ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 32, nº 4 (1980), p. 303-318
<http://www.numdam.org/item?id=AIHPA_1980__32_4_303_0>

*Ann. Inst. Henri Poincaré*, Section A :
Vol. XXXII, n° 4, 1980, p. 303-318. *Physique théorique.*

# Predictive extension of a singular lagrangian system

by

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ABSTRACT. — A predictive extension of the Dominici-Gomis-Longhi singular lagrangian system is obtained. We use the 4-acceleration given by this lagrangian as boundary conditions for the Droz-Vincent differential equations and we get the predictive 4-acceleration as a power expansion on the whole phase space. The harmonic oscillator and the kepler problem are treated as particular cases.

## I. INTRODUCTION

In a first tentative approach the interaction between quarks was explained by means of an infinite potential [*1*]. More realistic models are given by the harmonic oscillator approaches [*2*]. Among them we find the relativistic ones which supply the correct form factors [*3*]. However, undesired states with negative energy appear in these models.

These states are called ghost states and are associated to time-like oscillations. This problem can be avoided carrying on the quantization on the space-like hiperplane (P. r.) = 0, and then a positive energy spectrum is obtained [*4*].

A classical approach which exhibits all these properties is the singular lagrangian model of Dominici, Gomis and Longhi (D. G. L.) [*5*]. It is a covariant description of an interaction at a distance between two particles with the constraint (P. r.) = 0. In the opinion of these authors, this constraint

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distinguishes the center of mass frame among all the other inertial ones and it gives up the principle of special relativity.

In our understanding this principle has an important and clear physical meaning. And we believe that something more than a model is needed to abandon one of the fundamental principles of the physics.

In this paper we show that the equations of motion of D. G. L. are nothing else that a good boundary condition which determines a unique solution of the Droz-Vincent's differential equations [*12*]. So, with the help of the Predictive Relativistic Mechanics, we can construct a covariant model of action-at-a-distance between two particles which is instantaneous in any inertial frame and which reproduces the results of D. G. L. in the center of mass system.

We also show how and when a singular Lagrangian system (S. L. S.), defined on a submanifold of the phase space, can be extended to yield a predictive relativistic system on the whole phase space.

We know that, when we make predictive the Klein-Gordon equation, the positive and negative energies already split at the classical level [*6*]. We expect that something similar will hold in our predictively extended D. G. L. model.

In sections 2 and 3, we show those aspects of the S. L. S. of D. G. L. and of the predictive invariant systems (P. I. S.) under the Poincaré group which are of interest for this work. We also give a definition for the predictive extension of a S. L. S. Section 4 is used to make predictive the S. L. S. of D. G. L. through an expansion in some coupling constant. In section 5, we apply this result to a harmonic-oscillator type potential and to a Kepler type one. We obtain the equations of motion in both cases.

## II. THE D. G. L. TWO PARTICLE SYSTEM

In their paper Dominici *et al.* [*5*] use the following singular lagragian:

$$L = \varepsilon^{A}\sqrt{-\,U_{A}\dot{x}_{A}^{2}} \tag{2.1}$$

where $\varepsilon^{A} = 1$ and $U_{A} = m_{A}^{2} - V(x^{2})$ (1), in this expression $V(x^{2})$ is any $C^{1}$ function, the other symbols are $x^{\mu} = \eta^{A}x_{A}^{\mu}$, $\eta_{A} = \eta^{A} = (-\,1)^{A+1}$ and $\dot{x}_{A}^{2} = \eta_{\mu\nu}\dot{x}_{A}^{\mu}\dot{x}_{A}^{\nu}$. $\eta_{\mu\nu}$ holds for the Minkowsky metric tensor.

Using the Euler-Lagrange equation we obtain from (2.1):

$$\underset{A}{P}{}^{\mu}{}_{\nu}\ddot{x}_{A}^{\nu} = \underset{A}{P}{}^{\mu}{}_{\nu}\,.\,a_{A}^{\nu}(x_{B}, \dot{x}_{C}) \tag{2.2}$$

(1) The summation convention holds for the A, B, C, ... = 1, 2 and $\mu, \nu, \rho, \ldots = 0, 1, 2, 3$ indices, except if we unvalidate such convention using a bracket.

where:

$$\underset{\mathrm{A}}{\mathrm{P}}{}^{\mu}{}_{\nu} \equiv \eta^{\mu}{}_{\nu} - \frac{\dot{x}^{\mu}_{\mathrm{A}}\dot{x}_{\mathrm{A}\nu}}{\dot{x}^{2}_{\mathrm{A}}} \tag{2.3}$$

$$a^{\nu}_{\mathrm{A}} = \eta_{\mathrm{A}}\mathrm{V}'(x^{2})\cdot\frac{\sqrt{-\dot{x}^{2}_{\mathrm{A}}}}{\sqrt{\mathrm{U}_{\mathrm{A}}}}.\varepsilon^{\mathrm{B}}\frac{\sqrt{-\dot{x}^{2}_{\mathrm{B}}}}{\sqrt{\mathrm{U}_{\mathrm{B}}}}x^{\nu} \tag{2.4}$$

The equation (2.2) gives us the component of the 4-acceleration $\ddot{x}^{\mu}_{\mathrm{A}}$ perpendicular to the 4-velocity $\dot{x}^{\mu}_{\mathrm{A}}$. This relation only holds on the manifold $\mathscr{V} \subset \mathrm{T}^{+}(\mathrm{M}_4)^2$ given by:

$$\mathscr{V} \equiv \{ (x^{\mu}_{\mathrm{B}}, \dot{x}^{\nu}_{\mathrm{C}}) \in \mathrm{T}^{+}(\mathrm{M}_4)^2 / \mathrm{P}^{\mu}x_{\mu} = 0 \} \tag{2.5}$$

where:

$$\mathrm{P}^{\mu} \equiv \varepsilon^{\mathrm{B}}\sqrt{\mathrm{U}_{\mathrm{B}}}\,\frac{\dot{x}^{\mu}_{\mathrm{B}}}{\sqrt{-\dot{x}^{2}_{\mathrm{B}}}} \tag{2.6}$$

is the 4-momentum which is an integral of the motion. Obviously the Euler-Lagrange equations derived from the singular lagrangian are not compatible outside of $\mathscr{V}$.

In this problem, the goal of the singular Lagrangian theory is to derive a second order differential system:

$$\frac{d}{d\lambda} = \varepsilon^{\mathrm{A}}\left\{ \dot{x}^{\mu}_{\mathrm{A}}\frac{\partial}{\partial x^{\mu}_{\mathrm{A}}} + \xi^{\mu}_{\mathrm{A}}(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}})\frac{\partial}{\partial \dot{x}^{\mu}_{\mathrm{A}}} \right\} \tag{2.7}$$

where $\ddot{x}^{\mu}_{\mathrm{A}} = \xi^{\mu}_{\mathrm{A}}(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}})$ satisfy the conditions (2.2) on $\mathscr{V}$, sub-manifold of $\mathscr{V}$.

The solution was given in [5], and

*i*) $\tilde{\mathscr{V}} \subset \mathscr{V}$ is defined by the new constraint:

$$\mathrm{P}^{\mu}\dot{x}_{\mu} = 0 \quad , \quad \dot{x}^{\mu} \equiv \eta^{\mathrm{A}}\dot{x}^{\mu}_{\mathrm{A}} \tag{2.8}$$

*ii*) The 4-accelerations are defined up to a continuous arbitrary function $l(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}}; \lambda)$:

$$\xi^{\mu}_{\mathrm{A}}(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}}\,; \lambda) = a^{\mu}_{\mathrm{A}}(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}}) + l(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}}\,; \lambda).\dot{x}^{\nu}_{\mathrm{A}} \tag{2.9}$$

In this way, from the singular lagrangian (2.1) we obtain a family of second order differential systems $\{ \mathrm{D}_l \}_{l\in \mathrm{C}^0}$ on $\tilde{\mathscr{V}}$ which from now we shall call singular lagrangian system (S. L. S.) associated to (2.1).

Then for a given set of initial conditions $z_0 = (\underset{0}{x^{\mu}_{\mathrm{A}}}, \underset{0}{\dot{x}^{\nu}_{\mathrm{B}}}) \in \tilde{\mathscr{V}}$ and an arbitrary function $l(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}}; \lambda)$—*i. e.* for each $\mathrm{D}_l$ differential system—there is a unique world line for the system:

$$x^{\mu}_{\mathrm{A}} = \varphi^{\mu}_{\mathrm{A}}(\lambda, \underset{0}{z}\,; [l]), \qquad \mathrm{A} = 1, 2\,; \lambda \in ]\varepsilon_1, \varepsilon_2[ \,\equiv \mathrm{I} \subset \mathbb{R} \tag{2.10}$$

such that:

*i)*
$$\left.\begin{aligned} \varphi^\mu_A(0, \underset{0}{z}\,; [l]) &= \underset{0}{x}{}^\mu_A \\ \dot{\varphi}^\mu_A(0, \underset{0}{z}\,; [l]) &= \underset{0}{\dot{x}}{}^\mu_A \end{aligned}\right\} \tag{2.11}$$

where:

$$\dot{\varphi}^\mu_A \equiv \frac{\partial \varphi^\mu_A}{\partial \lambda}$$

*ii)* it is an integral of the system $D_l$:

$$\ddot{\varphi}^\mu_A(\lambda, \underset{0}{z}\,; [l]) = \xi^\mu_A(\varphi_B(\lambda, \underset{0}{z}\,; [l]), \dot{\varphi}_C(\lambda, \underset{0}{z}\,; [l])\,; \lambda) \quad , \quad \forall \lambda \in I \tag{2.12}$$

It has been shown [8] the following proposition:

If $\underset{0}{z} \in \tilde{\mathscr{V}}$, $l(x_B, \dot{x}_C; \lambda)$ are given and $\varphi^\mu_A(\lambda, \underset{0}{z}; [l])$, $\lambda \in I$ is the trajectory (2.10), then there is a reparametrization $f : I \to J \subset \mathbb{R}$ which allows us to write $\varphi^\mu_A$ as follows:

$$\varphi^\mu_A(\lambda, \underset{0}{z}\,; [l]) = \varphi^\mu_A(f(\lambda), \underset{0}{z}\,; [0]) \quad ; \quad \forall \lambda \in I \quad , \quad A = 1, 2 \tag{2.13}$$

## III. PREDICTIVE INVARIANT SYSTEMS (P. I. S.)

1. Definition. — Two given vector fields $\vec{H}_A$ defined on $T^+(M_4)^2$ [9], [10].

$$\vec{H}_A(x_B, \pi_C) = \pi^\mu_{(A)} \frac{\partial}{\partial x^\mu_{(A)}} + \theta^\mu_{(A)}(x_B, \pi_C) \cdot \frac{\partial}{\partial \pi^\mu_{(A)}} \tag{3.1}$$

is a two particle predictive system invariant under the Poincaré group if:

*i)*
$$\pi^\mu_{(A)}\theta_{(A)\mu}(x_B, \pi_C) = 0 \tag{3.2}$$

*ii)*
$$[\vec{H}_1, \vec{H}_2] = 0 \tag{3.3}$$

*iii)* the two vector fields $\vec{H}_1$, $\vec{H}_2$ are Poincaré invariant, *i. e.* $\theta^\mu_A(x_B, \pi_C)$ is a 4-vector with respect the Poincaré transformations.

The equation (3.3) is the integrability condition of the system (3.1) [11]. That is to say, there is one and only one 2-surface integral of the system for any point $\underset{0}{z} = (\underset{0}{x}{}^\mu_B, \underset{0}{\pi}{}^\nu_A) \in T^+(M_4)^2$:

$$x^\mu_A = \psi^\mu_A(\lambda_A, \underset{0}{z}) \quad ; \quad \pi^\mu_A = \psi^{\mu\prime}_A(\lambda_A, \underset{0}{z}) \quad ; \quad \lambda_A \in I_A \quad , \quad A = 1, 2 \tag{3.4}$$

where:

$$\psi^{\mu\prime}_A \equiv \frac{\partial \psi^\mu_A}{\partial \lambda_A}$$

such that:

$$\underset{0}{x}{}^{\mu}_{\mathrm{A}} = \psi^{\mu}_{\mathrm{A}}(0, \underset{0}{z}) \quad , \quad \underset{0}{\pi}{}^{\mu}_{\mathrm{A}} = \psi^{\mu\prime}_{\mathrm{A}}(0, \underset{0}{z}) \tag{3.5}$$

From condition (3.2) we see:

$$\frac{\partial}{\partial\lambda_{\mathrm{A}}}\{\psi^{\mu\prime}_{(\mathrm{A})}(\lambda_{\mathrm{A}}, z_0)\cdot\psi'_{(\mathrm{A})\mu}(\lambda_{\mathrm{A}}, \underset{0}{z})\} = 0 \quad ; \quad \lambda_{\mathrm{A}} \in \mathrm{I}_{\mathrm{A}} \quad , \quad \mathrm{A} = 1, 2 \tag{3.6}$$

The world line for each particle going through the $z_0$ initial conditions is

$$x^{\mu}_{\mathrm{A}} = \psi^{\mu}_{\mathrm{A}}(\lambda_{\mathrm{A}}, \underset{0}{z}) \quad ; \quad \lambda_{\mathrm{A}} \in \mathrm{I}_{\mathrm{A}} \quad , \quad \mathrm{A} = 1, 2 \tag{3.7}$$

2. Definition. — We call the P. I. S. system (3.1) a predictive extension of the S. L. S. (2.9) iff $\forall \underset{0}{z} = (\underset{0}{x}_{\mathrm{B}}, \underset{0}{\dot{x}}_{\mathrm{C}}) \in \tilde{\mathscr{V}}$, for any $\mathrm{C}^0$ function $l(x_{\mathrm{B}}, \dot{x}_{\mathrm{C}}; \lambda)$ and $\forall(\beta_1, \beta_2) \in \mathbb{R}^{+2}$ there are two reparametrizations—*i. e.* one for each world line:

$$\left.\begin{aligned} f_{\mathrm{A}}(\underset{0}{z}\,;\,[l]) : \mathrm{I} &\longrightarrow \mathrm{I}_{\mathrm{A}} \subset \mathbb{R} \\ \lambda &\longrightarrow f_{\mathrm{A}}(\lambda, \underset{0}{z}\,;\,[l]) \end{aligned}\right\} \quad \mathrm{A} = 1, 2$$

where $f_{\mathrm{A}}$ and $f_{\mathrm{A}}^{-1}$ are bijective maps, that allow us to write:

$$\varphi^{\mu}_{\mathrm{A}}(\lambda, \underset{0}{z}\,;\,[l]) = \psi^{\mu}_{\mathrm{A}}\left( f_{\mathrm{A}}(\lambda, \underset{0}{z}\,;\,[l]), \underset{0}{x}_{\mathrm{B}}, \beta_{\mathrm{C}}\frac{\underset{0}{\dot{x}}_{\mathrm{C}}}{\sqrt{-\underset{0}{\dot{x}}{}^2_{\mathrm{C}}}} \right) \tag{3.8}$$

From this condition and the initial conditions (2.11) and (3.5), we obtain:

$$f_{\mathrm{A}}(0, z\,;\,[l]) = 0 \quad ; \quad \dot{f}_{\mathrm{A}}(0, \underset{0}{z}\,;\,[l]) = \frac{\sqrt{-\underset{0}{\dot{x}}{}^2_{\mathrm{A}}}}{\beta_{\mathrm{A}}} \quad ; \quad \dot{f}_{\mathrm{A}} \equiv \frac{\partial f_{\mathrm{A}}}{\partial\lambda} \tag{3.9}$$

3. Proposition. — The necessary and sufficient condition for the P. I. S. (3.1) to be a predictive extension of the S. L. S. (2.9) is:

$$\begin{gathered} \forall z = (x_{\mathrm{B}}, \pi_{\mathrm{C}}) \in \mathscr{V} \quad , \quad \forall(\alpha_1, \alpha_2) \in \mathbb{R}^{+2}/(x_{\mathrm{B}}, \alpha_{\mathrm{C}}\pi_{\mathrm{C}}) \in \tilde{\mathscr{V}}, \\ \theta^{\mu}_{\mathrm{A}}(x_{\mathrm{B}}, \pi_{\mathrm{C}}) = \alpha_{(\mathrm{A})}^{-2}\underset{(\mathrm{A})}{\mathrm{P}}{}^{\mu}{}_{\nu}\cdot a^{\nu}_{(\mathrm{A})}(x_{\mathrm{B}}, \alpha_{\mathrm{C}}\pi_{\mathrm{C}}) \end{gathered} \tag{3.10}$$

*Proof.* — *a)* Necessary: let us suppose that the P. I. S. (3.1) is a predictive extension of the S. L. S. (2.9). $\forall \underset{0}{z} = (\underset{0}{x}_{\mathrm{B}}, \underset{0}{\pi}_{\mathrm{C}}) \in \mathscr{V}$ and $\forall(\alpha_1, \alpha_2) \in \mathbb{R}^{+2}$ verifying $\underset{0}{\tilde{z}} \equiv (\underset{0}{x}_{\mathrm{B}}, \alpha_{\mathrm{C}}\underset{0}{\pi}_{\mathrm{C}}) \in \tilde{\mathscr{V}}$ we have from definition (3.2) that:

$\forall(\beta_1, \beta_2) \in \mathbb{R}^{+2}$ and any $C^0$ function $l(x_B, \dot{x}_C; \lambda)$ there will exist two reparametrizations $f_A$ such that the condition:

$$\varphi^\mu_A(\lambda, \underset{0}{x}_B, \alpha_C \underset{0}{\pi}_C ; [l]) = \psi^\mu_A\left( f_A(\lambda, \underset{0}{\tilde{z}} ; [l]), \underset{0}{x}_B, \beta_C \cdot \frac{\underset{0}{\pi}_C}{\sqrt{-\underset{0}{\pi}^2_C}} \right) \quad (3.11)$$

is fulfilled.

In our case, we take $\beta_C = \sqrt{-\underset{0}{\pi}^2_C}$, $C = 1, 2$. Then from (3.9) we have:

$$f_A(0, \underset{0}{\tilde{z}} ; [l]) = 0 \quad , \quad \dot{f}_A(0, \underset{0}{\tilde{z}} ; [l]) = \alpha_A \quad (3.12)$$

Differentiating twice (3.11) with respect to $\lambda$ we immediately obtain for $\lambda = 0$:

$$\ddot{f}_{(A)}(0, \underset{0}{\tilde{z}} ; [l]) . \alpha_{(A)} . \underset{0}{\pi}^\mu_{(A)} + \alpha^2_{(A)} \theta^\mu_{(A)}(\underset{0}{x}_B, \underset{0}{\pi}_C) = \xi^\mu_A(\underset{0}{x}_B, \alpha_C \underset{0}{\pi}_C ; [l]) \quad (3.13)$$

where we have used the initial conditions (2.11) and (3.5). The orthogonal projection of (3.13) with respect to $\pi^\mu_A$ yields the equality (3.10).

*b*) Sufficient: given $(\underset{0}{x}_B, \underset{0}{\dot{x}}_C) \in \tilde{\mathscr{V}}$, $(\beta_1, \beta_2) \in \mathbb{R}^{+2}$ and a $C^0$ function $l(x_B, \dot{x}_C; \lambda)$, let us consider the solution of the S. L. S. (2.9):

$$\varphi^\mu_A(\lambda) \equiv \varphi^\mu_A(\lambda, \underset{0}{x}_B, \underset{0}{\dot{x}}_C; [l]) \quad , \quad \lambda \in I \quad , \quad A = 1, 2.$$

Then we can make a reparametrization:

$$\psi^\mu_A(\lambda_A) = \varphi^\mu_A(g_A(\lambda_A)) \quad , \quad A = 1, 2 \quad (3.14)$$

where the functions $g_A : I_A \to I$ $(A = 1, 2)$ verify the three following conditions:

*i*) The maps $g_A$ and $g_A^{-1} = f_A$ are bijective and differentiable.

*ii*) $$g_A(0) = 0 \quad , \quad g'_A(0) = \frac{\beta_A}{\sqrt{-\underset{0}{\dot{x}}^2_A}} \quad , \quad g'_A \equiv \frac{dg_A}{d\lambda_A} \quad (3.15)$$

*iii*) $f_A$ satisfy:

$$\ddot{f}_A + \dot{f}_A \cdot \frac{1}{\dot{\varphi}_A(\lambda)^2} \cdot \dot{\varphi}_{(A)\mu}(\lambda) . a^\mu_{(A)}(\varphi_B(\lambda), \dot{\varphi}_C(\lambda)) - l(\varphi_B(\lambda), \dot{\varphi}_C(\lambda) ; \lambda) = 0 \quad (3.16)$$

Then using (3.10) the world lines (3.4) are solutions of the S. P. I. (3.1) and:

$$\psi^\mu_A(0) = \underset{0}{x}^\mu_A \quad ; \quad \psi^{\mu'}_A(0) = \beta_A \cdot \frac{\underset{0}{\dot{x}}^\mu_A}{\sqrt{-\underset{0}{\dot{x}}^2_A}} \quad ; \quad \psi^{\mu'}_A \equiv \frac{d\psi^\mu_A}{d\lambda_A} \quad (3.17)$$

and the theorem is proved.

# IV. THE PREDICTIVE EXTENSION OF THE D. G. L. SINGULAR LAGRANGIAN SYSTEM

The condition (3.10) give us the $\theta^\mu_A(x_B, \pi_C)$ function for $(x_B, \pi_C) \in \mathscr{V}$ and it is taken as the boundary condition of the differential system (3.3), which can be also written as:

$$\pi^\alpha_{(A')} \cdot \frac{\partial \theta^\mu_A}{\partial x^\alpha_{(A')}} + \theta^\alpha_{(A')} \cdot \frac{\partial \theta^\mu_A}{\partial \pi^\alpha_{(A')}} = 0 \quad ; \quad A = 1, 2 \quad , \quad A' \neq A \tag{4.1}$$

If (3.10) should be a good boundary condition it must verify:

*i*) Being compatible with (4.1). This can be proved by construction, if we assume that $\theta^\mu_A$ is an analytical function of some coupling constant $g$, introduced in the potential taking:

$$V(x^2) \equiv g W(x^2) \tag{4.2}$$

*ii*) The left hand side of (3.10) must be a 4-vector under Poincaré transformations, as will be easily seen from (4.5).

*iii*) The left hand side of (3.10) must be independent of $(\alpha_1, \alpha_2) \in \mathbb{R}^{+2}$. This is accomplished in our problem since we give $(x_B, \pi_C) \in \mathscr{V}$, $\forall (\alpha_1, \alpha_2) \in \mathbb{R}^{+2}$ verifying $(x_B, \alpha_C \pi_C) \in \tilde{\mathscr{V}}$ they will satisfy:

$$\frac{\alpha_{A'}}{\alpha_A} = \frac{P_\mu \pi^\mu_A}{P_\mu \pi^\mu_{A'}} \quad , \quad A' \neq A \tag{4.3}$$

as it is shown in (2.8).

Then from (4.3) and (2.4) the left hand side of (3.10) can be written:

$$\alpha^{-2}_{(A)} \cdot \underset{(A)}{P^\mu}{}_\nu \cdot a^\nu_{(A)}(x_B, \alpha_C \pi_C)$$
$$= g \cdot \eta_{(A)} \cdot W'(x^2) \cdot \frac{\pi_A}{\sqrt{U_A}} \cdot \left\{ \frac{\pi_A}{\sqrt{U_A}} + \frac{\alpha_{A'}}{\alpha_A} \cdot \frac{\pi_{A'}}{\sqrt{U_{A'}}} \right\} \cdot \underset{(A)}{P^\mu}{}_\nu \cdot x^\nu \tag{4.4}$$

which depends only on $\dfrac{\alpha_{A'}}{\alpha_A}$ and then $\theta^\mu_A$ does not depend on $\alpha_B$:

$$\theta^\mu_A(x_B, \pi_C)$$
$$= g \cdot \eta_A \cdot W' \cdot \left\{ 1 + \sqrt{\frac{U_A}{U_{A'}}} \cdot \frac{\sqrt{U_A} \cdot \pi_A \cdot \pi_{A'} + k \cdot \sqrt{U_{A'}}}{\sqrt{U_A} \cdot k + \sqrt{U_{A'}} \cdot \pi_A \cdot \pi_{A'}} \right\} \frac{x^2_A}{U_A} \left( \pi^\mu + \frac{x \pi_A}{\pi^2_A} \pi^\mu_A \right) \tag{4.5}$$

where:

$$\pi_B \equiv \sqrt{- \pi^\mu_B \pi_{B\mu}} \quad , \quad k \equiv - \pi^\mu_1 \cdot \pi_{2\mu} \quad , \quad (x_B, \pi_C) \in \mathscr{V}$$

The equation (4.1) is easily transformed in an integrofunctional equation [9]. The problem is simplified using the new variables:

$$\left.\begin{array}{ll} t^\alpha_{\rm A} \equiv \pi^2_{\rm A'}.\pi^\alpha_{\rm A} - k\pi^\alpha_{\rm A'} & ; \quad \Lambda^2 \equiv k^2 - \pi^2_{\rm A}.\pi^2_{\rm A'} \\ z_{\rm A} \equiv \eta_{\rm A}\Lambda^{-2}.t^\alpha_{\rm A}x_\alpha & ; \quad h^\alpha \equiv x^\alpha - \eta^{\rm B}z_{\rm B}\pi^\alpha_{\rm B} \end{array}\right\} \tag{4.6}$$

if we define : $D_A \equiv \pi^\alpha_{(A)} \cdot \dfrac{\partial}{\partial x^\alpha_{(A)}}$, we have [13] :

$$\left.\begin{array}{ll} D_A z_B = \delta_{AB} & ; \quad D_A h^2 = D_A k = D_A \pi_B = 0 \\ D_A h^\alpha = D_A t^\alpha_B = 0 \quad , & \qquad\qquad A, B = 1, 2 \end{array}\right\} \tag{4.7}$$

In the new variables, the equation (2.5) defining $\mathscr{V}$ will be written as:

$$\left(\pi_{\rm A}.\pi_{\rm A'}\sqrt{{\rm U_A}} + k\sqrt{{\rm U_{A'}}}\right).\pi_{\rm A}.z_{\rm A} = \left(\pi_{\rm A}.\pi_{\rm A'}\sqrt{{\rm U_{A'}}} + k\sqrt{{\rm U_A}}\right).\pi_{\rm A'}z_{\rm A'} \tag{4.8}$$

and we can separate:

$$z_{\rm A'} = \psi_{\rm A}(z_{\rm A}, h^2, k, \pi_{\rm B}) \tag{4.9}$$

If we write also $\theta^\mu_A$ in the new variables and using $D_A = \dfrac{\partial}{\partial z_A}$ we obtain the equivalent integrofunctional system (4.10) from the equation (4.1) and the boundary conditions (4.5) on $\mathscr{V}$ (4.8):

$$\theta^\mu_{\rm A} = \underset{*}{\theta^\mu_{\rm A}} - \int_{\hat{z}_{\rm A'}}^{z_{\rm A'}} dz_{\rm A'} \cdot \left\{ \theta^\alpha_{\rm A'} \cdot \frac{\partial \theta^\mu_{\rm A}}{\partial \pi^\alpha_{\rm A'}} \right\} \tag{4.10}$$

where:

$$\hat{z}_{\rm A'} = \psi_{\rm A}(z_{\rm A}, h^2, k, \pi_{\rm B}) \qquad \text{and} \qquad \underset{*}{\theta^\mu_{\rm A}}(z_{\rm A}, h^2, k, \pi_{\rm B}, h^\alpha, t^\alpha_{\rm B})$$

is obtained when we substitute $\hat{z}_{A'}$ by $\psi_A(z_A, ...)$, A = 1, 2, in the equation (4.5). Now we can solve (4.10) if we assume that $\theta^\mu_A$ can be expanded in powers of $g$. First of all using the properties *i*) and *iii*) of definition (3.1) we write:

$$\theta^\mu_{\rm A} = \eta_{\rm A} a_{\rm A} h^\mu + l_{\rm AA'} t^\mu_{\rm A'} \tag{4.11}$$

where $a_A$ and $l_{AA'}$ are functions of $z_B$, $h^2$, $k$ and $\pi_B$, and we also assume that they can be expanded in $g$ powers.

$$a_{\rm A} = \sum_{n=1}^{\infty} g^n \overset{(n)}{a}_{\rm A} \quad , \quad l_{\rm AA'} = \sum_{n=1}^{\infty} g^n \overset{(n)}{l}_{\rm AA'} \tag{4.12}$$

We assume that $W(x^2)$ can be expanded too in powers of $g$. Then from (4.8), (4.9), we obtain:

$$\psi_{\rm A} = \sum_{n=0}^{\infty} g^n . \overset{(n)}{\psi}_{\rm A} \tag{4.13}$$

When we expand (4.10) in powers of $g$ we find an additional problem because $\widehat{z}_{\mathrm{A}}$, in lower integral limit, depends on the coupling constant $g$.

LEMMA. — If $f(x) = \int_{\alpha(x)}^{\beta} h(x, y).dy$ and we assume that $f$, $h$ and $\alpha$ can be expanded in powers of $x$, then

$$\overset{(n)}{f} = \int_{\overset{0}{\alpha}}^{\beta} \overset{(n)}{h}(y).dy - \sum_{m=1}^{\infty} \sum_{\substack{m_1+\ldots+m_p=m \\ m_i \geqslant 1}} \frac{1}{p\,!} \cdot \left( \frac{\partial^{p-1} \overset{(n-m)}{h}}{\partial y^{p-1}} \right)_{y=\overset{0}{\alpha}} \cdot \overset{(m_1)}{\alpha} \cdot \ldots \cdot \overset{(m_p)}{\alpha} \quad (4.14)$$

where:

$$f(x) = \sum_{n=0}^{\infty} \overset{(n)}{f} .x^n \quad , \quad \alpha(x) = \sum_{n=0}^{\infty} \overset{(n)}{\alpha} .x^n \quad , \quad h(x, y) = \sum_{n=0}^{\infty} \overset{(n)}{h}(y).x^n$$

*Proof.* — If $\mathrm{H}(x, y)$ verify $\dfrac{\partial \mathrm{H}(x, y)}{\partial y} = h(x, y)$ we obtain:

$$f(x) = \mathrm{H}(x, \beta) - \mathrm{H}(x, \alpha(x))$$

$$= \mathrm{H}(x, \beta) - \mathrm{H}(x, \overset{0}{\alpha}) - \sum_{p=1}^{\infty} \frac{1}{p\,!} \cdot \left( \frac{\partial^p \mathrm{H}}{\partial y^p} \right)_{(x,\overset{0}{\alpha})} \cdot \left( \sum_{m=1}^{\infty} \overset{(m)}{\alpha} .x^m \right)^p$$

and then (4.14) can be easily found. From the lemma and the equations (4.11), (4.13) we have

$$\overset{(n)}{\theta}{}^{\mu}_{\mathrm{A}} = \underset{*}{\overset{(n)}{\theta}}{}^{\mu}_{\mathrm{A}} + \int_{\overset{(0)}{\psi}_{\mathrm{A}}}^{z_{\mathrm{A}'}} \overset{(n)}{\mathrm{Z}}{}^{\mu}_{\mathrm{A}}.dz_{\mathrm{A}'}$$

$$- \sum_{m=1}^{n-2} \sum_{\substack{m_1+\ldots+m_p=m \\ m_i \geqslant 1}} \frac{1}{p\,!} (\mathrm{D}^{p-1}_{\mathrm{A}'} \overset{(n-m)}{\mathrm{Z}}{}^{\mu}_{\mathrm{A}})_{z_{\mathrm{A}'}=\overset{(0)}{\psi}_{\mathrm{A}}} \cdot \overset{(m_1)}{\psi}_{\mathrm{A}} \cdot \ldots \cdot \overset{(m_p)}{\psi}_{\mathrm{A}} \quad (4.15)$$

where $\overset{(n)}{\mathrm{Z}}{}^{\mu}_{\mathrm{A}}$ is the coefficient of $g^n$ when we expand the term $- \theta^{\alpha}_{\mathrm{A}'} . \dfrac{\partial \theta^{\mu}_{\mathrm{A}}}{\partial \pi^{\alpha}_{\mathrm{A}'}}$ (Appendix 1); the sum are extended to $m = n - 2$ because $\overset{(p)}{\mathrm{Z}}{}^{\mu}_{\mathrm{A}} = 0$ if $p < 2$. In Appendix 1 we can also find the $\overset{(n)}{\theta}{}^{\mu}_{\mathrm{A}}$ terms.

1. PROPOSITION. — The boundary condition (4.5) on $\mathscr{V}$ give us a unique solution $\theta^{\mu}_{\mathrm{A}}$ of the equation (4.1) if we assume that $\theta^{\mu}_{\mathrm{A}}$ can be expanded in powers of $g$ and $\mathrm{W}(x^2)$ is an analytic function except perhaps in a finite number of points.

*Proof.* — From (4.15) we obtain $\overset{(1)}{\theta}{}^{\mu}_{\mathrm{A}} = \underset{*}{\overset{(1)}{\theta}}{}^{\mu}_{\mathrm{A}}$ and $\overset{(n)}{\theta}{}^{\mu}_{\mathrm{A}}$ is determined from the lower terms $\overset{(p)}{\theta}{}^{\mu}_{\mathrm{A}}$, $p < n$ (Appendix 1). Then we can find order by order the $\overset{(n)}{\theta}{}^{\mu}_{A}$ when we know the terms $\underset{*}{\overset{(n)}{\theta}}{}^{\mu}_{A}$ and $\overset{(n)}{\psi}_{A}$ of the $g$-expansion of (4.5) and (4.9).

Now we are going to compute the $\overset{(n)}{\psi}_{A}$ terms to the lower orders. In the equation (4.8) the functions $U_B$ depends on $x^2$:

$$x^2 = h^2 - \pi_A^2 z_A^2 - \pi_{A'}^2 z_{A'}^2 + 2k.z_A.z_{A'} \tag{4.16}$$

on the submanifold $\mathscr{V}$, but $z_{A'} = \psi_A(z_A, h^2, k, \pi_B)$. Then if W is analytica we obtain:

$$\left.\begin{aligned} \overset{(0)}{\psi}_{A} &= \frac{k+\pi_A^2}{k+\pi_{A'}^2}\cdot z_A \\ \overset{(1)}{\psi}_{A} &= -\frac{\Lambda^2 z_A(\pi_A^2-\pi_{A'}^2)}{2.\pi_A^2.\pi_{A'}^2(k+\pi_{A'}^2)^2}\cdot W(\widehat{x}_A^2) \end{aligned}\right\} \tag{4.17}$$

where:

$$\widehat{x}_A^2 \equiv h^2 + \frac{\Lambda^2.(2k+\pi_A^2+\pi_{A'}^2)}{(k+\pi_{A'}^2)^2} z_A^2 \tag{4.18}$$

In the same way we can expand $\underset{*}{\theta}{}^{\mu}_{A}$. The lower term is:

$$\left.\begin{aligned} \underset{*}{\overset{(1)}{a}}_{A} &= \left(1+\frac{k+\pi_A^2}{k+\pi_{A'}^2}\right)\cdot W'(\widehat{x}_A^2) \\ \underset{*}{\overset{(1)}{l}}_{AA'} &= -\frac{k+\pi_A^2}{k+\pi_{A'}^2}\cdot\left(1+\frac{k+\pi_A^2}{k+\pi_{A'}^2}\right)\cdot\frac{z_A}{\pi_A^2}\cdot W'(\widehat{x}_A^2) \end{aligned}\right\} \tag{4.19}$$

where W′ is the first derivative of $W(x^2)$ with respect to $x^2$. The second term in the expansion of $\underset{*}{\theta}{}^{\lambda}_{A}$ is given in Appendix 2.

The 4-acceleration $\theta^{\mu}_{A}$ to first order in $g$ is:

$$\theta^{\mu}_{A} = g\cdot\left(1+\frac{k+\pi_A^2}{k+\pi_{A'}^2}\right)\cdot W'(\widehat{x}_A^2)\cdot\left\{\eta_A h^{\mu} - \frac{k+\pi_A^2}{k+\pi_{A'}^2}\cdot\frac{z_A}{\pi_A^2}\cdot t^{\mu}_{A'}\right\} + O(g^2) \tag{4.20}$$

and the second order coefficients are:

$$\left.\begin{aligned} \overset{(2)}{a}_{A} &= \underset{*}{\overset{(2)}{a}}_{A} + \int_{\overset{(0)}{\psi}_{A}}^{z_{A'}} \overset{(2)}{A}_{A}(z_{A'}).dz_{A'} \\ \overset{(2)}{l}_{AA'} &= \underset{*}{\overset{(2)}{l}}_{AA'} + \int_{\overset{(0)}{\psi}_{A}}^{z_{A'}} \overset{(2)}{L}_{AA'}(z_{A'}).dz_{A'} \end{aligned}\right\} \tag{4.21}$$

where the functions: $\underset{*}{\overset{(2)}{a}}_{A}$, $\underset{*}{\overset{(2)}{l}}_{AA'}$, $\overset{(2)}{A}_{A}$, $\overset{(2)}{L}_{AA'}$ can be found in Appendix 2; the integrals depend on the potential $W(x^2)$ used.

## V. APPLICATION

First of all, let us especify our result when both particles have equal masses:

$$\overset{(0)}{\psi}_{\mathrm{A}} = z_{\mathrm{A}} \quad ; \quad \overset{(n)}{\psi}_{\mathrm{A}} = 0 \quad ; \quad \forall n \geqslant 1 \tag{5.1}$$

and then:

$$\overset{(n)}{\theta}{}^{\mu}_{\mathrm{A}} = 2.\mathrm{W}'(\widehat{x_{\mathrm{A}}^2})\cdot\left(\frac{\mathrm{W}(\widehat{x_{\mathrm{A}}^2})}{m^2}\right)^{n-1}\cdot\left(\eta_{\mathrm{A}} h^{\mu} - \frac{z_{\mathrm{A}}}{\pi_{\mathrm{A}}^2} t^{\mu}_{\mathrm{A}'}\right) + \int_{z_{\mathrm{A}}}^{z_{\mathrm{A}'}} \overset{(n)}{\mathrm{Z}}{}^{\mu}_{\mathrm{A}}.dz_{\mathrm{A}'} \tag{5.2}$$

where $\overset{(2)}{\mathrm{A}}_{\mathrm{A}}$ and $\overset{(2)}{\mathrm{L}}_{\mathrm{AA}'}$ can be found in Appendix 2.

We give this result because this system is perhaps related with a realistic quark interaction.

For different masses we try two potentials: the oscillator $\mathrm{W} \propto x^2$, used in quark models [4], and $\mathrm{W} \propto (x^2)^{-1/2}$ as a relativistic approximation to the Kepler problem.

In order to obtain the proportionality we study the lagrangian (2.1) in the Newtonian approximation in the C. M. system $-\, \mathrm{P}^{\mu} = (\mathrm{P}, \mathrm{O}, \mathrm{O}, \mathrm{O})$. Now the condition (2.5) reads $x_1^0 = x_2^0 = t$, $x^2 = |\vec{x}_1 - \vec{x}_2|^2$.

With the $t$ parameter, in the newtonian approximation

$$\frac{\mathrm{V}(x^2)}{\vec{v}_{\mathrm{A}}^{\,2}} \sim 1 \quad , \quad \vec{v}_{\mathrm{A}}^{\,2} \ll 1 \quad , \quad \vec{v}_{\mathrm{A}}^{\,2} \equiv \delta_{ij}\dot{x}^i\dot{x}^j$$

we have:

$$\mathscr{L} = -\,(m_1 + m_2) + \frac{1}{2}\varepsilon^{\mathrm{A}} m_{\mathrm{A}} \vec{v}_{\mathrm{A}}^{\,2} + \frac{1}{2\mu}\mathrm{V}(|\vec{x}_1 - \vec{x}_2|^2) + \mathrm{O}(\vec{v}_{\mathrm{A}}^{\,4}) \tag{5.3}$$

Then if we compare with the oscillator and Kepler lagrangians we obtain respectively:

$$\mathrm{V}_{\mathrm{osc}}(x^2) = -\,\mu^2\omega^2 x^2 \quad ; \quad \mathrm{V}_k(x^2) = 2\mathrm{GM}\mu^2(x^2)^{-1/2} \tag{5.4}$$

Where: $\mu \equiv \dfrac{m_1 m_2}{\mathrm{M}}$ is the reduced mass, $\mathrm{M} = m_1 + m_2$ the total mass, $\omega$ the newtonian frequency and G the gravitatorial constant.

In the oscillator problem we take from (5.4) and (4.2):

$$g = \omega^2 \quad , \quad \mathrm{W}(x^2) = -\,\mu^2 x^2 \tag{5.5}$$

The 3-accelerations for any initial system are [13]:

$$a^i_{\mathrm{A}}(\vec{x}_{\mathrm{B}}, \vec{v}_{\mathrm{C}}\,;\, t) = m_{\mathrm{A}}^{-2}\gamma_{\mathrm{A}}^{-2}(\delta^i_j - v^i_{\mathrm{A}} v^j_{\mathrm{A}}).\theta^j_{\mathrm{A}}(x_1^0 = x_2^0 = t) \tag{5.6}$$

where:

$$\pi^{\mu}_{\mathrm{A}} = m_{\mathrm{A}}\gamma_{\mathrm{A}}(1, \vec{v}_{\mathrm{A}}) \quad , \quad \gamma_{\mathrm{A}} = (1 - \vec{v}_{\mathrm{A}}^{\,2})^{-1/2}$$

Then in a $1/c$ expansion, we obtain:

$$\vec{a}_{A} = -\eta_{A}\frac{\mu\omega^2}{m_A}\left(1 - \frac{\vec{v}_A^{\,2}}{c^2} + \frac{m_A(m_{A'} - m_A)}{M^2}\cdot\frac{\vec{v}^{\,2}}{2c^2}\right)\cdot\vec{r}$$
$$+ \frac{\mu\omega^2}{m_A c^2}\cdot\frac{m_A(\vec{r}\cdot\vec{v}_A) + m_{A'}(\vec{r}\cdot\vec{v}_{A'})}{M}\vec{v} + O\left(\frac{\omega^2}{c^4}\right) + O(\omega^4) \quad (5.7)$$

this formula give us the relativistic corrections to the harmonic oscillator in this model.

For equal masses and up to second order in the expansion we have:

$$\left.\begin{aligned}
\overset{(2)}{a}_A &= \frac{m^2}{8}\{h^2 + 2(k - m^2)z_A^2\} + \frac{m^4}{8}(z_A - z_{A'})^2 \\
\overset{(2)}{l}_{AA'} &= -\frac{1}{8}\{h^2 + 2(k - m^2)z_A^2\} z_A \\
&\quad + \frac{m^2}{4}\left\{\frac{h^2}{k + m^2}(z_{A'} - z_A) - \frac{1}{3}(z_{A'}^3 - z_A^3) + \frac{k z_A}{2m^2}(z_{A'}^2 - z_A^2)\right\}
\end{aligned}\right\} \quad (5.8)$$

In the Kepler potential we have:

$$g = G \quad ; \quad W(x^2) = 2M\mu^2(x^2)^{-1/2} \quad (5.9)$$

The 3-acceleration in first order in the coupling constant G and to order $1/c^2$ is:

$$\vec{a}_A = -\eta_A G m_{A'}\left\{1 - \frac{\vec{v}_A^{\,2}}{c^2} + \frac{m_A.(m_{A'} - m_A).v^2}{2M^2c^2}\right\}\cdot\frac{\vec{r}}{r^3}$$
$$+ \frac{G m_{A'}}{c^2 M}\cdot\frac{m_A(\vec{r}\cdot\vec{v}_A) + m_{A'}(\vec{r}\cdot\vec{v}_{A'})}{r^3}\cdot\vec{v} \quad (5.10)$$
$$+ \eta_A\frac{3Gm_{A'}}{2c^2}\left\{(\vec{r}\cdot\vec{v}_{A'})^2 - \frac{m_A^2}{M^2}(\vec{r}\cdot\vec{v})^2\right\}\frac{\vec{r}}{r^5} + O\left(\frac{G}{c^4}, \frac{G^2}{c^2}\right)$$

and as we see they differ from the Einstein, Infeld and Hoffmann lagrangian results [14]:

$$\vec{a}_A^{\,EIH} = -\eta_A.G.m_{A'}\left\{1 - \frac{\vec{r}_A^{\,2}}{c^2} + \frac{2\vec{v}^{\,2}}{c^2}\right\}\cdot\frac{\vec{r}}{r^3} + \eta_A\frac{3Gm_{A'}}{2c^2}(\vec{r}\cdot\vec{v}_{A'})^2\frac{\vec{r}}{r^5}$$
$$+ \frac{Gm_{A'}}{c^2}\{4(\vec{r}\cdot\vec{v}_A) - 3(\vec{r}\cdot\vec{v}_{A'})\}\frac{\vec{v}}{r^3} + O\left(\frac{G}{c^4}, \frac{G^2}{c^2}\right) \quad (5.11)$$

But this is not surprising, because we can get a lot of relativistic generalizations from the same newtonian result, depending on the criterium and method followed to obtain the generalization.

## VI. CONCLUSION AND OUTLOOK

In this paper we showed against the opinion of some authors [5] how the S. L. S. theory is not in contradiction with the predictive invariant mechanics, but it give us in some problems the good boundary conditions from which we obtain a unique solution of the Droz-Vincent equations. This is exactly what also happens in classical field theory, where for the scalar and vector short range and for the electromagnetic interaction we obtain again a unique solution. It should be noticed nevertheless that the boundary conditions obtained for the S. L. S. are Cauchy-data whereas this is not so for classical field theory. Thus in section 4 we obtained using the S. L. S. of D. G. L. a unique P. I. S. in a power expansion in some coupling constant $g$. We have finally gotten the predictive equations of motion for the harmonic oscillator and Kepler potential comparing this last one with the Lagrangian of Einstein-Infeld-Hoffmann.

This work allows us to extend the equivalence in the $n$-interacting particle problem.

We can also now study the two interacting quark system in the case of the harmonic-oscillator type potential. This problem can be solved constructing a hamiltonian and quantizing the obtained predictive invariant system. Then we will compare the obtained energy spectrum for the bounded states and the one usually given in the literature.

ACKNOWLEDGMENTS

We would like to thank Prof. L. Bel for suggesting us the problem and Dr. J. Gomis for many useful discussions.

# APPENDIX 1

$$\overset{(n)}{Z}{}^{\mu}_{A} = \eta_A \overset{(n)}{A}_A . h^{\mu} + \overset{(n)}{L}_{AA'} t^{\mu}_{A'}$$

$$\overset{(n)}{A}_A = \sum_{m=1}^{n-1} \{ z_{A'} \overset{(m)}{a}_A \overset{(n-m)}{a}_{A'} + |\pi_A|^2 \overset{(m)}{a}_{A'} \overset{(n-m)}{l}_{AA'} - a_{A'}(N_{A'} \overset{(n-m)}{a}_A) - l_{A'A}(Q_{A'} \overset{(m)}{\;} \overset{(n-m)}{a}_A) \} \quad , \quad n \geqslant 2$$

$$\overset{(n)}{L}_{AA'} = \sum_{m=1}^{n-1} \{ k \overset{(m)}{l}_{A'A} \overset{(n-m)}{l}_{AA'} - \Lambda^{-2} h^2 \overset{(m)}{a}_A \overset{(n-m)}{a}_{A'} - \overset{(m)}{a}_{A'}(N_{A'} \overset{(n-m)}{l}_{AA'}) - \overset{(m)}{l}_{A'A}(Q_{A'} \overset{(n-m)}{l}_{AA'}) \} \quad , \quad n \geqslant 2$$

$$\overset{(n)}{A}_A = 0 \qquad \overset{(n)}{L}_{AA'} = 0 \quad , \quad n < 2$$

$$\overset{(n)}{a}_A = \overset{(n)}{\underset{*}{a}}_A + \int_{\underset{\psi_A}{(0)}}^{z_{A'}} \overset{(n)}{A}_A(z_{A'}) dz_{A'} - \sum_{m=1}^{n-2} \sum_{\substack{m_1+\ldots+m_p=m \\ m_i \geqslant 1}} \frac{1}{p\,!} (D^{p-1}_{A'} \overset{(n-m)}{A}_A)_{\underset{(\psi_A)}{(0)}} \cdot \overset{(m_1)}{\psi}_A \cdot \ldots \cdot \overset{(m_p)}{\psi}_A$$

$$\overset{(n)}{l}_{AA'} = \overset{(n)}{\underset{*}{l}}_{AA'} + \int_{\underset{\psi_A}{(0)}}^{z_{A'}} \overset{(n)}{L}_{AA'}(z_{A'}) dz_{A'} - \sum_{m=1}^{n-2} \sum_{\substack{m_1+\ldots+m_p=m \\ m_i \geqslant 1}} \frac{1}{p\,!} (D^{p-1}_{A'} \overset{(n-m)}{L}_{AA'})_{\underset{(\psi_A)}{(0)}} \cdot \overset{(m_1)}{\psi}_A \cdot \ldots \cdot \overset{(m_p)}{\psi}_A$$

ADDENDUM

Where the operators $N_A$, $Q_A$ are :

$$N_A = \eta_{(A)} h^{\mu} \frac{\partial}{\partial \pi^{\mu}_{(A)}} \qquad ; \qquad Q_A = t^{\mu}_{A'} \frac{\partial}{\partial \pi^{\mu}_{A}}$$

# APPENDIX 2

$$
\overset{(2)}{\underset{*}{a}}_{A} = \frac{W(\widehat{x_A^2}).W'(\widehat{x_A^2})}{2m_A^2 m_{A'}^2 k_{A'}^2}\{4k^2 m_{A'}^2 + k(m_A^4 + 2m_A^2 m_{A'}^2 + 5m_{A'}^4) + 2m_{A'}^2(m_A^4 + m_{A'}^4)\}
$$
$$
+ \left(1 + \frac{k_A}{k_{A'}}\right) \cdot \frac{\Lambda^4 z_A^2(m_{A'}^2 - m_A^2)}{m_A^2 m_{A'}^2 k_{A'}^3} W(\widehat{x_A^2}).W''(\widehat{x_A^2})
$$

$$
\overset{(2)}{\underset{*}{l}}_{AA'} = -\frac{W(\widehat{x_A^2}).W'(\widehat{x_A^2})z_A^2}{2m_A^4 m_{A'}^2 k_{A'}^3}\{2k^3(3m_{A'}^2 - m_A^2) + 6m_{A'}^2 k^2(m_A^2 + m_{A'}^2)
$$
$$
+ (m_A^6 + 6m_A^4 m_{A'}^2 + 3m_A^2 m_{A'}^4 + 2m_{A'}^6) + m_A^2 m_{A'}^2(3m_A^4 + m_{A'}^4)\}
$$
$$
+ \frac{k_A}{k_{A'}}\left(1 + \frac{k_A}{k_{A'}}\right) \cdot \frac{\Lambda^4 z_A^3(m_A^2 - m_{A'}^2)}{m_A^4 m_{A'}^2 . k_{A'}^3} W(\widehat{x_A^2}).W''(\widehat{x_A^2})
$$

where :

$$
k_A \equiv k + m_A^2 \qquad \text{and} \qquad m_A \equiv |\pi_A|
$$

$$
\overset{(2)}{A}_{A} = \left\{\frac{(k_A + k_{A'})(k_A m_{A'}^2 + k_{A'} m_A^2)}{m_{A'}^2 k_A^2} z_{A'} - \left(\frac{k_A}{k_{A'}} + 1\right)^2 z_A\right\} W'(\widehat{x_A^2}).W'(\widehat{x_{A'}^2})
$$
$$
+ \left\{\frac{2h^2(k_A + k_{A'})^2}{k_A k_{A'}}\left(z_{A'} - \frac{k(k_A + k_{A'})}{k_{A'}^2} z_A\right) + \frac{2\Lambda^2(k_A + k_{A'})^3}{k_A^2 . k_{A'}^2} z_A z_{A'}\right.
$$
$$
\left. \cdot \left(z_{A'} - z_A \cdot \frac{k . k_{A'}^2 + k_A^2 m_{A'}^2}{m_{A'}^2 k_{A'}(k_A + k_{A'})}\right)\right\} W'(\widehat{x_{A'}^2}).W''(\widehat{x_A^2})
$$

$$
\overset{(2)}{L}_{AA'} = \left\{\frac{(k_A + k_{A'})^2 h^2}{m_A^2 k_A k_{A'}^2} + \frac{(k_A + k_{A'})[kk_A k_{A'}(k_A + k_{A'}) + \Lambda^2(m_A^2 - m_{A'}^2)(2k_A + k_{A'})]}{m_A^2 m_{A'}^2 k_A^2 k_{A'}^2} z_A z_{A'}\right.
$$
$$
\left. - \frac{(k_A + k_{A'})^2 z_{A'}^2}{m_A^2 k_A k_{A'}}\right\} W'(\widehat{x_A^2}).W'(\widehat{x_{A'}^2}) + \left\{\frac{2k(k_A + k_{A'})^3 h^2}{m_A^4 k_{A'}^4} z_A^2\right.
$$
$$
- \frac{2h^2(k_A + k_{A'})^2}{m_A^2 k_{A'}^2} z_A z_{A'} + \frac{2\Lambda^2(k_A + k_{A'})^2[kk_{A'}(k_A + k_{A'}) - \Lambda^2 k_A]}{m_A^2 m_{A'}^2 k_A k_{A'}^4} z_A^3 z_{A'}
$$
$$
\left. - \frac{2\Lambda^2(k_A + k_{A'})^3}{m_A^2 k_A k_{A'}^3} z_A^2 z_{A'}^2\right\} W'(\widehat{x_A^2}).W''(\widehat{x_A^2})
$$

for equal masses we can write

$$
\overset{(2)}{A}_{A} = 4(z_{A'} - z_A)W'(\widehat{x_A^2})W'(\widehat{x_{A'}^2}) + 8W'(\widehat{x_{A'}^2})W''(\widehat{x_A^2})
$$
$$
\times \left\{h^2 z_{A'} - \frac{2kh^2 z_A}{k + m^2} - \Lambda^2\frac{z_A z_{A'}}{m^2} + 2(k - m^2)z_A z_{A'}\right\}
$$

$$
\overset{(2)}{L}_{AA'} = \frac{4}{m^2}\left(\frac{h^2}{k + m^2} - z_{A'}^2 + \frac{kz_A z_{A'}}{m^2}\right)W'(\widehat{x_A^2})W'(\widehat{x_{A'}^2})
$$
$$
+ \frac{8z_A}{m^2}\left\{\frac{2kh^2 z_A}{k + m^2} - h^2 z_{A'} + \frac{\Lambda^2 z_A^2 z_{A'}}{m^2} - 2(k - m^2)z_A z_{A'}^2\right\}W'(\widehat{x_{A'}^2})W''(\widehat{x_A^2})
$$

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*(Manuscrit reçu le 14 avril 1980).*