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Infrared singularities, vacuum structure and pure phases in local quantum field theory

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ABSTRACT. — The occurrence of infrared singularities of the confining type imply that the associated quantum field theory cannot satisfy the positivity condition and therefore one has a strong departure from standard (positive metric) QFT's. The general structure properties of indefinite metric local QFT's (which include gauge QFT's) are investigated. In particular we discuss the problem of associating a Hilbert space structure to a given set of Wightman functions, the properties of maximal Hilbert space structures, the connection between the occurrence of infrared singularities and the existence of more than one translation invariant state (θ -vacua), the definition and uniqueness of the vacuum state and its relation with the irreducibility of the local field algebra (pure phases).

RÉSUMÉ. — L'existence de singularités infrarouges du type confinant implique que la théorie des champs associée ne peut pas satisfaire la condition de positivité et par conséquent on a un fort départ par rapport aux TCQ's standards (métrique positive). Les propriétés générales de structure des théories des champs quantiques ayant une métrique indéfinie (qui comprennent les théories des champs de jauge) sont étudiées. En particulier nous discutons le problème d'associer une structure Hilbertienne à un

ensemble donné de fonctions de Wightman, les propriétés des structures Hilbertiennes maximales, la connection entre l'existence de singularités infrarouges et l'existence de plus qu'un état invariant par translations (θ -vacua), la définition et l'unicité de l'état du vide et sa relation avec l'irréductibilité de l'algèbre locale des champs (phases pures).

1. INTRODUCTION

For a long time the main problem of local quantum field theory (QFT) has been the control and the elimination of ultraviolet divergences, first as a crucial practical problem for computing finite higher order contributions in perturbation theory [1] and then as a question of whether the theory was internally consistent [2] and mathematically acceptable [3]. With the advent of gauge QFT's [4] it has been suggested that it might be better to have a good ultraviolet behaviour at the price of a bad infrared structure, with the hope that the infrared problem would be solved by a correct identification of the physical states (confinement mechanism). The properties of such theories has been the subject of many investigations during the last years both at the level of the perturbation theory [4], with improvements which take the non linear effects of the classical solutions into account [5], and at the level of constructive QFT [6] in the lattice field theory approach. It seems however that the main infrared problems, connected with the construction of the « charged » states, are still open, apart from the abelian case [7]. The deep physical reason is that in such theories there exist « phases » or « sectors » which cannot be characterized in terms of expectation values of local observables or by local order parameters, but one must use observables of the type of charges which obey a Gauss' law [8], and characterize such sectors by the expectation value of a loop or a flux at infinity [5]. A direct consequence of such phenomena is that not all the physical states, in particular the charged states and in general states with non trivial « topological » numbers like the θ -vacua [5], cannot be described in terms of local excitations of the vacuum [9], [8] nor by local morphisms of the algebra of observables. Therefore, a by far non trivial step is involved in going from the Green's or Wightman's functions of the local fields to the construction of all the physical states, in particular the charged states.

From a more technical point of view, the above difficulties are strictly related to the fact that such theories cannot be described in terms of an irreducible set of local fields without giving up the positivity condition and one is rather naturally lead to indefinite metric QFT's. The necessity of

indefinite metric can be proved quite generally when there are charges which obey a Gauss' law [9] [8] (gauge QFT's) and it is also unavoidable whenever the Green's or Wightman's functions exhibit infrared singularities associated with a « confining » potential, since such singularities are incompatible with positivity, in a local QFT [10]. In all such cases we have a strong departure from standard (positive metric) QFT [11] where the quantum mechanical interpretation of the theory, equivalently the identification of the physical states is uniquely fixed by the *local states*, whereas in indefinite metric QFT apparently a large arbitrariness is involved.

In Sect. 2, 3 we identify the condition (*Hilbert space structure condition*) which replaces the axiom of positivity and allows the construction of a Hilbert space associated to the given set of Wightman functions (reconstruction theorem for indefinite metric QFT's). One of the main points of our analysis is to emphasize that when the Wightman functions do not satisfy the positivity condition, one may associate with them different Hilbert space structures, leading in general to completely different spaces of states. In particular, very important structure properties like the existence of more than one translation invariant state (mixed phase), spontaneous symmetry breaking, existence of θ -vacua, reducibility of the field algebra, crucially depend on the Hilbert space structure one associates to the given Wightman functions.

Among all the possible Hilbert space structures, it is shown that maximal Hilbert space structures (Krein spaces) identify maximal sets of states to the given Wightman functions and exhibit very important properties. In particular, in maximal Hilbert space structures one may establish a connection between the occurrence of infrared singularities and the existence of more than one translation invariant state (θ -vacua) (Sect. 4), a phenomenon which is not governed in general by the cluster property as in the standard (positive metric) case [12]. As illustrative examples the massless scalar field in two dimensions and the dipole field in four dimensions are discussed.

The question of the irreducibility of the field algebra (pure phases) in connection with the cyclicity and uniqueness of the vacuum is solved in Sect. 5. We also analyze the definition and uniqueness of the vacuum, a concept which requires much more care when the space time translations are not described by unitary operators and one cannot associate with them spectral projectors, in general (Sect. 5). Many of the basic results of standard (positive metric) QFT [11] are generalized to the case of indefinite metric. In our opinion, the emerging picture is that the general structure of indefinite metric QFT's is much richer than standard QFT's since it can naturally account for important phenomenon like confining infrared singularities, θ -vacua, absence of local order parameters, Higgs' phenomenon, etc. In particular the use of an irreducible set of local fields (in particular the charged fields) may allow the construction of the charged states and/or of other

topological sectors which are not directly available in terms of the algebra of observables and whose construction appears to be far from trivial starting from the vacuum sector.

2. COVARIANCE, LOCALITY AND SPECTRAL CONDITION

As clarified in the mid-fifties, a field theory is defined by a set of Green or Wightman functions [3]. Actually, this is the way a field theory is obtained either by perturbation theory or by constructive field theory methods. Therefore an analysis of the structure properties of a (positive or indefinite) quantum field theory (QFT) always reduces to a study of its Wightman functions.

In this paper we will focus our attention on local and covariant QFT, namely to theories whose Wightman distributions [11] satisfy the following properties (which for simplicity we specify only in the case of an hermitian scalar field).

I. COVARIANCE

For any Poincaré transformation $\{a, \Lambda\}$ the n -point functions are invariant

$$\mathcal{W}(\Lambda x_1 + a, \dots, \Lambda x_n + a) = \mathcal{W}(x_1, \dots, x_n) \quad (2.1)$$

II. LOCALITY

If $x_i - x_{i+1}$ is spacelike

$$\mathcal{W}(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = \mathcal{W}(x_1, \dots, x_{i+1}, x_i, \dots, x_n) \quad (2.2)$$

III. SPECTRAL PROPERTIES

The Fourier transforms $\tilde{\mathcal{W}}(q_1, \dots, q_{n-1})$ of the distributions

$$\mathcal{W}(\xi_1, \dots, \xi_{n-1}) \equiv \mathcal{W}(x_1, \dots, x_n), \quad \xi_j = x_j - x_{j+1}$$

have support contained in the cones $q_i^2 \geq 0$.

The hermiticity conditions reads

$$\mathcal{W}(x_1, \dots, x_n) = \overline{\mathcal{W}(x_n, \dots, x_1)}. \quad (2.3)$$

For the motivations of I-III we refer to [11]. Even in the positive metric case, I-III involve in general an extrapolation with respect to the physical requirements, as it must be since a Wightman field theory has a richer structure than the algebraic formulation of Araki Haag and Kastler, based exclusively on the algebra of observables [13]. The richer Wightman structure, which involves physical as well as non physical fields (e. g. the

fermion fields) has indeed proved to be an advantage in the actual construction of field theories. The usefulness of introducing unphysical fields to allow a simple and practical definition of interactions, with the automatic validity of basic physical properties like microscopic causality and positivity of the energy, has been repeatedly recognized in the past. For these reasons we maintain I-III also in the indefinite metric case. This choice also leads to very important technical advantages. We recall the basic role played by locality and covariance in the development and the very formulation of renormalization theory [14]. Even the recent developments built on the functional integral formulation of QFT are crucially based on the analytic continuation of the Wightman functions from Minkowski to Euclidean space, a property which relies on I-III.

In the positive metric case two further properties are added to I-III, namely positivity and the cluster property, and these allow to recover a quantum mechanical interpretation of the theory in the terms of states, transition probabilities and *irreducible* field operators. For indefinite metric QFT this strategy requires modifications and the main purpose of this paper is to discuss them in detail.

With only I-III at our disposal we can recover only a set of vector states which have a linear structure [15].

THEOREM 1. — Given a set of Wightman distributions $\mathcal{W}$ satisfying I-III, one can construct a linear space $\mathbb{W}$ a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{W}$ and operator valued distributions $\varphi(f)$, $f \in \mathcal{S}(\mathbb{R}^4)$, acting on $\mathbb{W}$ such that

- a) there is a vector $\Psi_0 \in \mathbb{W}$, cyclic with respect to the polynomial algebra $\mathcal{P}$ generated by the field operators $\varphi(f)$, with $\langle \Psi_0, \Psi_0 \rangle > 0$
- b) the field operators satisfy locality, namely

$$[\varphi(f), \varphi(g)] = 0, \quad (2.4)$$

if $\text{supp } f$ is spacelike with respect to $\text{supp } g$

- c) there is a linear representation $U(a, \Lambda)$ of the Poincaré group on $\mathbb{W}$ defined by (P denotes a polynomial)

$$U(a, \Lambda)P(\varphi(f))\Psi_0 = P(\varphi(f_{(a, \Lambda)}))\Psi_0, \quad (2.5)$$

with $f_{(a, \Lambda)}(x) = f(\Lambda^{-1}x - a)$, so that Ψ_0 is a Poincaré invariant vector

- d) the sesquilinear form is hermitean, non degenerate, Poincaré invariant and

$$\langle \Psi_0, \varphi(x_1) \dots \varphi(x_n)\Psi_0 \rangle = \mathcal{W}(x_1, \dots, x_n). \quad (2.6)$$

Proof. — We start from the Borchers [16] algebra $\underline{\mathcal{S}}$ whose elements are finite sequences

$$\underline{f} = (f_0, f_1, \dots), \quad (2.7)$$

with $f_n \in \mathcal{S}(\mathbb{R}^{4n})$. The Wightman functional $\underline{\mathcal{W}}$ defines a linear functional on $\underline{\mathcal{S}}$ through

$$\underline{\mathcal{W}}(\underline{f}) = \sum \mathcal{W}_n(f_n). \quad (2.8)$$

Denoting by $\times$ the tensor production $\underline{\mathcal{S}}$

$$(\underline{f} \times \underline{g})_n = \sum_{l+k=n} f_l g_k \quad (2.9)$$

We get the following sesquilinear form on $\underline{\mathcal{S}}$

$$\langle \underline{f}, \underline{g} \rangle \equiv \mathcal{W}(\underline{f}^\star \times \underline{g}), \quad (2.10)$$

which is hermitean as a consequence of condition (2.3). To obtain a non degenerate form we consider the set L of elements $\underline{g} \in \underline{\mathcal{S}}$ such that

$$\langle \underline{f}, \underline{g} \rangle = 0, \quad \forall \underline{f} \in \underline{\mathcal{S}}. \quad (2.11)$$

L is clearly a linear space and it is invariant by left multiplication

$$\mathcal{W}(\underline{f}^\star \times (\underline{h} \times \underline{g})) = \mathcal{W}(\underline{h}^\star \times \underline{f})^\star \times \underline{g} = 0. \quad (2.12)$$

We then define as linear space $\mathbb{W}$ the set of equivalence classes $[\underline{f}] \in \underline{\mathcal{S}}/L$.

On $\mathbb{W}$ the field operators are defined by

$$\varphi(f)[\underline{g}] = [\underline{f} \times \underline{g}], \quad (2.13)$$

with $\underline{f} = (0, f, 0, \dots)$. Equation (2.13) is well defined since, by eq. (2.12), the right hand side does not depend on the choice of $\underline{g}$ within its equivalence class. Clearly, the vector $\Psi_0 = [f_0]$, $f_0 = (f_0, 0, 0, \dots)$ is non zero and it is cyclic with respect to $\mathcal{F}$. Furthermore

$$\begin{aligned} \langle \Psi_0, \varphi(f^{(1)}) \dots \varphi(f^{(n)}) \Psi_0 \rangle \\ = \mathcal{W}(\underline{f}^{(1)} \times \underline{f}^{(2)} \times \dots \times \underline{f}^{(n)}) \\ = \int dx_1 \dots dx_n \mathcal{W}(x_1, \dots, x_n) f^{(1)}(x_1) \dots f^{(n)}(x_n) \end{aligned} \quad (2.14)$$

i. e. eq. (2.6) holds. The completion of the proof then follows easily.

3. HILBERT SPACE STRUCTURE CONDITIONS

As we have seen in the previous section the definition of a QFT in terms of its Wightman functions leads to a representation in terms of field operators and local states in a vector space. In general, it is not guaranteed that one may introduce a (pre) Hilbert space structure in such a vector space $\mathbb{W}$ and provide $\mathbb{W}$ with a quantum mechanical interpretation and one might envisage the situation in which such a quantum mechanical structure will emerge only at the level of asymptotic states.

The need for a convenient Hilbert topology in order to define the asymptotic limit and strongly motivated physical considerations suggest to consider the case in which a quantum mechanical structure and a quantum mechanical interpretation emerges also for field configurations at finite times.

As we will see in more detail later, the physical interpretation of the theory (or equivalently the identification of the physical states associated to the given Wightman functions) crucially relies on the specification of a Hilbert space structure. In the standard case [11] the QM structure is required to emerge already at the level of local states; more precisely one assumes that local states have a physical interpretation as quantum mechanical states and therefore the Wightman functions are required to satisfy the positivity condition. This amounts to consider QFT's whose physical (phase) structure can be read off at the local level, i. e. there exist *local* observables which take different values in different phases (*local order parameters*).

This does not cover the case in which different pure phases cannot be completely characterized in terms of expectation values of local observables, but one must use observables of the type of charges which obey Gauss' law [17]. This is the case of phases which are characterized by the expectation value of a loop or a flux at infinity. In all such cases, not all the physical states can be described in terms of local excitations of the vacuum, nor by local morphisms [18]. The discussion of such states requires the introduction of a space of states the structure of which is related to the local structure of the theory (i. e. to the local space $\mathcal{L}$) in a more complicated way than in the standard case. One has therefore to solve this problem first in order to get a physical interpretation of the theory and to exhibit its QM structure.

To this purpose we suggest adopting the approach in which such a Hilbert space structure is related to the Wightman functions (in a local theory) and therefore it provides for a concrete and natural method for constructing representations of the algebra of observables which cannot be obtained in terms of local morphisms.

As a first step we have to specify necessary and sufficient conditions in order that a given set of Wightman functions may be given a Hilbert space structure. More precisely we have to specify the conditions under which there exists a Hilbert space $\mathcal{H}$ such that the local vectors are dense in $\mathcal{H}$ ($\mathcal{L} = \mathcal{H}$) and the Wightman functions can be written in terms of an indefinite scalar product, defined for all vectors of $\mathcal{H}$. This means that we look for (*Hilbert space structure*)

i) a Hilbert space $\mathcal{H}$, with scalar product $(\cdot, \cdot)$, such that $\mathcal{L}$ is dense in $\mathcal{H}$

ii) and with the property that there exists a bounded self adjoint operator η such that

$$\begin{aligned} \mathcal{W}(x_1, \dots, x_n) &= (\Psi_0, \eta\varphi(x_1) \dots \varphi(x_n)\Psi_0) \\ &= (\varphi(x_j) \dots \varphi(x_1)\Psi_0, \eta\varphi(x_{j+1}) \dots \varphi(x_n)\Psi_0) \\ &\equiv \langle \varphi(x_j) \dots \varphi(x_1)\Psi_0, \varphi(x_{j+1}) \dots \varphi(x_n)\Psi_0 \rangle \quad (3.1) \end{aligned}$$

or, equivalently,

$$\mathcal{W}(\underline{f}^\star \times \underline{g}) = \langle \underline{f}, \underline{g} \rangle = (\underline{f}, \underline{g}). \quad (3.2)$$

We have then the following results [19].

THEOREM 2. — Given a set of Wightman functions a necessary and sufficient condition for the validity of *i*), *ii*) is that there exists a set $\{p_n\}$ of Hilbert seminorms p_n defined on $\mathcal{S}(\mathbb{R}^{4n})$ such that

$$|\mathcal{W}_{n+m}(f_n^\star \times g_m)| \leq p_n(f_n)p_m(g_m) \quad (3.3)$$

Proof. — The necessity of condition (3.3) follows from Schwartz inequality with $p_n(f_n) = (f_n, f_n)^{\frac{1}{2}} \|\eta\|^{\frac{1}{2}}$.

On the other hand if eq. (3.3) holds, then one may introduce a scalar product in $\mathcal{L}$

$$(\underline{f}, \underline{g}) = \Sigma(n+1)^2 (f_n, g_n)_n$$

where $(\ , \)_n$ is the Hilbert scalar product defined by p_n .

Clearly

$$\begin{aligned} |\langle \underline{f}, \underline{g} \rangle| &= |\mathcal{W}(\underline{f}^\star \times \underline{g})| = \left| \sum \mathcal{W}(f_n^\star \times g_m) \right| \\ &\leq \sum_{n,m} |\mathcal{W}(f_n^\star \times g_m)| \leq \sum_{n,m} p_n(f_n)p_m(g_m) = \left(\sum_n p_n(f_n) \right) \left(\sum_m p_m(g_m) \right) \\ &\leq \left(\sum_n (n+1)p_n(f_n) \frac{1}{n+1} \right) \left(\sum_m (m+1)p_m(g_m) \frac{1}{m+1} \right) \\ &\leq (\Sigma(n+1)^2 p_n(f_n)^2)^{\frac{1}{2}} \left(\Sigma \left(\frac{1}{n+1} \right)^2 \right)^{\frac{1}{2}} \left(\sum_m (m+1)^2 p_m(g_m)^2 \right)^{\frac{1}{2}} \left(\Sigma \left(\frac{1}{m+1} \right)^2 \right)^{\frac{1}{2}} \\ &\leq C(\underline{f}, \underline{f})^{\frac{1}{2}} (\underline{g}, \underline{g})^{\frac{1}{2}} \end{aligned}$$

so that by a redefinition of the scalar product we can satisfy eq. (3.2).

Remark. — Since $\mathcal{S}_n$ is a locally convex nuclear topological space, it is always possible to describe its topology by Hilbert seminorms. The non trivial point is the existence of Hilbert seminorms which satisfy eq. (3.3); this is in general not guaranteed unless the Wightman functionals $\mathcal{W}(\underline{f}^\star \times \underline{g})$ are jointly continuous in $\underline{f}$ and $\underline{g}$ [20].

THEOREM 3. — A sufficient condition for the validity of condition (3.3) is that the Wightman functions satisfy the following regularity condition: when smeared in the variables $x_{j+1}, \dots, x_n$ the distributions $\mathcal{W}_n(x_1, \dots, x_j, g)$ have an order which is bounded by a number N_j independent of n and of the test function g .

Proof. — By standard arguments, the validity of the condition of the theorem implies that one can construct Sobolev type seminorms p_j such that

$$|\mathcal{W}_n(f_j^\star \times g_{n-j})| \leq C_g^{j, n-j} p_j(f_j).$$

By using the Hermiticity conditions one can also find a seminorm p_{n-j} such that

$$|\mathcal{W}_n(\zeta_j^\star \times g_{n-j})| \leq C^{j,n-j} p_j(f_j) p_{n-j}(g_{n-j}).$$

Thus, by introducing the new seminorms

$$p'_j(f_j) = (\sup_{k \leq j} C^{j,k}) p_j(f_j)$$

we get

$$|\mathcal{W}_n(f_j^\star \times g_{n-j})| \leq C p'_j(f_j) p'_{n-j}(g_{n-j}).$$

The condition of theorem 3 is also necessary and sufficient for the field operator $\varphi(f)$ to be strongly continuous in f in the $\mathcal{S}$ topology; actually it can be shown that in this case the weak continuity implies the strong continuity so that the condition of theorem 3 must be satisfied if one wants the continuity of the matrix elements $\langle \Phi, \varphi(f)\Psi \rangle$, $\forall \Psi \in \mathcal{S}$, $\forall \Phi \in \mathcal{K}$. (Clearly the continuity of the matrix elements $\langle \Phi, \varphi(f)\Psi \rangle$, $\Phi, \Psi \in \mathcal{S}$ is already guaranteed by theorem 1). The continuity of the matrix elements $\langle \Phi, \varphi(f)\Psi \rangle$ for any $\Phi \in \mathcal{K}$ might be a redundant and not necessary requirement.

In general, a Hilbert space structure satisfying i), ii) defines a self adjoint operator η which may be degenerate, i. e. there may be non zero vectors $\Psi \in \mathcal{K}$, such that

$$\langle \Psi, \Phi \rangle = 0 \quad \forall \Phi \in \mathcal{K}$$

However one may always reduce to the case in which η is non-degenerate by the following argument.

Let

$$\mathcal{K}_0 \equiv \{ \Psi, \langle \Psi, \Phi \rangle = 0 \quad \forall \Phi \in \mathcal{K} \}$$

Then

$$\eta \mathcal{K}_0 = 0$$

and if P_0 denote the projector on $\mathcal{K}_0$, the new scalar product

$$(\underline{f}, \underline{g})' = (\underline{f}, (1 - P_0)g),$$

$\forall \underline{f}, \underline{g} \in \mathcal{S}$, still provides a Hilbert majorant of $\langle \cdot, \cdot \rangle$ and the Hilbert space $\mathcal{K}$ obtained by completion of $\mathcal{S}$ with respect to $(\cdot, \cdot)'$, still satisfies i), ii) and therefore it provides a Hilbert space with a non degenerate metric operator.

In particular, such removal of the degeneracy of η automatically takes care of non trivial ideals of the Borchers algebra $\mathcal{S}$ arising from specific properties of the Wightman functions, like locality and spectral conditions.

We can now state the axiom which replaces the positivity condition of the standard case.

IV. HILBERT SPACE STRUCTURE CONDITION

There exists a set of Hilbert seminorm $\{p^n\}$, p_n defined on $\mathcal{S}(\mathbb{R}^{4n})$ such that

$$|\underline{\mathcal{W}}(f_n^\star \times g_m)| \leq p_n(f_n) p_m(g_m) \quad (3.3)$$

(We will say that the Hilbert space structure condition is satisfied in a *strong form* if p_n are continuous seminorms on $\mathcal{S}(\mathbb{R}^{4n})$).

The results discussed in Sect. 2, 3 then lead to the following reconstruction theorem.

THEOREM 4 (*Reconstruction theorem for indefinite metric QFT*). — Given a set of Wightman functions satisfying I-IV one can construct

a) a separable Hilbert space $\mathcal{H}$, with scalar product $(\cdot, \cdot)$ and a metric operator η which is bounded, self adjoint non degenerate

b) a representation $U(a, \Lambda)$ of $P_+^\uparrow$ in $\mathcal{H}$, where the operators $U(a, \Lambda)$ have a common dense domain $D \supset D_0 \equiv \mathbb{W}$ and are η -unitary [22] i. e.

$$(U(a, \Lambda)\Phi, \eta U(a, \Lambda)\Psi) = (\Phi, \eta\Psi) \equiv \langle \Phi, \Psi \rangle, \forall \Phi, \Psi \in D_0$$

c) a translation invariant vector Ψ_0 with $\langle \Psi_0, \Psi_0 \rangle > 0$

d) a local (hermitian) field operator $\varphi(f)$, $f \in \mathcal{S}(\mathbb{R}^4)$ with a dense domain D_0 , such that

$$\begin{aligned} \mathcal{W}(x_1, \dots, x_n) &= (\Psi_0, \eta\varphi(x_1) \dots \varphi(x_n)\Psi_0) \\ &= (\varphi(x_1) \dots \varphi(x_n)\Psi_0, \eta\varphi(x_{j+1}) \dots \varphi(x_n)\Psi_0) \end{aligned}$$

Furthermore if the seminorms are invariant under $U(a)$ (and/or $U(\Lambda)$) then $U(a)$ (and/or $U(\Lambda)$) can be extended from $\mathbb{W}$ to all of $\mathcal{H}$ and the extended operators are unitary operators on $\mathcal{H}$.

Finally, if the seminorms P_n are continuous seminorms on $\mathcal{S}(\mathbb{R}^{4n})$, not only the matrix elements $\langle \Phi, \varphi(f)\Psi \rangle$, $\Phi, \Psi \in D_0$ are tempered distributions, since they are finite sums of Wightman functions, but also the matrix elements $(\Phi, \varphi(f)\Psi)$, $\Psi \in D_0$, $\Phi \in \mathcal{H}$ are tempered distributions.

As discussed before, the physical interpretation of a theory defined by a set of Wightman functions crucially relies on the introduction of a Hilbert space structure and it is natural to ask how much arbitrariness is involved in this procedure. In the standard case, the Hilbert space structure has an intrinsic meaning since it is directly given by the set of Wightman functions, via the positivity condition. In the indefinite metric case, such a connection is much less tight and, in fact, to a given set of Wightman functions, one may associate completely different Hilbert space structures [23], leading to completely different space of states [25].

Therefore, whereas in the standard (positive metric) case the set of local states uniquely fix their closure, in the indefinite metric case different closures are available, corresponding to different Hilbert space topologies. A more systematic investigation of this problem is deferred to a subsequent paper. For the present paper it is enough to introduce the following notion of *maximality*.

DEFINITION. — A Hilbert space structure $(\eta, \mathcal{H})$ associated to a given set of Wightman functions, with non degenerate metric operator η , is

maximal if there is no other Hilbert space structure $(\tilde{\eta}, \tilde{\mathcal{H}})$, associated to the given set of Wightman functions, with a non degenerate metric, operator $\tilde{\eta}$, such that $\mathcal{H}$ is properly contained in $\tilde{\mathcal{H}}$.

Clearly, since we are interested in obtaining as much information as possible from the set of Wightman functions $\{W\}$, it is natural to look for Hilbert space structures which are maximal, i. e. such that they associate to $\{W\}$ a maximal set of states. We have then

THEOREM 5. — A Hilbert space structure $(\eta, \mathcal{H})$ associated to a set of Wightman functions $\{W\}$ is maximal iff η^{-1} is bounded.

LEMMA. — Given a Hilbert space structure $(\eta, \mathcal{H})$ with η^{-1} bounded, one may redefine the metric, without changing $\mathcal{H}$, in such a way that the new metric $\tilde{\eta}$ satisfies $\tilde{\eta}^2 = 1$.

Proof. — Let $(\cdot, \cdot)_1$ be the scalar product defined by $(\eta, \mathcal{H})$. Then we define $(\cdot, \cdot) = (\cdot, |\eta| \cdot)_1$ so that

$$\langle \cdot, \cdot \rangle = (\cdot, \eta \cdot)_1 = (\cdot, \text{sign } \eta \cdot) \equiv (\cdot, \tilde{\eta} \cdot)$$

Proof of theorem 5. — Given a Hilbert space $(\eta, \mathcal{H})$, with scalar product $(\cdot, \cdot)_1$, one can always introduce a new Hilbert space structure $(\tilde{\eta}, \tilde{\mathcal{H}})$ in such a way that $\tilde{\eta}^{-1}$ is bounded. We put

$$(\cdot, \cdot) = (\cdot, |\eta| \cdot)_1$$

The new scalar product defines a Hilbert topology $\tilde{\tau}$ on $\mathbb{W}$, which is weaker than the topology τ defined by $(\cdot, \cdot)_1$ since

$$\|x_n - x_m\| \leq \|\eta\| \|x_n - x_m\|_1$$

The Hilbert space $\mathcal{H}$ is complete with respect to $\tilde{\tau}$ iff η^{-1} is bounded. Therefore if η^{-1} is not bounded the above construction shows that $(\eta, \mathcal{H})$ is not maximal. On the other hand if η^{-1} is bounded and $(\eta_1, \mathcal{H}_1)$ is a Hilbert space structure such that $\mathcal{H}_1 \supset \mathcal{H}$, then $(\cdot, \cdot)_1 \leq C(\cdot, \cdot)$, which implies $(x, y)_1 = (x, Ay)$ with $A = A^\star$, $A \geq 0$, $\|A\| < \infty$. It then follows that $A\eta_1 = \eta$ and since η^{-1} is bounded so is A^{-1} and the two scalar products define the same Hilbert topology.

Remark. — By the above result given a Hilbert space structure $(\eta, \mathcal{H})$ there is always a new Hilbert space structure $(\tilde{\eta}, \tilde{\mathcal{H}})$ which is maximal and such that $\tilde{\mathcal{H}} \supseteq \mathcal{H}$; therefore given a Hilbert space structure $(\eta, \mathcal{H})$ it is always possible to construct a new metric $\tilde{\eta}$ such that $\tilde{\eta}^2 = 1$. Inner product space with the property that the metric operator $\tilde{\eta}$ satisfies $\tilde{\eta}^2 = 1$ are also called Krein spaces and they have been extensively studied in the literature [24] [26].

The above maximality condition has a direct consequence in terms of physical states. In fact given a Hilbert space structure $(\eta, \mathcal{H})$, for the

physical interpretation of the theory one has to extract a physical vector space $\mathcal{H}' \subset \mathcal{H}$ on which the indefinite inner product is non-negative. The physical states are then identified with the rays of the Hilbert space $\mathcal{H}_{\text{phys}} \equiv \mathcal{H}'/\mathcal{H}''$, with scalar product induced by $\langle \cdot, \cdot \rangle$, where $\mathcal{H}'' \equiv \{x, x \in \mathcal{H}', \langle x, x \rangle = 0\}$. In general a completion is necessary since $\mathcal{H}'/\mathcal{H}''$ might only be a pre-Hilbert space with respect to $\langle \cdot, \cdot \rangle$. $\mathcal{H}'/\mathcal{H}''$ has in fact two topologies; as the quotient of two closed subspaces of $\mathcal{H}$ it has a topology $\tau_{(\cdot)}$ induced by $\sqrt{\langle x, x \rangle}$ in $\mathcal{H}$, with respect to which it is complete, and furthermore it has a topology $\tau_{\langle \cdot, \cdot \rangle}$ induced by $\sqrt{\langle x, x \rangle}$, (since $\langle x, x \rangle \geq 0$ on $\mathcal{H}'$), with respect to which it need not to be complete. If the two topologies $\tau_{(\cdot)}$ and $\tau_{\langle \cdot, \cdot \rangle}$ are equivalent on $\mathcal{H}'/\mathcal{H}''$, then $\mathcal{H}'/\mathcal{H}''$ is complete with respect to the scalar product induced by $\langle \cdot, \cdot \rangle$; this means that all the physical states are already present in $\mathcal{H}$ and that the process of taking the quotient $\mathcal{H}'/\mathcal{H}''$ does not require a further completion. The physical content of the theory is therefore readable in $\mathcal{H}$. Under general conditions, the boundedness of η^{-1} turns out to be a necessary condition for the completeness of $\mathcal{H}'/\mathcal{H}''$.

4. INFRARED SINGULARITIES, CLUSTER PROPERTY AND VACUUM STRUCTURE

The role and physical implications of infrared singularities in QFT's has attracted much attention lately especially in connection with gauge quantum field theories, the confinement mechanism, the phenomenon of θ -vacua, etc.

In the standard case, in which the Wightman functions satisfy the positivity condition, the situation is well understood at the general level: the translation invariance of the Wightman functions implies that the space-time translations are described by unitary operators $U(a)$ and therefore the infrared singularities of the theory are rather mild. One can show in fact that for any two local states Ψ, Φ the Fourier transform of $\langle \Psi, U(x)\Phi \rangle$ is a (complex) measure. This property has strong consequences for the phase or vacuum structure of the theory as shown by Araki, Hepp and Ruelle [27]: the uniqueness (or non uniqueness) of the vacuum, which guarantees the irreducibility of the field algebra, is equivalent to the validity (or the non validity) of the cluster property. Thus, given a set of Wightman functions which do not satisfy the cluster property, the structure of pure phases (i. e. theories with unique vacuum) is obtained by decomposing the given Wightman functional into positive invariant functionals which satisfy the cluster property.

The situation appears less clear in the indefinite metric case. It has sometimes been claimed that even in this case the non validity of the cluster property should be interpreted as a sign of the non uniqueness of the

vacuum, but the arguments offered are not convincing. A basic point is that in the indefinite metric case the state content of the theory is not determined by the Wightman functions alone, since in general different Hilbert space structures are available and therefore the states which are obtained by taking the closure of the local states strongly depend on the Hilbert topology one chooses. In the general case the question of existence of more than one translation invariant state cannot be answered without reference to a precise Hilbert space structure. Actually, as it will turn out, the connection between the cluster property and the uniqueness of the vacuum does not hold in general and the problem has to be investigated anew.

To make the discussion more precise it is convenient to classify the infrared singularities in two classes.

DEFINITION. — We will say that a set of Wightman functions have *non confining infrared singularities* if for any two local states Ψ, Φ the Fourier transform of $\langle \Psi, U(x)\Phi \rangle$ is a measure in the neighborhood of the light cone $\{q^2 = 0\}$. A set of Wightman functions is said to exhibit *infrared singularities of the confining type* if there are local states Ψ, Φ such that the Fourier transform of $\langle \Psi, U(x)\Phi \rangle$ is not a measure in the neighborhood of the light cone $\{q^2 = 0\}$.

One can prove that in theories with non confining infrared singularities the cluster property may fail by a constant

$$\lim_{\substack{a = \text{spacelike} \\ \lambda \rightarrow \infty}} [\langle B_1(x_1)B_2(x_1 + \lambda a) \rangle - \langle B_1(x_1) \rangle \langle B_2(x_1) \rangle] = \text{const} \neq 0$$

and that this implies the existence of more than one translation invariant state [27].

When confining singularities are present in the Wightman functions, the positivity condition cannot be satisfied and therefore the metric operator η cannot be trivial [10]. In this case the translation invariance of the Wightman functions only implies that the operators $U(a)$ are η -unitary [22], but they cannot be unitary [10]. Furthermore, they are in general unbounded operators [22], with a common dense domain which contains the local states. These properties make the analysis of the connection between the cluster property and the uniqueness of the vacuum much more delicate. For example if for a local state Φ the Fourier transform of $\langle \Phi, U(x)\Phi \rangle$ has support at the origin, one cannot conclude that the state Φ is translation invariant [28]. This unjustified and in general wrong conclusion underlies most of the discussions of the Schwinger model, where the cluster property fails and nevertheless one can find a Hilbert space structure of Sobolev type in which the vacuum state is unique [29]. Actually, this result can be viewed as a special case of a general theorem.

DEFINITION 3. — A Hilbert space structure $(\eta, \mathcal{H})$, with a possibly degenerate metric operator η , is said to be of *Sobolev type* if $\eta\Psi_0 = \Psi_0$ and there

is a linear correspondence between a dense subspace $D \subset \mathcal{H}$ and the space $\mathcal{S}$ such that

1) for any $\underline{f}_n = (0, \dots, f_n, 0, \dots) \in \underline{\mathcal{S}}$, $p(\underline{f}_n) \equiv (\Psi_{f_n}, \Psi_{f_n})^{\frac{1}{2}}$ is a Sobolev semi-norm [30] on $\underline{\mathcal{S}}_n$ in momentum space

2) for any sequence $\{\Psi_{f^{(k)}}, \underline{f}^{(k)} \in \underline{\mathcal{S}}\}$ converging in $\mathcal{H}$, the sequences $\Psi_{f_n^{(k)}}$, of the n -th components, are also converging [31] in $\mathcal{H}$.

If the metric η is degenerate one can always define a new Hilbert space structure $(\tilde{\eta}, \tilde{\mathcal{H}})$, $\tilde{\eta} = (1 - P_0)\eta$, $\tilde{\mathcal{H}} = (1 - P_0)\mathcal{H}$, where P_0 is the projector onto the subspace $\mathcal{H}_0 = \{x \in \mathcal{H}; (y, \eta x) = 0 \ \forall y \in \mathcal{H}\}$ (see sect. 3).

PROPOSITION. Let $(\eta, \mathcal{H})$ be a Hilbert space structure of Sobolev type, with a possibly degenerate metric operator η , and $\mathcal{H}_0 \subset \mathcal{H}$ be the subspace $\{x \in \mathcal{H}, (y, \eta x) = 0, \forall y \in \mathcal{H}\}$. If either of the following conditions holds

- a) $U(a)$ are unitary operators on $\mathcal{H}$
- b) $U(a)\mathcal{H}_0^\perp \subset \mathcal{H}_0^\perp$

then there is only one translation invariant state in $\tilde{\mathcal{H}} \equiv \mathcal{H} / \mathcal{H}_0$ (uniqueness of the vacuum).

Proof. — We first show that a) $\Rightarrow$ b). In fact, if $U(a)$ are not only η -unitary but also unitary operators one has $[U(a), \eta] = 0$ and therefore $[U(a), 1 - P_0] = 0$, i. e. $U(a)\mathcal{H}_0^\perp \subset \mathcal{H}_0^\perp$.

Now, to prove that b) implies the uniqueness of the vacuum in $\tilde{\mathcal{H}}$, let Ψ be a vector of $\mathcal{H}$, which gives rise to a translation invariant vector $\tilde{\Psi}$ in $\tilde{\mathcal{H}}$.

Then, one must have

$$\langle \chi, U(a)\Psi - \Psi \rangle = 0, \quad \forall \chi \in \mathcal{H},$$

i. e.

$$U(a)\Psi - \Psi \in \mathcal{H}_0. \quad (4.1)$$

Since, by definition $\mathcal{H}_0$ is invariant under translations, the above equation is equivalent to

$$U(a)\Phi - \Phi \in \mathcal{H}_0, \quad \Phi \equiv (1 - P_0)\Psi. \quad (4.2)$$

By condition b), $U(a)\Phi \in \mathcal{H}_0^\perp$ and therefore eq. (3.2) can only be satisfied if

$$U(a)\Phi - \Phi = 0. \quad (4.3)$$

We will show that eq. (4.3) has only the solution $\Phi = (\Phi_0, 0, 0, \dots)$. In fact, by condition 2), eq. (4.3) implies $(U(a)\Phi - \Phi)_n = 0$, or, for the corresponding f_n

$$e^{ia_0 \Sigma q_0^{(1)}} e^{-i\vec{a} \cdot \Sigma \vec{q}^{(1)}} f_n(q_1, \dots, q_n) = f_n(q_1, \dots, q_n). \quad (4.4)$$

Since all $f = (0, 0, \dots, f_n, 0, \dots)$, belong to Sobolev spaces they also belong to $L^2(\mathbb{R}^{4n})$. Now it is well known that eq. (3.4) has no solution in $L^2(\mathbb{R}^{4n})$ ($n \geq 1$), different from zero.

REMARK 1. — Clearly, the same conclusion holds if $(\eta, \mathcal{H})$ is a Hilbert space structure such that in momentum space for any $\Psi_{\underline{f}}$,

$$\underline{f} = (f_0, f_1, \dots, f_m, \dots),$$

each component f_n belongs to a suitable Sobolev space in the variable $Q = (\Sigma q_0^{(i)}, \Sigma \vec{q}^{(i)})$, (restricted on the forward cone).

REMARK 2. — As discussed in sect. 3 a sufficient and, under general assumptions, necessary condition for the existence of a Hilbert space structure, associated to a given set of Wightman functions is the existence of Sobolev seminorms satisfying eq. (3.3) (theorem 3). It turns out that, under rather general conditions, to a given set of Wightman functions one can associate a Hilbert space structure which, if *b*) holds, implies a unique vacuum state even if the cluster property may fail [29]. The point is that a Hilbert space structure of Sobolev type defines a rather strong topology on $\mathbb{W}$ and therefore the closure of the local states is very small. Much interesting information contained in the Wightman functions may then be lost in this procedure. In the positive metric case, a Hilbert space structure of Sobolev type would not in general satisfy *b*) if there are $\delta^4(p)$ singularities in the truncated Wightman functions [32].

The above discussion should have made clear that the uniqueness of the vacuum crucially depends on the Hilbert space structure one associates to the given set of Wightman functions. Since Hilbert space structures of Sobolev type look rather narrow, it is natural to investigate the problem of uniqueness of the vacuum in larger Hilbert space structures, containing the given Hilbert space structure of Sobolev type. This leads to consider maximal Hilbert space structure (see sect. 3).

In this Section we will discuss the relation between the existence of infrared singularities of the confining type in a given set of Wightman functions and the occurrence of more than one translation invariant state in maximal Hilbert space structures associated to the given Wightman functions. Under very general conditions we will in fact show that such Hilbert space structure have always more than one translation invariant state (*Θ-vacua phenomenon*).

We first consider a η -hermitean operator A with the vacuum in its domain and with an $\mathcal{S}'$ two point function

$$A(x - y) = \langle A_x \Psi_0, A_y \Psi_0 \rangle, \quad (4.5)$$

where

$$A_x \equiv U(x)AU(x)^{-1}.$$

The above two point function induces an inner product on $\mathcal{S}(\mathbb{R}^4)$ by

$$\langle f, f \rangle_A \equiv \langle A(f)\Psi_0, A(f)\Psi_0 \rangle = (A(f)\Psi_0, \eta A(f)\Psi_0),$$

where

$$A(f) = \int d^4x A_x f(x).$$

Such an inner product is dominated by the Hilbert scalar product

$$(f, f)_A \equiv (A(f)\Psi_0, A(f)\Psi_0) \quad (4.6)$$

and furthermore

$$\langle f, f \rangle = (f, \eta_A f), \quad (4.7)$$

where η_A is the restriction of the quadratic form η to

$$\mathcal{H}_A \equiv \overline{\{A(f)\Psi_0, f \in \mathcal{S}(\mathbb{R}^4)\}}. \quad (4.8)$$

The following analysis is based on the following assumptions

a) There exists a η -hermitean operator A such that η_A is invertible and it has a bounded inverse. Then, by the results of sect. 3 one can always reduce oneself to the case in which η_A satisfies $\eta_A^2 = 1$, and we will for simplicity consider this case in the following.

b) As a vector space $\mathcal{S}(\mathbb{R}^4)$ admits the following decomposition

$$\mathcal{S} = K_0 + V_c + V_d, \quad (4.9)$$

where K_0 is a vector space invariant under translations such that

$$\langle f, f \rangle_A > 0 \quad \forall f \in K_0,$$

V_c is a vector space whose elements define continuous functionals on K_0 in the following sense: if $f \in V_c$, there exists a constant C_f such that

$$|\langle f, g \rangle| \leq C_f \langle g, g \rangle^{\frac{1}{2}}, \quad \forall g \in K_0, \quad (4.10)$$

and V_d is a non trivial finite dimensional space

$$0 < \dim V_d < \infty \quad (4.11)$$

whose elements define discontinuous functionals on K_0 .

c) The spectrum of the restriction of the form η to $\bar{K}_0$, $\sigma(\eta|_{\bar{K}_0})$, (the closure being taken with respect to the Hilbert scalar product) has a gap over zero.

REMARK 3. — Condition a) is a maximality condition for the space $\mathcal{H}_A$. It reflects the maximality of whole Hilbert space $\mathcal{H}$ plus a regularity condition for the subspace $\mathcal{H}_A$. For the geometric meaning of such structure see e. g. Bognar's book or Krein's paper [24] [26].

REMARK 4. — The decomposition (4.9) or the identification of K_0 essentially corresponds to the selection of those test functions which give rise to states, (associated to the operator A), which do not exhibit bad infrared singularities (« gauge invariant states »). For example K_0 may be the vector space of test functions f whose Fourier transform vanishes

at the origin together with its derivatives up to order N . In this case the condition $\dim V_d < \infty$ is trivially satisfied since

$$\mathcal{S} = K_0 + V, \quad K_0 = \mathcal{S}_0^N \equiv \{f, \tilde{f}^{(k)}(0) = 0, k = 1, \dots, N\},$$

with V finite dimensional.

As it is clearly suggested by this example, K_0 plays in general the role of the physical vector space associated to the operator A and in this perspective condition $c)$ has a simple physical meaning as shown by the following Proposition 4.1 (See also Remark 7).

REMARK 5. — The physical meaning of the condition $\dim V_d < \infty$ is essentially that the theory requires only a finite number of infrared regularizations and therefore after a finite number of subtractions the states become free of infrared singularities. A more concrete and rather common situation is that discussed in Remark 4.

PROPOSITION 4.1. — The condition that $\sigma(\eta|_{\bar{K}_0})$ has a gap over zero is equivalent to the condition that quotient space

$$\mathcal{H}_1 \equiv \bar{K}_0/V_0, \quad (4.12)$$

where $V_0 \equiv \{f \in \bar{K}_0, \langle f, f \rangle_A = 0\}$, is complete with respect to the scalar product induced by $\langle \cdot, \cdot \rangle_A$.

Proof. — The proof follows easily from the inequality

$$\langle f, f \rangle_A \leq (f, f)_A \leq (\inf \sigma(\eta|_{\bar{K}_0 \ominus V_0}))^{-1} \langle f, f \rangle_A$$

which states the equivalence of the two Hilbert norms.

REMARK 6. — Assumptions $b)$ and $c)$ can be weakened but we prefer to make here the discussion in this less general case in order not to obscure the main ideas. For the same reasons we will also assume that $U(a)$ is a bounded operator on $\bar{\mathcal{S}}$, the closure being taken with respect to the Hilbert scalar product $(\cdot, \cdot)$. Actually this property holds under rather general conditions (see Remark 7).

In the following we will often omit the subscript A , whenever it does not rise to confusion and we will usually identify the subspaces of $\bar{\mathcal{S}}(\mathbb{R}^4)$ and the corresponding subspaces of $\mathcal{H}_A$.

THEOREM 6. — Under the assumptions $a)$, $b)$ and $c)$, $\bar{K}_0$ contains at least one translation invariant state different from Ψ_0 .

LEMMA 1. — The space

$$V_0 = \{f \in \bar{K}_0, \langle f, f \rangle = 0\} \quad (4.13)$$

is invariant under translations

Proof. — If $f \in V_0$, $\langle U(a)f, g \rangle = \langle f, U(-a)g \rangle = 0 \quad \forall g \in \bar{K}_0$, which implies $U(a)f \in V_0$, since $U(a)\bar{K}_0 \subset \bar{K}_0$.

LEMMA 2. — Let

$$H \equiv \bar{\mathcal{P}} \ominus \bar{K}_0 \quad (4.14)$$

and

$$H = H_+ + H_- + H_0 \quad (4.15)$$

be the decomposition of H according to the positive, negative and null spectrum of the restriction of η to H , then

$$\eta H_0 = V_0. \quad (4.16)$$

Proof. — We first show that $\eta H_0 \subset V_0$. In fact, for any $f \in H_0$ ηf must belong to $\bar{K}_0$ since the matrix elements of η between H_0 and $H_{\pm}$ or H_0 vanish. On the other hand

$$\langle \eta f, \eta f \rangle = \langle f, \eta f \rangle = 0$$

so that $\eta f \in V_0$.

Conversely, if $f \in V_0$, then $\eta f \in H$, because

$$\langle \eta f, g \rangle = \langle f, g \rangle = 0, \quad \forall g \in K_0,$$

as a consequence of $\eta \geq 0$ on $\bar{K}_0$ and $V_0 \subset \bar{K}_0$. Then, if ηf is decomposed according to H_+ , H_- , H_0

$$\eta f = g_+ + g_- + g_0,$$

one has $\eta(g_+ + g_-) \in V_0$, since $\eta g_0 \in V_0$ by the first part of the proof. Hence $(\eta g_+ + \eta g_-, H_{\pm}) = 0$, which requires $g_{\pm} = 0$.

We will now show that the elements of H_0 define discontinuous functionals on K_0 , through the product $\langle g, \cdot \rangle$, $g \in H_0$. To this purpose we define

$$L_r = \{ f \in \bar{\mathcal{P}}; \langle f, \cdot \rangle \text{ is a continuous functional on } K_0 \text{ in the sense of eq. (4.10)} \}. \quad (4.17)$$

LEMMA 3. — As a consequence of assumption c), it follows that

$$L_r = \bar{\mathcal{P}} \ominus H_0 \quad (4.18)$$

Proof. — We start by showing that $f \in L_r$ implies $f \in H_0^{\perp}$. In fact if $f \in L_r$ then there a constant C_f such that

$$\langle f, g \rangle \leq C_f \langle g, g \rangle^{\frac{1}{2}} \quad \forall g \in \bar{K}_0,$$

namely

$$\sup_{g \in K_0} \frac{\langle f, g \rangle}{\langle g, g \rangle^{\frac{1}{2}}} < \infty,$$

which implies

$$\langle \eta f, g \rangle = \langle f, g \rangle = 0, \quad \forall g \in V_0.$$

Then $\eta f \in V_0^{\perp}$ and consequently

$$f \in \eta V_0^{\perp} = H_0^{\perp}.$$

Conversely, if $f \in H_0^\perp$, then $f \in L_r$. In fact if $f \in H_0^\perp$, then it can be decomposed in the form

$$f = f_+ + f_- + f_0, \quad f_\pm \in H_\pm, \quad f_0 \in \bar{K}_0.$$

Since f_0 obviously defines a continuous functional on $\bar{K}_0$, it suffices to prove that

$$\sup_{g \in \bar{K}_0} \frac{\langle f_+ + f_-, g \rangle}{\langle g, g \rangle^{\frac{1}{2}}} < \infty.$$

To this purpose we note that $\eta(f_+ + f_-)$ can be written as

$$\eta f_+ + \eta f_- = h_+ + h_- + h,$$

where $h_\pm \in H_\pm$ and $h \in \bar{K}_0 \ominus V_0$. No component in H_0 and V_0 appears in the decomposition (4.16) because $(H_\pm, \eta H_0) = 0$ and

$$(\eta H_\pm, V_0) = (H_\pm, \eta V_0) = (H_\pm, H_0) = 0.$$

We then have

$$\sup_{g \in \bar{K}_0} \frac{(h_+ + h_- + h, g)}{\langle g, g \rangle^{\frac{1}{2}}} = \sup_{g \in \bar{K}_0} \frac{(h, g)}{\langle g, g \rangle^{\frac{1}{2}}} = \sup_{g \in \bar{K}_0 \ominus V_0} \frac{(h, g)}{\langle g, g \rangle^{\frac{1}{2}}} \leq C_h \sup_{g \in \bar{K}_0 \ominus V_0} \frac{(g, g)^{\frac{1}{2}}}{\langle g, g \rangle^{\frac{1}{2}}}$$

Furthermore, by assumption the spectrum $\sigma(\eta|_{\bar{K}_0})$ has a gap over zero, so that

$$= \sup_{g \in \bar{K}_0 \ominus V_0} \frac{(g, g)^{\frac{1}{2}}}{(g, \eta g)^{\frac{1}{2}}} = (\inf \sigma(\eta|_{\bar{K}_0 \ominus V_0}))^{-1} < \infty.$$

LEMMA 4. — $\dim V_d = \dim H_0 = \dim V_0$.

Proof. — Let $f_1, \dots, f_n$ be n linearly independent vectors of V_d , then $Pf_1, \dots, Pf_n$ (P being the projector vector of H_0 , are n linearly independent vectors of H_0 . In fact, if $\sum \lambda_i Pf_i = 0$ then $P \sum \lambda_i f_i = 0$, which implies $\sum \lambda_i f_i \in H_0^\perp$ and by Lemma 3, $\sum \lambda_i f_i$ defines a continuous functional on K_0 . This is possible only if $\lambda_i = 0$.

Conversely if $\dim H_0 \geq n$, since $P\mathcal{S}$ is dense in H_0 one can find n linearly independent vectors $h_1, \dots, h_n$ belonging to $P\mathcal{S}$, $h_i = Pg_i$, $g_i \in \mathcal{S}$. We will show that any non trivial combination $\sum \lambda_i g_i$ defines a discontinuous functional on K_0 and therefore $\dim V_d \geq n$. In fact, if $g \equiv \sum \lambda_i g_i$ defines a continuous functional on K_0 , then by Lemma 3, $Pg = 0 = \sum \lambda_i Pg_i$ in contrast with the linear independence of the Pg_i 's.

Proof of theorem 6. — By the above Lemmas, $\bar{K}_0$ contains a non trivial finite dimensional subspace V_0 , which is invariant under translations. Since the translations $U(a)$ are bounded operators on $\bar{K}_0$ and are therefore well defined on V_0 , to the commutative algebra generated by the $U(a)$'s we can apply the analysis of Dunford and Schwartz [33] and therefore conclude that, by standard results, V_0 contains one translation invariant vector.

For the applications it may be useful to have simple criteria which guarantee the validity of assumption c) given a) :

PROPOSITION 4.2. — Any of the following properties guarantees that condition c) holds:

$$i) \quad \dim H_+ + \dim H_- < \infty \quad (4.19)$$

$$ii) \quad \mathcal{S} = K_0 + V, \quad \text{with } \dim V < \infty \quad (4.20)$$

iii) the operator $P_0 P_- P_0$, where P_0 is the projector on $\bar{K}_0 \ominus V_0$ and P_- is the projector on the negative spectrum of η , is a compact operator.

Proof. — Clearly $ii) \Rightarrow i)$, since if $\dim V < \infty$ then

$$\dim H_{\pm} < \dim H = \dim (\bar{\mathcal{S}} \ominus \bar{K}_0) \leq \dim V < \infty.$$

To prove that $i)$ implies $c)$ we show that $\sigma(\eta|_{\bar{K}_0 \ominus V_0})$ is a finite point spectrum. We have

$$\eta|_{\bar{K}_0 \ominus V_0} = P_0 - 2P_0 P_- P_0$$

and therefore it suffices to prove that $P_0 P_- P_0$ is a symmetric operator with a finite dimensional range.

We first notice that as sum of vector space

$$G \equiv H_+ + H_- + \eta H_+ + \eta H_-$$

is mapped into itself by $\eta : \eta G \subset G$. Then, putting $G^{\perp} \equiv (\bar{\mathcal{S}} \ominus V_0 \ominus H_0) \ominus G$ we obtain, using Lemma 2,

$$\eta G^{\perp} \subset G^{\perp}$$

and since η is positive on $G^{\perp}$ we have

$$P_- P_{G^{\perp}} = 0$$

where $P_{G^{\perp}}$ is the projector on $G^{\perp}$. Hence

$$P_0 P_- P_0 = P_0 P_- (1 - P_{G^{\perp}}) P_0$$

and since by assumption G is finite dimensional so is $(1 - P_{G^{\perp}}) P_0$ and $P_0 P_- P_0$.

Finally, if $P_0 P_- P_0$ is a compact operator $P_0 - 2P_0 P_- P_0$ has a point spectrum which can at most have 1 as accumulation point and therefore $\sigma(\eta|_{\bar{K}_0 \ominus V_0})$ is a finite point spectrum.

REMARK 7. — As already pointed out in Remark 4, 5 a rather common situation is when the construction of the « gauge invariant states » requires only a finite number of infrared subtractions in the sense that

$$\mathcal{S} = K_0 + V, \quad \dim V < \infty. \quad (4.21)$$

In this case it is easy to see that the assumption of theorem 6 are satisfied. In particular we note that the boundedness of the operators $U(a)$ on $\bar{\mathcal{S}}$ is automatically guaranteed as it is whenever $\dim P_- < \infty$. It is worthwhile to stress that condition (4.21) is directly given in terms of the space of test

functions and it does not involve any assumption on the Hilbert space closure $\overline{\mathcal{S}}$. Actually, in this case there is only one maximal Hilbert space, uniquely fixed by the indefinite inner product on $\mathcal{S}$.

REMARK 8. — We recall that by Naimark's theorem [34], if $\mathcal{K}$ is a Pontrjagin space (that is $0 < \dim P_- < \infty$), it always contains a finite dimensional (non trivial) space invariant under the abelian group of space time translations. (Clearly if $\mathcal{S} = K_0 + V$, the condition $0 < \dim P_- < \infty$ is satisfied). As pointed out in Remark 7, one of the main features of Theorem 6, even in the simple case $\dim P_- < \infty$, is to relate the translation invariant space to the structure of the space of test functions $\mathcal{S}$.

As already mentioned the existence of translation invariant states in $\overline{\mathcal{S}}$ is not uniquely related to the assumptions of Theorem 6. We defer to a subsequent paper a more general analysis of the mathematical structures leading to such phenomenon. Here, we mention at least another case of such structures, which appears suitable for discussing the implications of the existence of infrared singularities of the type $\delta'(p^2)$ in four space time dimensions (corresponding to the confining potential $V(r) \underset{r \rightarrow \infty}{\sim} cr$).

THEOREM 7. — Let

$$\mathcal{S} = K_0 + V \quad (4.22)$$

where K_0 is a translation invariant space equipped with a positive inner product $[\cdot, \cdot]$ such that

$$i) \quad \forall x, y \in K_0 \quad \langle x, y \rangle = [x, \eta_0 y] \quad (4.23)$$

with η_0 and η_0^{-1} bounded (maximal Hilbert space structure on K_0),

ii) V is a finite dimensional vector space, all the elements of which define discontinuous functionals on $K_0 \langle x, \cdot \rangle, x \in V$, with respect to the scalar product $[\cdot, \cdot]$.

Then:

A) There exists a Hilbert space structure defined by a positive inner product $(\cdot, \cdot)$ on $\mathcal{S}$ such that

$$[x, x] \leq C(x, x) \quad \forall x \in K_0 \quad (4.24)$$

and $\overline{\mathcal{S}}$ is a Krein space ($\eta^2 = 1$).

B) If $(\cdot, \cdot)_1, (\cdot, \cdot)_2$ are two positive inner products on $\mathcal{S}$, both leading to Krein spaces ($\eta_1^2 = 1, \eta_2^2 = 1$) and satisfying eq. (4.24), then they define the same Hilbert topology on $\mathcal{S}$ (uniqueness of the closure $\overline{\mathcal{S}}$).

C) $\overline{\mathcal{S}}$ contains a subspace V_0 invariant under translations with

$$\dim V_0 = \dim V$$

and consequently $\overline{\mathcal{S}}$ contains translation invariant states.

The proof of B) and C) follows a pattern similar to that of Theorem 6, apart from a heavy use of condition (4.24). The construction of the maximal

Hilbert space structure of point A) is an abstraction and a generalization of the construction discussed in Example 2 below. (The proof of Theorem 7 will be presented elsewhere).

EXAMPLE 1 (*Massless Schwinger model*)

This model has been extensively discussed, especially as a prototypic quantum field theory exhibiting the confinement, the existence of θ -vacua, etc. Most of its crucial properties are essentially related to the properties of the massless scalar field, in two dimensions, in terms of which the solution is constructed. Therefore, to simplify the discussion we will refer to this simpler case (for a more detailed discussion we refer to Ref. 25). Even for this simpler case, a careful discussion of the Hilbert space structures which can be associated to the Wightman functions of the local massless scalar field in two dimensions does not seem to exist in the literature, at least to our knowledge. Therefore, as repeatedly emphasized in the previous Sections, questions like the uniqueness of the translation invariant state, confinement mechanism, θ -vacua, etc., are not well posed.

One can realize the theory in a Hilbert space of Sobolev type (see Sect. 4, Remark 1) and in this case the vacuum is unique. A richer structure emerges if one considers a maximal Hilbert space structure. One can actually prove that in this case such maximal Hilbert space structure is essentially unique and it can be explicitly constructed. The two point function

$$W(x) = -\frac{a}{4\pi} \log [-x^2 + i\epsilon x_0], \quad x \in \mathbb{R}^2,$$

defines an indefinite inner product in $\mathcal{S}(\mathbb{R}^2)$

$$\langle f, g \rangle = \left(\frac{1}{p_+} \right) [\bar{f}(p_+)g(p_+)] + \left(\frac{1}{p_-} \right)_+ [\bar{f}(p_-)g(p_-)] \quad (4.25)$$

where

$$f(p_{\pm}) \equiv f(k_0, k_1) |_{k_0 \pm k_1 = 0, k_0 \pm k_1 = p_{\pm}}$$

and

$$\left(\frac{1}{x} \right)_+ [f(x)] \equiv - \int_0^{\infty} \log x \frac{df}{dx} dx.$$

The above indefinite inner product can be dominated by the Sobolev type product

$$(f, g) = (f, g)_+ + (f, g)_-, \quad (4.26)$$

with

$$(f, g)_{\pm} = \int_0^{\infty} dp_{\pm} (\log |p_{\pm}| + 1) [\bar{f}(p_{\pm})g(p_{\pm}) + \bar{f}'(p_{\pm})g'(p_{\pm})].$$

The corresponding Hilbert space structure has one translation invariant state (*uniqueness of the vacuum*).

A maximal Hilbert space structure is constructed in the following way

(see Theorem 7). We chose in momentum space a test function $\chi \in \mathcal{S}(\mathbb{R}^4)$ with $\chi(0) = 1$ and to simplify the discussion we further require

$$\langle \chi, \chi \rangle = 0 \quad (4.27)$$

(such a χ can always be found). The test function space $\mathcal{S}$ in momentum space is decomposed according to

$$\mathcal{S} = \mathbf{K}_0 + \mathbf{V},$$

with $\mathbf{K}_0 = \{f \in \mathcal{S}, f(0) = 0\}$, and one introduces the positive scalar product

$$[f, g] = (f_1, g_1) + \langle f_1, \chi \rangle \langle \chi, g_1 \rangle + \bar{f}(0)g(0), \quad (4.28)$$

with $f(p) = f(0)\chi + f_1$ and

$$(f_1, g_1) \equiv \int \frac{1}{p} \bar{f}_1(p)g_1(p)dp. \quad (4.29)$$

One can show that $[\cdot, \cdot]$ defines a maximal Hilbert space structure ($\eta^2 = 1$), that such a maximal Hilbert space $\mathcal{H}$ is unique (see Theorem 7) and that the element $v = \eta\chi \in \bar{\mathbf{K}}_0$ is translation invariant. Thus the one particle space contains a translation invariant state

$$\Psi_v = \varphi(v)\Psi_0 = \eta\varphi(\chi)\Psi_0. \quad (4.30)$$

In such a framework, θ like vacua correspond to the combinations

$$\Psi_\theta = \int e^{i\theta\lambda} \Omega_\lambda d\lambda, \quad (4.31)$$

where Ω_λ are the coherent superpositions

$$\Omega_\lambda = e^{i\varphi(iv)\lambda} \Psi_0. \quad (4.32)$$

For a rigorous discussion and interpretation of eq. (4.31) we refer to Ref. 25.

The « operator » $e^{i\varphi(iv)\lambda} = e^{iQ\lambda}$ generates the transformation

$$\varphi \rightarrow \varphi + 2\lambda. \quad (4.33)$$

Such a transformation is not implementable in the Sobolev type Hilbert structure defined by eq. (4.26), whereas it is implementable in the maximal Hilbert space $\mathcal{H}$.

Clearly the uniqueness of the vacuum is lost in $\mathcal{H}$, however there does not exist a strictly positive subspaces made of vectors invariant under translations, with dimension greater one (*essential uniqueness of the vacuum*, see Sect. 5).

EXAMPLE 2 (*Dipole field model*)

The model is defined by a scalar field satisfying

$$\square^2 \varphi = 0,$$

more precisely by the following two point function

$$W(x) = -\frac{a}{(4\pi)^2} \ln(-x^2 + i\epsilon x_0). \quad (4.34)$$

Such a two point function defines an indefinite inner product in $\mathcal{S}(\mathbb{R}^4)$ in momentum space [36]

$$\langle f, g \rangle = \frac{a}{2} \int \frac{d^3k}{(2\omega)^3} \{ \bar{F}_1(k)G_1(k) - \bar{F}_2(k)G_2(k) + 2\bar{f}(0)\bar{\chi}(k)G_1(k) + 2g(0)\bar{F}_1(k)\chi(k) \} \quad (4.35)$$

where

$$F_1 = \left[2f(k) - \left(k^0 \frac{\partial}{\partial k_0} k^i \frac{\partial}{\partial k^i} \right) f(k) - 2f(0)\chi(k) \right]_{k_0=\omega}$$

$$F_2 = \left[\left(k^0 \frac{\partial}{\partial k_0} - k^i \frac{\partial}{\partial k^i} \right) f(k) \right]_{k_0=\omega},$$

and $\chi(k) \in \mathcal{S}(\mathbb{R}^4)$ satisfies $\chi(0) = 1$ and is also chosen in such a way that $[(k^0 \partial / \partial k_0 - k^i \partial / \partial k^i) \chi]_{k_0=\omega} = 0$.

The unique maximal Hilbert space structure is constructed in the following way (Theorem 7)

$$[f, g] = \frac{|a|}{2} \int \frac{d^3k}{(2\omega)^3} (\bar{F}_1 G_1 + \bar{F}_2 G_2) + \langle f, \chi \rangle \langle \chi, g \rangle + \bar{f}(0)g(0).$$

One can easily check that the conditions of Theorem 7 are satisfied with $K_0 = \{ f \in \mathcal{S}, f(0) = 0 \}$ and $V = \{ c\chi \}$.

Similarly to the case of the massless scalar field in two dimensions one can construct a translation invariant state in the one particle space, θ like vacua, etc. [39].

5. VACUUM STRUCTURE AND PURE PHASES

In the case of positive metric QFT the basic question « when is a relativistic quantum theory a field theory? » has been discussed by Wightman and Garding [37] and answered by the condition of cyclicity of the vacuum. The main support to this requirement is that it implies the irreducibility of the field algebra if the vacuum is unique. Thus a theory with a cyclic unique vacuum corresponds to what is called a « pure phase » in Statistical Mechanics.

It is worthwhile to recall that the proof of the irreducibility of the field algebra in a standard QFT is crucially based on the property of locality and positivity. Since gauge QFT's cannot satisfy all the standard (positive metric) Wightman axioms (more specifically they cannot satisfy locality and positivity at the same time) [8], the question « when is a relativistic quantum theory a field theory? » has to be reconsidered in a wider frame-

work. We will discuss this problem in a local and covariant formulation, i. e. in the case in which the standard Wightman axioms are satisfied, except positivity as discussed in Sect. 2, 3.

The interest in this problem is motivated by recent investigations on non-abelian gauge QFT's and by the suggestion that a characteristic feature of such theories is to describe a « mixed phase » i. e. the field algebra is reducible. It is then natural to ask the following question: under which conditions the uniqueness of the vacuum implies the irreducibility of the field algebra?

We will first discuss the definition of the vacuum state. In indefinite metric QFT's, the space-time translations $U(a)$ are only required to be η -unitary operators and in general they are not unitary operators. Actually, they cannot be unitary operators in GQFT's like the abelian Higgs-Kibble or in general in theories with infrared singularities of the confining type [10]. In those cases the standard (positive metric) characterization of the vacuum as the (only) normalizable eigenvector of the space-time translation [38] requires therefore some comment or generalization, since very little is known about the spectral theory of η -unitary operators.

Clearly an essential requisite for the vacuum state is to be translation invariant; this means that the corresponding vector Ψ_0 should be an eigenvector of the space-time translations

$$U(a)\Psi_0 = \lambda_a\Psi_0, \quad \lambda_a \in \mathbb{C}. \quad (5.1)$$

Now, in principle one could not exclude the existence of vectors Ψ , satisfying eq. (5.1), which correspond to some sort of long range « collective » excitations or to background fields, with some property of homogeneity in space or in space time. The occurrence of such phenomena has in fact been suggested in $(\text{QED})_2$ in connection with the existence of a *constant* background electric field. One may then ask whether such eigenstates Ψ may exist, corresponding to eigenvalues $\lambda_a \neq 1$; the answer to it is important in order to resolve a possible ambiguity in a (precise) definition of vacuum state in indefinite metric QFT's.

THEOREM 8. — Let $U(a, \Lambda)$ be a weakly continuous representation of the Poincaré group in a Hilbert space $\mathcal{H}$ with indefinite metric $\langle \cdot, \cdot \rangle = (\cdot, \eta \cdot)$, $(\cdot, \cdot)$ denoting the ordinary (positive) scalar product in $\mathcal{H}$, such that η is a bounded self-adjoint operator and $U(a, \Lambda)$ are η -unitary operators with a common dense domain D_U . Then, if $\mathcal{H}$ is a separable Hilbert space, the only eigenvectors Ψ of $U(a)$, with the properties

$$i) \langle \Psi, \Psi \rangle \neq 0,$$

$$ii) \Psi \in D_{U(0, \Lambda)}, \forall \Lambda,$$

have eigenvalue $\lambda_a = 1$.

Proof. — Let Ψ satisfies *i*), *ii*) and

$$U(a)\Psi = \lambda_a \Psi, \quad \lambda_a \in \mathbb{C}.$$

The η -unitarity of $U(a)$ implies $\bar{\lambda}_a \lambda_a = 1$ or

$$|\lambda_a| = 1,$$

so that $a \rightarrow \lambda_a = \langle \Psi, U(a)\Psi \rangle / \langle \Psi, \Psi \rangle$ is a continuous function of modulus equal to 1. Moreover, by the group law

$$\begin{aligned} \lambda_{a+b} &= \langle \Psi, U(a+b)\Psi \rangle = \langle \Psi, U(a)U(b)\Psi \rangle \\ &= \bar{\lambda}_{-a} \lambda_b. \end{aligned}$$

Hence

$$\lambda_a = e^{ipa}, \quad (5.2)$$

with p a four vector which may be chosen to label the corresponding eigenvectors

$$U(a)\Psi_p = e^{ipa}\Psi_p. \quad (5.3)$$

On the other hand, by the group law:

$$U(a)U(0, \Lambda)\Psi_p = U(0, \Lambda)U(\Lambda^{-1}a)\Psi_p = e^{i(\Lambda p)a}U(0, \Lambda)\Psi_p \quad (5.4)$$

i. e. $U(0, \Lambda)\Psi_p \equiv \Psi_{\Lambda p}$ is an eigenvector of $U(a)$ with eigenvalue $e^{i(\Lambda p)a}$.

We will now show that a separable Hilbert space cannot contain the vectors $\Psi_{\Lambda p}$, $p \neq 0$. We remark that by the η -unitarity of $U(0, \Lambda)$

$$\langle \Psi_{\Lambda p}, \Psi_{\Lambda p} \rangle = \langle \Psi_p, \Psi_p \rangle \neq 0. \quad (5.5)$$

Now, the subspace $\mathcal{L}$ generated by the vectors $\Psi_{\Lambda p}$, $\Lambda \in L^+$, closed with norm induced by $(\cdot, \cdot)$, has the property that the restriction of the bilinear form $\langle \cdot, \cdot \rangle$ to $\mathcal{L}$, defines a (bounded) self-adjoint non negative operator $\eta_{\mathcal{L}}$, as a consequence of eq. (5.5).

Therefore, we may define the vectors

$$\Phi_{\Lambda p} = \sqrt{\eta_{\mathcal{L}}} \Psi_{\Lambda p} / \langle \Psi_{\Lambda p}, \Psi_{\Lambda p} \rangle \in \mathcal{K}.$$

For them, we have $(\Lambda \in L^+)$

$$\begin{aligned} (\Phi_{\Lambda p}, \Phi_{\Lambda' p}) &= 0 && \text{if } \Lambda \neq \Lambda', \\ &= 1 && \text{if } \Lambda = \Lambda', \end{aligned}$$

and this contradicts the separability of $\mathcal{K}$, unless $p = 0$.

We may now define.

DEFINITION (Uniqueness of the vacuum). — In a (relativistic) quantum theory defined on a separable Hilbert space $\mathcal{K}$, a normalizable eigenvector Ψ of $U(a)$ with $\langle \Psi, \Psi \rangle > 0$, is called a proper *vacuum* vector. The vacuum is *unique* if there is only one normalizable eigenvector of $U(a)$. The vacuum is said to be *essentially unique* if one cannot find strictly positive subspaces made of vectors invariant under $U(a)$, with dimension greater than one.

Remark. — In the above definition of vacuum vector only the translation operators $U(a)$ occur; if, furthermore, the Wightman functions are invariant under the Poincaré group, then there exist operators $U(a, \Lambda)$ which satisfy the assumptions of Theorem 8 and therefore the vacuum state is an eigenvector of $U(a)$ corresponding to the eigenvalue one.

PROPOSITION 5.1. — The essential uniqueness of the vacuum is equivalent to the non existence of two translation invariant states Ψ_{01}, Ψ_{02} which are positive and η -orthogonal

$$\langle \Psi_{01}, \Psi_{01} \rangle > 0, \langle \Psi_{02}, \Psi_{02} \rangle > 0 \quad (5.6)$$

$$\langle \Psi_{01}, \Psi_{02} \rangle = 0 \quad (5.7)$$

Proof. — Clearly if the vacuum is essentially unique one cannot find two invariant states satisfying eqs (5.6), (5.7) because, otherwise, they would generate a space of invariant states of dimension greater than one.

Conversely, if the vacuum is not essentially unique there exist at least two positive invariant states satisfying eq. (5.6). Then the metric operator, restricted to the space generated by such vectors, is a positive hermitean matrix. Its diagonalization leads to two eigenvectors corresponding to positive eigenvalues, which would then satisfy eq. (5.6) (5.7).

The following Theorem 9 provides a useful criterium for deciding when a vector Φ is an eigenvector of $U(a)$.

THEOREM 9. — We consider an indefinite metric quantum field theory with a (proper) vacuum vector Ψ_0 which is cyclic with respect to the polynomial algebra $\mathcal{F}$ of the smeared fields. If Φ belongs to the domain of $U(a)$ for all a 's and for any $\Psi \in \mathcal{F}\Psi_0$ the Fourier transform of $\langle \Psi, U(x)\Phi \rangle$ has support contained in $\{p = 0\}$, then if the vacuum Ψ_0 is the only translation invariant vector (uniqueness of the vacuum)

$$U(a)\Phi = \Phi, \quad c\Psi_0, \quad c \in \mathbb{C} \quad (5.8)$$

Proof. — $U(x)\Phi$ defines a vector valued (tempered) distribution and we may consider its Fourier transform $\tilde{U}(\tilde{f})\Phi$. If its support is not contained in $\{p = 0\}$, then there exists a function $\tilde{f}$ with support not containing the origin, such that

$$\tilde{U}(\tilde{f})\Phi \neq 0 \quad (5.9)$$

Since, by cyclicity of Ψ_0 , $D_0 \equiv \mathcal{F}\Psi_0$ is dense in $\mathcal{H}$ and so is ηD_0 , eq. (5.9) implies that there must exist a vector $\Psi_1 \in D_0$ such that

$$\langle \Psi_1, \tilde{U}(\tilde{f})\Phi \rangle \neq 0$$

and this contradicts the assumption that for any $\Psi \in D_0$, $\langle \Psi, \tilde{U}(\tilde{f})\Phi \rangle$ has

support contained in $\{p = 0\}$. Therefore $U(x)\Phi$ is a polynomial in x , whose coefficients are fixed vectors $\Psi_{(j)}$ (independent of x):

$$U(x)\Phi = \sum_{n_j}^N x_0^{n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} \Psi_{(n)} \quad (5.10)$$

The linear space X generated by taking linear combinations of the vectors $U(x)\Phi$, as x varies over $\mathbb{R}^4$, is invariant under $U(a)$ and it is finite dimensional since it is contained in the linear span of the $\Psi_{(n)}$, whose number is finite. Furthermore, the norm $\|U(x)\Psi\|^2$, $\Psi \in X$ is a polynomial in x , whose degree is bounded by N^2 , N being the order of the vector valued distribution $U(x)\Phi$. Then, for fixed x , the spectral radius of $U(x)$, restricted to the space X , is one and therefore the spectrum of $U(x)$ restricted to X lies on a circle of radius one, it is a point spectrum since X is finite dimensional and by the uniqueness of the vacuum it consists of only one point λ of multiplicity one:

$$U(a)\Psi_0 = \lambda\Psi_0, \quad |\lambda| = 1.$$

Therefore $U(a) - \lambda$ is a nilpotent of order n [33], $n \equiv \dim X$, i. e.

$$(U(a) - \lambda)^n = 0$$

and one has

$$\text{Range } (U(a) - \lambda)^{n-1} = c\Psi_0, \quad c \in \mathbb{C} \quad (5.11)$$

If $n > 1$, eq. (5.11) implies that $\langle \Psi_0, \Psi_0 \rangle = 0$. In fact, $\forall \Psi \in \mathcal{H}$ one has

$$0 = \langle (U(a)^{-1} - \lambda^{-1})\Psi_0, \Psi \rangle = \langle \Psi_0, (U(a) - \lambda)\Psi \rangle$$

and since Ψ_0 is of the form $(U(a) - \lambda)\Psi$, it follows

$$\langle \Psi_0, \Psi_0 \rangle = 0.$$

This is impossible since Ψ_0 is a proper vacuum vector and therefore it must be $n = 1$, which implies $N = 0$ and

$$U(x)\Phi = \text{const } \Psi_0,$$

or

$$\Phi = \text{const } \Psi_0.$$

We will now discuss the irreducibility of the field operators in indefinite metric QFT's, under the assumption that there is a (proper) vacuum vector which is a cyclic vector with respect to the polynomial algebra $\mathcal{F}$ of the smeared fields, a property which is essentially guaranteed by the reconstruction theorem if the theory is defined by a set of Wightman functions. To this purpose we distinguish the following notions of irreducibility.

DEFINITION (Irreducibility). — We will say that the field algebra $\mathcal{F}$ is *reducible* if there exists a bounded operator C which commutes with $\mathcal{F}$ and it is not a multiple of the identity. The field algebra $\mathcal{F}$ is said to be *irreducible* if any bounded operator C which commutes with $\mathcal{F}$ is a multiple

of the identity. The field algebra $\mathcal{F}$ is said to be *essentially irreducible* if the Wightman functional over the Borchers algebra $\mathcal{S}$ cannot be decomposed into translation invariant functionals generated by positive vectors.

THEOREM 10. — Let E_0 be the projection operator onto one proper vacuum vector Ψ_0 , which is supposed cyclic with respect to the field algebra $\mathcal{F}$. Then for any open set $\mathcal{O}$, the set of operators $\{E_0, \mathcal{F}(\mathcal{O})\}$ where $\mathcal{F}(\mathcal{O}) \equiv$ polynomial algebra of fields smeared with functions with support in $\mathcal{O}$, is irreducible i. e. any bounded operator C such that $\forall \Phi, \Psi \in D_0$,

$$\langle \Phi, CA\Psi \rangle = \langle A^\star \Phi, C\Psi \rangle, \quad \forall A \in \mathcal{F}(\mathcal{O}), \\ CE_0 = E_0 C,$$

is a multiple of the identity.

Proof. — As in the standard (positive metric case) $\forall \Phi \in D_0, \Psi = A\Psi_0, A \in \mathcal{F}(\mathcal{O})$

$$\begin{aligned} \langle \Phi, C\Psi \rangle &\equiv \langle \Phi, CA\Psi_0 \rangle = \langle A^\star \Phi, C\Psi_0 \rangle = \langle A^\star \Phi, CE_0\Psi_0 \rangle \\ &= \langle A^\star \Phi, E_0 C\Psi_0 \rangle \\ &= (\eta A^\star \Phi, E_0 C\Psi_0) \\ &= \langle \Phi, \Psi \rangle (\Psi_0, C\Psi_0) \end{aligned}$$

Since D_0 and $\mathcal{F}(\mathcal{O})\Psi_0$ are dense in $\mathcal{H}$ [10] and C is continuous

$$C = (\Psi_0, C\Psi_0)$$

Remark. — It is not difficult to see that the irreducibility of $\{E_0, \mathcal{F}(\mathcal{O})\}$ still holds if E_0 is only required to be the projector on a vector which is cyclic with respect to the field algebra $\mathcal{F}$.

THEOREM 11. — In an indefinite metric QFT with a cyclic (proper) vacuum Ψ_0 which is the only translation invariant vector, the field algebra $\mathcal{F}$ is irreducible.

Proof. — If C is a bounded operator such that $\forall \Phi, \Psi \in \mathcal{F}\Psi_0, A \in \mathcal{F}$

$$\langle \Phi, CA\Psi \rangle = \langle A^\star \Phi, C\Psi \rangle, \quad (5.12)$$

as in the positive metric case we get

$$\langle \Psi_0, CU(a)\varphi(f_1) \dots \varphi(f_n)\Psi_0 \rangle = \langle \varphi(f_n)^\star \dots \varphi(f_1)^\star \Psi_0, U(-a)C\Psi_0 \rangle.$$

By the spectral condition eq. (5.12) implies that the Fourier transform of $\langle \varphi(f_n)^\star \dots \varphi(f_1)^\star \Psi_0, U(x)C\Psi_0 \rangle$ has support contained in $\{p = 0\}$ and therefore by Theorem 9 the vector $C\Psi_0$ is invariant under $U(a)$:

$$U(a)C\Psi_0 = C\Psi_0$$

Since the vacuum is unique

$$C\Psi_0 = c\Psi_0$$

and one has from eq. (5.12)

$$\begin{aligned} \langle \Phi, CA\Psi_0 \rangle &= c \langle A^*\Phi, \Psi_0 \rangle = c \langle \Phi, A\Psi_0 \rangle \\ \text{i. e.} \qquad \qquad \qquad C &= cI. \end{aligned}$$

THEOREM 12. — In an indefinite metric QFT with a cyclic (proper) vacuum Ψ_0 , which is essentially unique, the field algebra $\mathcal{F}$ is essentially irreducible.

Proof. — It suffices to prove that if the Wightman functional can be decomposed in invariant functionals generated by positive states Ψ_{0i} , $i = 1, \dots$, then these states must be η -orthogonal and by proposition 5.1 this is not possible if the vacuum is essentially unique.

To see this, let

$$\Psi_0 = \Psi_{01} + \Psi_{02}$$

with

$$\langle \Psi_{01}, \Psi_{01} \rangle > 0, \quad \langle \Psi_{02}, \Psi_{02} \rangle > 0$$

and for any $A \in \mathcal{F}$

$$\langle \Psi_0, A\Psi_0 \rangle = \langle \Psi_{01}, A\Psi_{01} \rangle + \langle \Psi_{02}, A\Psi_{02} \rangle$$

Then taking $A = \text{identity}$ we get

$$0 < \langle \Psi_0, \Psi_0 \rangle = \langle \Psi_{01}, \Psi_{01} \rangle + \langle \Psi_{02}, \Psi_{02} \rangle$$

which implies

$$\langle \Psi_{01}, \Psi_{02} \rangle = 0$$

NOTES AND REFERENCES

- [1] J. SCHWINGER, *Quantum Electrodynamics*. Dover.
- [2] L. D. LANDAU, *Niels Bohr and the Development of Physics*. Mc Graw-Hill, New York, 1955.
- G. KALLEN, *Consistency problems in Quantum Electrodynamics*. Cern Rep. 57-43 and article in *Handuch der Physik*.
- H. LEHMAN, K. SYMANZIK and W. ZIMMERMANN, *Nuovo Cimento*, t. **1**, 1955, p. 1425; t. **2**, 1955, p. 425; t. **6**, 1957, p. 319.
- [3] A. S. WIGHTMAN, Les Problèmes Mathématiques de la Théorie Quantique des Champs, C. N. R. S., Paris, 1959. *Phys. Rev.*, t. **101**, 1956, p. 860.
- [4] See S. WEINBERG, *Rev. Mod. Phys.*, t. **46**, 1974, p. 255.
- [5] K. WILSON, *Phys. Rev.*, t. **D 10**, 1974, p. 2445.
- A. M. POLYAKOV, *Phys. Letters*, t. **59 B**, 1975, p. 82.
- A. A. BELAVIN *et al.*, *Phys. Letters*, t. **59 B**, 1975, p. 85.
- G. 'T HOOFT, *Phys. Rev. Letters*, t. **37**, 1976, p. 8.
- R. JACKIW and C. REBBI, *Ibid.*, t. **37**, 1976, p. 172.
- C. CALLAN, R. DASHEN and D. GROSS, *Phys. Letters*, t. **63 B**, 1976, p. 334.
- [6] K. OSTERWALDER and E. SEILER, *Ann. Phys. N. Y.*, t. **110**, 1978, p. 440.
- D. BRIDGE, J. FRÖHLICH and T. SPENCER, *I. H. E. S.*, preprints, 1979.
- [7] See J. FRÖHLICH, G. MORCHIO and F. STROCCHI, *Ann. Phys. N. Y.*
- [8] F. STROCCHI, *Gauss' law in local QFT, Invited talk at the International Symposium on Elementary Particle Physics*. Reinhardsbrenn, October 1978.
- F. STROCCHI. *Phys. Rev.*, t. **D 17**, 1978, p. 2010.

- [9] R. FERRARI, F. E. PICASSO and F. STROCCHI, *Commun. Math. Phys.*, t. **35**, 1974, p. 25; *Nuovo Cimento*, t. **39 A**, 1977, p. 1.
- [10] F. STROCCHI, *Phys. Letters*, t. **62 B**, 1976, p. 60; *Phys. Rev.*, t. **D 17**, 1978, p. 2010.
- [11] R. F. STREATER and A. S. WIGHTMAN, *PCT Spin and Statistics and All That*. Benjamin, 1964.
- [12] In indefinite metric QFT's also when the WIGHTMAN functions do not satisfy the cluster property one may nevertheless associate to the WIGHTMAN functions a Hilbert space structure which lead to a unique vacuum.
- [13] R. HAAG and D. KASTLER, *Journ. Math. Phys.*, t. **5**, 1964, p. 848.
H. ARAKI, in *Local Quantum Field Theory*. Varenna School, 1968, R. Jost, ed.
- [14] N. N. BOGOLUBOV and D. W. SHIRKOV, *Introduction to the Theory of Quantized Fields*. Interscience, New York, 1959.
H. EPSTEIN and V. GLASER, in *Les Houches Lectures*, 1970, *Statistical Mechanics and Quantum Field Theory*. C. De Witt and R. Stora eds., Gordon and Breach, New York, 1971.
K. HEPP, *Ibid.*
- [15] E. SHEIBE, unpublished notes Oct. 1960. I am grateful to Prof. H. Reeh for giving me a copy of these notes.
- [16] H.-J. BORCHERS, Algebraic Aspects of Wightman's field theory, in *Statistical Mechanics and Field Theory*. Haifa Summer School, R. N. Sen and C. Weil eds., New York, 1972.
- [17] The existence of charges which obey Gauss' law represents a very distinctive property for a QFT, even with respect to QFT's whose physical states are characterized by observables at infinity, but are still obtainable by local morphisms (Ref. 18).
- [18] The analysis of S. DOPLICHER, R. HAAG and J. ROBERTS, *Commun. Math. Phys.*, t. **23**, 1971, p. 199; t. **28**, 1972, p. 331, is therefore not applicable in this case.
- [19] These results were discussed with J. YNGVASON, *Reports on Math. Physics*, t. **12**, 1977, p. 57, *Proposition 2*, and at the *Aspen Workshop on Quantum Electrodynamics*, Aspen, June 7-27, 1976 (F. STROCCHI, *Phys. Rev.*, t. **D 17**, 1978, p. 2010). (We do not share the philosophy of Yngvason's Proposition 3). An earlier discussion of the representation of the Borchers algebra by field operators was given in Ref. (20) under the assumption (see e. g. Lemma 2 of Ref. 20) that such a representation satisfies:

$$a) A(f^*) = A(f)^* \text{ on } D,$$

$$b) (\varphi, A(f)\psi) \text{ is continuous in } f \text{ for all } \varphi, \psi \text{ in } D \text{ (with } (\cdot, \cdot) \text{ the Hilbert space product).}$$

Within that framework the author proves the equivalence between the existence of such a representation and the joint continuity of the Wightman functionals. The application of this result to the indefinite metric case is however not obvious because the construction of a Hilbert space structure leading necessarily to a representation in which *a*) and *b*) hold implies that the metric operator η commutes weakly with the representation and therefore the field algebra is not irreducible if η is not trivial. In our opinion that structure strongly depends on the requirements *a*), *b*) which do not in general hold in the indefinite metric formulations of QFT's (a simple example is the Gupta-Bleuler formulation of QED). This difficulty underlies also the results of the subsequent paper (Reports on Math. Physics, t. **12**, 1977, p. 57) where the construction of a Hilbert space structure is based on a positive functional and obtained from it through a GNS construction. This procedure necessarily leads to *a*), which instead is not shared by other Hilbert space structures obtained via different constructions.

In addition it is worth-while to mention that our discussion is not based on the assumption that the Wightman functionals are jointly continuous with respect to the $\mathcal{S}$ topology. This property is automatically satisfied if they satisfy the positivity

condition (positive metric case) or when the seminorms of eq. (3.3) are assumed to be continuous on $\mathcal{S}(\mathbb{R}^{4n})$.

[20] See J. YNGVASON, *Commun. Math. Phys.*, t. **34**, 1973, p. 315.

[21] We sketch briefly the argument of the equivalence of weak and strong continuity, which applies also to cases in which the property a) $A(f^*) = A(f)^*$ on D , does not hold. Suppose there exists a sequence $f_n \rightarrow 0$ in $\mathcal{S}$ such that for some $\Psi \in \mathcal{S}$, $\|\varphi(f_n)\Psi\| \rightarrow 0$ so that one can extract a subsequence, again called f_m with the property $\|\varphi(f_m)\Psi\| > \varepsilon > 0$. Then, if $K(N)$ is the integer such that, in some ordering of the seminorms of $\mathcal{S}$, indexed by α , $\|f_m\|_\alpha < \frac{1}{N}$ for all $\alpha \leq N$, $m \geq K(N)$, the sequence $g_N \equiv f_{K(N)}$ satisfies $\|g_N\|_\alpha < \frac{1}{N}$, $\alpha \leq N$ and the sequence $h_N \equiv N^{\frac{1}{2}}g_N \rightarrow 0$ in $\mathcal{S}$; then if weak continuity holds $\varphi(h_N)\Psi \xrightarrow{w} 0$ and therefore $\|\varphi(h_N)\Psi\| < L$. On the other hand $\|\varphi(h_N)\Psi\| = N^{\frac{1}{2}}\|\varphi(g_N)\Psi\| \geq N^{\frac{1}{2}}\varepsilon \rightarrow \infty$.

[22] For mathematical properties of these operators see L. BRACCI, G. MORCHIO and F. STROCCHI, *Commun. Math. Phys.*, t. **41**, 1975, p. 289.

[23] The lack of uniqueness of the Hilbert space structure is essentially due to the fact that the Wightman functions provide us with a topological vector space $\mathcal{S}$ and an indefinite inner product $\langle \cdot, \cdot \rangle$, but no completeness property of $\mathcal{S}$ with respect to majorant Hilbert space topologies is guaranteed. Thus the interesting theorems discussed in Bognàr's book (Ref. 24) are in general not applicable.

[24] J. BOGNÀR, *Indefinite Inner Product Spaces*. Springer Verlag, 1974.

[25] G. MORCHIO and F. STROCCHI, paper on the Schwinger model, in preparation.

[26] G. KREIN, Second Mathematical Summer School, Part I. *Naukova Dumka Kiev*, 1965, translated in *A. M. S. Transl.*, t. **93**, 2, 1970, p. 103, and references therein.

[27] H. ARAKI, K. HEPP and D. RUELLE, *Helv. Phys. Acta*, t. **35**, 1962, p. 164.

[28] The conditions under which such conclusion holds will be discussed in Sect. 5, Theorem 9.

[29] See Example I and Ref. (25).

[30] This means that $p(f_n)$ is of the form

$$p(f) = \int d\mu(p) \sum_{k=0}^n |D^k f_n|^2$$

with $d\mu(p)$ absolutely continuous with respect to the Lebesgue measure.

[31] Condition 2) is equivalent to the property that for any $\Psi_g \in \mathcal{H}$,

$$g = (g_0, g_1, \dots, g_m, \dots),$$

$p(g_n)$ is finite.

[32] Clearly if such a $\delta^4(p)$ appear in the truncated Wightman functions, the above Sobolev type structure would necessarily lead to a non trivial metric operator and the translations $U(a)$ would only be η -unitary. We are grateful to Prof. A. S. Wightman for clarifying remarks.

[33] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators*, Part I. Interscience, New York, 1958, Chap. VII.

[34] M. A. NAIMARK, *Soviet Mat.*, t. **IV**, 1, 1963, p. 543.

[35] J. H. LOWENSTEIN and J. A. SWIECA, *Ann. Phys. N. Y.*, t. **68**, 1971, p. 172.

S. COLEMAN, R. JACKIW and L. SUSSKIND, *Ann. Phys. N. Y.*, t. **93**, 1975, p. 267.

[36] D. ZWANZINGER, *Phys. Rev.*, t. **D 17**, 1978, p. 457.

[37] A. S. WIGHTMAN and L. GARDING, *Arkiv for Fysik*, t. **28**, 1964, p. 129, p. 153-156.

[38] Ref. 11, p. 93.

[39] The details will be discussed in a subsequent paper.

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