y)$ and has as generator the differential operator

$$A = \sum_{n=1}^{N} a_n(x) \frac{d^n}{dx^n}.$$

Then if

$$\int e^{i\lambda x} P_t(dx, y) = \exp \sum_{n=1}^{\infty} \frac{i^n \lambda^n}{n!} b_n^t(y)$$

$$\left.\frac{\partial b_n^t(y)}{\partial t}\right|_{t=0} \begin{aligned} &= a_n(y) \quad \text{for} \quad n \leqslant N \\ &= 0 \quad \text{for} \quad n > N. \end{aligned}$$

Suppose now that T_t is a completely positive semigroup on the CCR algebra W(H) and let R_n^t be the truncated correlation functions corresponding to T_t i. e.

$$T_t W(h) = \exp \sum_{n=1}^{\infty} \frac{i^n}{n!} R_n^t(h, \ldots, h)$$

and let

$$A_n(h, \ldots, h) = \frac{dR_n^t}{dt}(h, \ldots, h)\big|_{t=0},$$

then by analogy with the classical case we shall say that T_t is a diffusion semigroup for the CCR if $A_n = 0$ for $n > 2$.

Note that the class of quasi-free completely positive semigroups studied in [*5*] [*6*] and [*7*] are of this type. In particular if

$$T_t(W(h)) = W(S_t h) \exp - \frac{1}{4}\{ \| h \|^2 - \| S_t h \|^2 \}$$

where $S_t = e^{-tG}$ is a semigroup of contractions.

Then

$$R_1^t(h) = R(S_t h)$$

$$R_2^t(h) = \frac{1}{2}\{ \| h \|^2 - \| S_t h \|^2 \}$$

$$R_n^t(h, \ldots, h) = 0 \qquad \text{for} \qquad n > 2\,.$$

and

$$A_1(h) = - R(Gh)\,, \qquad A_2(h) = \frac{1}{2}(h, (G + G^*)h)\,.$$

To make these definitions exact we suppose that the following technical conditions are satisfied. For each $h \in H$ the map $\lambda \to W(\lambda h)$, $\lambda \in \mathbb{R}$ is strongly continuous and therefore $W(h) = e^{iR(h)}$ where $R(h)$ is self-adjoint.

Let $T_t : W(H) \to W(H)$ be a completely positive semigroup and

$$T_t W(h) = e^{iR^t(h)}$$

where for each $h \in H$ and $t \in \mathbb{R}$, $iR^t(h)$ is a closed operator with domain $\mathcal{D}(R^t(h)) \subseteq \mathcal{H}$ which generates a semigroup of contractions on $\mathcal{H}$. We assume that for each $h \in H$ there is a dense subspace of $\mathcal{H}$, $\mathcal{D}(h)$ such that

i) $\mathcal{D}(h)$ is a core for $R(h)$,

ii) for each $\phi \in \mathcal{D}(h)$ and $t \in \mathbb{R}$ small enough

$$W(\lambda h)\phi \in \mathcal{D}(R^t(h)), \qquad \sup_{\lambda \in [0,1]} \| (R^t(h) - R(h))W(\lambda h)\phi \| \leqslant tc(h, \phi) < \infty\,,$$

and

$$\tilde{A}(h)W(\lambda h)\phi \equiv \lim_{t \to 0} \frac{(R^t(h) - R(h))}{t} W(\lambda h)\phi$$

exists,

iii) there is $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$ and $K > 0$ such that for t small enough $\| (R^t(h) - z)^{-1} \| < K$.

Then we have the following expression for the generator L of the semigroup T_t:

LEMMA 7. — For each $\phi \in \mathcal{D}(h)$, $\dfrac{T_t W(h) - W(h)}{t}\phi$ converges weakly as $t \to 0$ to $LW(h)\phi$ where

$$LW(h)\phi = \left[\int_0^1 W(\lambda h)\tilde{A}(h)W(-\lambda h)d\lambda\right]W(h)\phi$$

Proof. — Because of *ii*) $R^t(h)$ tends strongly to $R(h)$ on $\mathcal{D}(h)$ and this together with *iii*) and (VIII, Th. 1.5) [*13*] implies that $R^t(h) \to R(h)$ in the strong resolvent sense. Thus by (IX, Th. 2.16) [*13*] $e^{i\lambda R^t(h)} \to e^{i\lambda R(h)}$ strongly. We also have for $\phi \in \mathcal{D}(h)$ and $\psi \in \mathcal{H}$,

$$\left(\psi, \frac{T_t W(h) - W(h)}{t}\phi\right) = \int_0^1 \left(\psi, e^{i\lambda R^t(h)}\frac{R^t(h) - R(h)}{t}W((1-\lambda)h)\phi\right)d\lambda .$$

The integrand is bounded by $\|\psi\|\, c(h, \phi)$ and converges to

$$(\psi, W(\lambda h)\tilde{A}W((1-\lambda)h)\phi)$$

and therefore the lemma follows. ■

We shall say that L is of order N if

a) there is a function A_1 mapping H into the self-adjoint operators on $\mathcal{H}$ such that for each $\lambda \in \mathbb{R}$, $h \in H$ $A_1(\lambda h) = \lambda A_1(h)$ and for $h_1, h_2 \in H$, $A_1(h_1 + h_2) = \overline{A_1(h_1) + A_1(h_2)}$,

b) for $n = 2, \ldots, N$ there are real linear maps

$$A_n : (H \underset{n \text{ times}}{\times \ldots \times} H) \to \mathcal{L}(\mathcal{H})$$

c) if A is the closure of $\tilde{A}$

$$A(h) = \sum_{n=1}^{N} \frac{i^n}{n!} A_n(h, \ldots, h) \qquad \text{and} \qquad A_n \neq 0 .$$

DEFINITION. — The semigroup of completely positive maps

$$T_t : W(H) \to W(H)$$

is a diffusion semigroup for W(H) if the corresponding L is at most of second order.

THEOREM 8. — If L is as above and of finite order (i. e. $N < \infty$) then L is at most of second order.

Proof. — Let $h \in H$ and $x, y \in \mathbb{R}$ let

$$K(x, y) = L[W(-xh)W(yh)] - [LW(-xh)]W(yh) - W(-xh)[LW(yh)]$$

Since T_t is completely positive, for $x_1, \ldots, x_n$ real and $\phi_1, \ldots, \phi_n \in \mathscr{D}(h)$

$$\sum_{i,j} (\phi_i, T_t W((x_j - x_i)h)\phi_j) - (\phi_i, T_t W(-x_i h) T_t W(x_j h)\phi_j)$$
$$= \left\| \sum_j \tilde{W}(x_j h) V_t \phi_j \right\|^2 - \left\| V_t^* \sum_j \tilde{W}(x_j h) V_t \phi_j \right\|^2$$
$$\geqslant 0 \qquad \text{since} \qquad \| V_t^* \| \leqslant 1 .$$

Therefore by differentiating at $t = 0$ we obtain

$$\sum_{i,j} (\phi_i, K(x_i, x_j)\phi_j) \geqslant 0$$

and therefore

$$\sum_{i,j} c_i \bar{c}_j (\phi, W(x_i h) K(x_i, x_j) W(-x_j h)\phi) \geqslant 0 \qquad \text{for} \qquad \phi \in \mathscr{D}(h) .$$

But

$$LW(xh) = \sum_{n=1}^{N} \frac{(ix)^n}{n!} \int_0^1 W(\lambda x h) A_n(h, \ldots, h) W((1-\lambda)xh) d\lambda$$
$$= \sum_{n=1}^{N} \frac{(ix)^n}{n!} \frac{1}{x} \int_0^x W(\mu h) A_n(h, \ldots, h) W(x-\mu)h) d\mu = \sum_{n=1}^{N} A_n(h, x) \frac{(ix)^n}{n!} W(xh)$$

where

$$A_n(h, x) = \frac{1}{x} \int_0^x W(\mu h) A_n W(-\mu h) d\mu .$$

$$LW((y-x)h) = \sum_{n=1}^{N} i^n \frac{(y-x)^{n-1}}{n!} \int_0^{y-x} W(\mu h) A_n(h, \ldots, h) W(y-x-\mu)h) d\mu$$
$$= \sum_{n=1}^{N} i^n \frac{(y-x)^{n-1}}{n!} \left\{ \int_0^{-x} W(\mu h) A_n(h, \ldots, h) W((-x-\mu)h) d\mu W(yh) \right.$$
$$\left. + W(-xh) \int_0^y W(\mu h) A_n(h, \ldots, h) W((-\mu + y)h) d\mu \right\}$$
$$= \sum_{n=1}^{N} i^n \frac{(y-x)^{n-1}}{n!} (-x) W(-xh) A_n(h, -x) W(yh)$$
$$+ i^n \frac{(y-x)^{n-1}}{n!} y W(-xh) A_n(h, y) W(yh) .$$

Therefore

$$\mathrm{K}(x, y) = \sum_{n=2}^{\mathrm{N}} i^n \left\{ \frac{(y-x)^{n-1}(-x) - (-x)^n}{n\,!} \right\} \mathrm{W}(-xh)\mathrm{A}_n(h, -x)\mathrm{W}(yh)$$

$$+ \sum_{n=2}^{\mathrm{N}} i^n \left\{ \frac{(y-x)^{n-1}y - y^n}{n\,!} \right\} \mathrm{W}(-xh)\mathrm{A}_n(h, y)\mathrm{W}(yh)$$

$$= \mathrm{W}(-xh) \sum_{n=2}^{\mathrm{N}} \frac{1}{n\,!} \mathrm{K}_n(x, y)\mathrm{W}(yh)$$

where

$$\mathrm{K}_n(x, y) = i^n \{ (y-x)^{n-1}(-x) - (-x)^n \} \mathrm{A}_n(h, -x) + i^n \{ (y-x)^{n-1}y - y^n \} \mathrm{A}_n(h, y), \qquad n \geqslant 2.$$

Thus

$$\sum_{i,j} c_i \bar{c}_j \left(\phi, \sum_{n=2}^{\mathrm{N}} \frac{1}{n\,!} \mathrm{K}_n(x_i, xj)\phi \right) \geqslant 0 \qquad \text{for} \qquad \phi \in \mathscr{D}(h).$$

Since all operators involved in this expression are bounded and $\mathscr{D}(h)$ is dense this is true for all $\phi \in \mathscr{H}$.

By taking Riemann sums we obtain

$$\iint f(x) \left(\phi, \sum_{n=2}^{\mathrm{N}} \frac{1}{n\,!} \mathrm{K}_n(x, y)\phi \right) \overline{f(y)} d\mu(x) d\mu(y) \geqslant 0$$

for $f \in \mathrm{L}^2(\mathbb{R}, d\mu)$, $\phi \in \mathscr{H}$ where $d\mu(x) = e^{-x^2}dx$.

If we let $\phi_j(x) = (ix)^j$, $\psi_j^n(x) = (ix)^j(\phi, \mathrm{A}_n(h, x)\phi)$ then

$$\mathscr{R}\left\{ \sum_{n=2}^{\mathrm{N}} \frac{1}{n\,!} \sum_{r=1}^{n-1} {}^{n-1}\mathrm{C}_r(f, \phi_{n-r})(\psi_r^{(n)}, f) \right\} \geqslant 0 \tag{8}$$

where

$$(f, g) = \int \bar{f}(x) g(x) d\mu(x).$$

Now we can chose f such that $(f, \phi_{\mathrm{N}-1}) = 1$ and $(f, \phi_j) = 0$ $j = 1, 2, \ldots, \mathrm{N}-2$ so that $\mathscr{R}(\psi_1^{\mathrm{N}}, f) \geqslant 0$ which means that ψ_1^{N} depends linearly on $\phi_1, \ldots, \phi_{\mathrm{N}-1}$ otherwise f can be chosen so that $\mathscr{R}(\psi_1^{\mathrm{N}}, f) < 0$. But $\dfrac{\psi_1^{\mathrm{N}}(x)}{x}$ is bounded and therefore $\psi_1^{\mathrm{N}} = \alpha_{\mathrm{N}}\phi_1$.

Thus $(\phi, \mathrm{A}_{\mathrm{N}}(h, x)\phi) = \alpha_{\mathrm{N}} = (\phi, \mathrm{A}_{\mathrm{N}}(h, \ldots, h)\phi)$.

If $N > 2$ we can choose f such that

$$(f, \phi_{N-2}) = 1, \qquad (f, \phi_j) = 0 \qquad j = 1, \ldots, N-3, N-1$$

to obtain $\mathscr{R}(\psi_1^{N-1}, f) \geqslant 0$.

By repeating this procedure we prove that

$$(\phi, A_n(h, x)\phi) = (\phi, A_n(h, \ldots, h)\phi) \equiv \alpha_n \qquad \text{for} \qquad n = 2, \ldots, N$$

so that (8) becomes

$$\sum_{n=2}^{N} \frac{1}{n!} \alpha_n \sum_{r=1}^{n-1} {}^{n-1}C_r(f, \phi_{n-r})(\phi_r, f) \geqslant 0 .$$

If $N > 2$ we choose $(f, \phi_j) = 0, j = 2, \ldots, N-1, (f, \phi_1) = 1, (f, \phi_j) = A$ (arbitrary) to obtain

$$\alpha_2 + \frac{\alpha_N}{N-1} \mathscr{R}A \geqslant 0 \qquad \text{and so} \qquad \alpha_N = 0 . \qquad \blacksquare$$

For the final theorem we assume that: (α) for $h_1, \ldots, h_n \in \mathscr{H}, \bigcap_{k=1}^{n} \mathscr{D}(h_k)$ is dense and (β) if $h, g \in H$, $\phi \in \mathscr{D}(h_1) \cap \mathscr{D}(h_2)$ then $W(h)\phi \in \mathscr{D}(A_1(g))$ and $\lambda \to A(g)W(\lambda h)\phi$ is continuous.

THEOREM 9. — If T_t is a diffusion satisfying (α) and (β) then for each $h \in H$, $A_2(h, h)$ commutes with $W(h)$ and the kernel $A_2 : H \times H \to \mathscr{L}(\mathscr{H})$ is positive definite.

Proof. — We can see from the proof of Theorem 8 that

$$(\phi, A_2(h, x)\phi) = (\phi, A_2(h, h)\phi) \qquad \text{for all} \qquad x \in \mathbb{R} .$$

Therefore

$$\int_0^x W(\mu h)A_2(h, h)W(-\mu h)d\mu = xA_2(h, h)$$

and thus $W(xh)A_2(h, h) = A_2(h, h)W(xh)$, so that for $\phi \in \mathscr{D}(h)$,

$$LW(h)\phi = \int_0^1 W(\lambda h)A_1(h)W((1-\lambda)h)\phi d\lambda + A_2(h, h)W(h)\phi . \qquad (9)$$

For $h, g \in H$ let

$$K(h, g) = L(W(-h)W(g)) + L(W-g)W(h)) - (LW(-h))W(g) - W(-h)(LW(g)) - (LW(-g))W(h) - W(-g)(LW(h)) .$$

For $\phi, \psi \in \mathscr{D}(h) \cap \mathscr{D}(g)$, we find using (9) and (β)

$$\lim_{x \to 0} \frac{1}{x^2} (\phi, K(xh, xg)\phi) = 2A_2(h, g) .$$

But if $h_1, \ldots, h_n \in \mathrm{H}$ and $\phi_1, \ldots, \phi_n \in \bigcap_{k=1}^{n} \mathscr{D}(h_k)$ then

$$\frac{1}{x^2} \sum_{i,j} (\phi_i, \mathrm{K}(xh_i, xh_j)\phi_j) \geqslant 0$$

and therefore

$$\sum_{i,j} (\phi_i, \mathrm{A}_2(h_i h_j)\phi_j) \geqslant 0 .$$

Since $\bigcap_{k=1}^{n} \mathscr{D}(h_k)$ is dense this is true for any finite set $\{ \phi_1, \ldots, \phi_n \} \subseteq \mathscr{H}$. ■

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