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De Sitter-invariant field equations (*)

by

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ABSTRACT. — Field equations, of the type considered by Naimark [*1*] and Gel'fand-Minlos-Shapiro [*2*] in euclidean and Minkowski spaces, are explicitly written in De Sitter space. The conditions for their invariance, with respect to the De Sitter group, are derived. In the limit of vanishing curvature the results agree with the known results in the flat case.

1. INTRODUCTION

In Special Relativity a free field is represented by a vector function $\psi(x^i)$, with values in a suitable representation space $\mathbb{R}$ of the homogeneous Lorentz group $\mathscr{L}$, satisfying an equation of the form

$$\text{(1)} \qquad L^{i\alpha}_{\beta} \frac{\partial \psi^{\beta}}{\partial x^i} + i\chi\psi^{\alpha} = 0\,(^1),$$

$(x^i) = (x^1, x^2, x^3, x^4)$ being cartesian coordinates in Minkowski space $\mathscr{M}$, $L^{i\alpha}_{\beta}(i = 1, 2, 3, 4)$ a set of four linear operators associated with the representation ρ of $\mathscr{L}$ acting on $\mathbb{R}$, χ a fixed constant and $i = \sqrt{-1}$. Equation (1)

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(1) The Einstein summation convention on indices will be understood. Indices i, j, k, ... will refer to cartesian coordinates in flat space or to orthonormal tetrads in curved space ; the indices a, b, c, ... will refer to local coordinates in curved space. Greek indices refer to a basis in $\mathbb{R}$.

is required to be Lorentz-invariant, in the sense that its form remains unchanged under the combined effect of a space-time coordinate transformation $x^i \rightarrow x^{i'} = \Lambda^{i'}_i x^i + a^{i'}$ and a change of basis such that

$$\psi^\alpha \rightarrow \psi^{\alpha'} = \rho(\Lambda)^{\alpha'}_\alpha \psi^\alpha$$

in $\mathbb{R}$, where Λ is a homogeneous Lorentz transformation and $\rho(\Lambda)$ is its representative in ρ.

The conditions for Lorentz-invariance (cf. [*1*] and [*2*]) are given by:

$$[\mathrm{L}^{i\varepsilon}_\beta, \mathrm{A}^{[rs]\alpha}_\varepsilon] = \eta^{ir}\mathrm{L}^{s\alpha}_\beta - \eta^{is}\mathrm{L}^{r\alpha}_\beta \quad (2)$$

(2)

where $\eta^{ij} = \begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ 0 & & 1 & \\ & & & -1 \end{bmatrix}$ is the metric tensor in Minkowski space, $\mathrm{A}^{[rs]\alpha}_\beta$ are the components of the infinitesimal generators of $\mathscr{L}$ in the representation ρ, and

$$[\mathrm{L}^{i\varepsilon}_\beta, \mathrm{A}^{[rs]\alpha}_\varepsilon] = \mathrm{L}^{i\varepsilon}_\beta \mathrm{A}^{[rs]\alpha}_\varepsilon - \mathrm{A}^{[rs]\varepsilon}_\beta \mathrm{L}^{i\alpha}_\varepsilon$$

denotes the commutator of the matrices L^i and $\mathrm{A}^{[rs]}$.

The extension of the concept of invariance of field equations of type (1) to flat space with higher dimension and arbitrary signature is straightforward, and the corresponding conditions (2) are analogous. At contrary the generalization to curved spaces requires the preliminary extension of the equation via the operators of covariant differentiation of fields associated with arbitrary (possibly infinite-dimensional) representations of the Lorentz group [*3*] and the determination of the conditions which ensure the invariance of the generalized equation with respect to the whole isometry group admitted by the manifold [*4*].

To the Dirac's pionier work [*5*], many other follow. We recall in particular [*6*] in which equations of Klein-Gordon type are obtained as eigenvalue equations of the Casimir operators for spin 0, $\frac{1}{2}$ and 1.

After a summary of the results of [*4*] which are relevant in this connection, we shall explicitly write the equations which correspond to (1) in De Sitter space, following a different approach of that given in [*6*], and we derive the conditions which must satisfied for their invariance under the whole De Sitter group.

2. FIELD EQUATIONS AND INVARIANCE

Let $\mathscr{V}_4$ be a riemannian manifold of dimension 4 with signature + 2, $(x^a) = (x^1, x^2, x^3, x^4)$ a local coordinate system with domain $\mathscr{U}$, $\{ e_i \}$ a tetrad field.

(2) The simbol $[rs]$ denotes a pair of indices r and s with $r < s$.

In correspondence with any representation ρ of the homogeneous Lorentz group $\mathscr{L}$ acting on the representation space $\mathbb{R}$ one can define a « Lorentz field » by associating a vector $\psi(P)$ of a space R_P isomorphic with $\mathbb{R}$ to every point $P \in \mathscr{U}$; see [3].

On assumption of suitable smoothness of tetrads, we will consider field equations of the form

$$L^{i\alpha}_{\beta}\nabla_{e_i}\psi^{\beta} + i\chi\psi^{\alpha} = 0\,, \tag{3}$$

where $L^{i\alpha}_{\beta} = L^{i\alpha}_{\beta}(P)$ represents a field of four linear operators acting on R_P and ∇_{e_i} denotes the covariant differentiation in the direction e_i.

Let w be a vector field and φ_t the local one-parameter group of transformations generated by w. If, as we do not exclude, the linear representation ρ of the pseudo-orthogonal group $\mathscr{L}$ cannot be extended to the full linear group, in order to be able to give a good definition of invariance it is necessary that φ_t be an isometry, i. e. that w be a Killing vector :

$$(\nabla_{e_i}w)_h + (\nabla_{e_h}w)_i = 0\,. \tag{4}$$

Under these assumptions, if, we interpret φ_t, as it is possible, as a change of the anholonomous reference frame and not as a point transformation in $\mathscr{V}_4$, we can say that equations (3) are « invariant with respect to the local group φ_t » if they remain form-invariant under the combined application of the Lorentz transformation Λ which maps the old tetrad $\{ e_i(P) \}$ on the new one $\{ e_{i'}(P) \}$ in each tangent space $\mathscr{C}_P$, and of the corresponding transformation $\rho(\Lambda)$ which maps the old basis $\{ E_\alpha(P) \}$ onto the new one $\{ E_{\alpha'}(P) \}$ in R_P.

3. INVARIANCE CONDITIONS

We can write the matrix $\Lambda^i_{i'}$ connecting $\{ e_i(P) \}$ to $\{ e_{i'}(P) \}$ in the form [3]

$$\Lambda^i_{i'} = \delta^i_{i'} + tM(w)^i_j\delta^j_{i'} + 0(t^2) \tag{5}$$

where $M(w)_{ij}$ is the antisymmetric tensor given by

$$M(w)^i_j = e^i_b\left(w^a\frac{\partial e^b_j}{\partial x^a} - e^a_j\frac{\partial w^b}{\partial x^a}\right), \tag{6}$$

the corresponding matrix $\Lambda^\alpha_{\alpha'}$ connecting $\{ E_\alpha(P) \}$ to $\{ E_{\alpha'}(P) \}$ is expressed by

$$\Lambda^\alpha_{\alpha'} = \delta^\alpha_{\alpha'} + tM(w)_{[rs]}A^{[rs]\alpha}_{\beta}\delta^\beta_{\alpha'} + 0(t^2)\,. \tag{7}$$

The form of equation (3) in the new reference system is

$$L^{i'\alpha'}_{\beta'}\nabla_{e_{i'}}\psi^{\beta'} + i\chi\psi^{\alpha'} = 0$$

or, taking the meaning of $\Lambda^i_{i'}$ and $\Lambda^\alpha_{\alpha'}$ into account :

$$L^{i'\alpha'}_{\beta'}\nabla_{\Lambda^i_{i'}e_i}(\Lambda^{\beta'}_{\beta}\psi^{\beta}) + i\chi(\Lambda^{\alpha'}_{\alpha}\psi^{\alpha}) = 0$$

and therefore

$$L^{i'\alpha'}_{\beta'}\Lambda^{i}_{i'}\Lambda^{\alpha}_{\alpha'}\Lambda^{\beta'}_{\beta}\nabla_{e_i}\psi^{\beta} + i\chi\psi^{\alpha} + L^{i'\alpha'}_{\beta'}\Lambda^{i}_{i'}\Lambda^{\alpha}_{\alpha'}(\nabla_{e_i}\Lambda^{\beta'}_{\beta})\psi^{\beta} = 0 \tag{8}$$

Comparing (8) with (3) we get invariance if the following relations are verified:

$$L^{i\alpha}_{\beta} = L^{i'\alpha'}_{\beta'}\Lambda^{i}_{i'}\Lambda^{\alpha}_{\alpha'}\Lambda^{\beta'}_{\beta} \tag{9}$$

$$L^{i'\alpha'}_{\beta'}\Lambda^{i}_{i'}\Lambda^{\alpha}_{\alpha'}(\nabla_{e_i}\Lambda^{\beta'}_{\beta}) = 0\,. \tag{10}$$

From (9) and (10), differentiating with respect to t and taking (5) and (7) in to account we get, in the limit $t \to 0$:

$$\partial_{w}L^{i\alpha}_{\beta} + [L^{i\varepsilon}_{\beta},\ M(w)_{[rs]}A^{[rs]\alpha}_{\varepsilon}] + M(w)^{i}_{j}L^{j\alpha}_{\beta} = 0 \tag{11}$$

$$L^{i\alpha}_{\varepsilon}\nabla_{e_i}(M(w)_{[rs]}A^{[rs]\varepsilon}_{\beta}) = 0 \tag{12}$$

where $\partial_{w}L^{i\alpha}_{\beta}$ denotes the derivative of $L^{i\alpha}_{\beta}$, regarded as a scalar field, in the direction of w.

If the tetrad field is chosen arbitrarily on a hypersurface intersected by the lines of the vector field w, and at every other neighbouring point the local tetrad is constructed by Lie transport from the points in the hypersurface, the left-hand side of equation (12) vanishes everywhere with $M(w)^{j}_{i}$. On account of its tensor character, equation (12) is therefore identically verified, and the condition (11) alone guarantees the invariance of equation (3). The left-hand side of (11) is just the Lie-derivative of $L^{i\alpha}_{\beta}$ in the direction of w; see [*3*]. On the other hand the invariance of (3) leads to the equality of the matrices L which enter in (9), so that, by using in $\mathscr{C}_{P}$ the method given in [*1*] and [*2*], whe find

$$[L^{i\varepsilon}_{\beta},\ A^{[rs]\alpha}_{\varepsilon}] = \eta^{ir}L^{s\alpha}_{\beta} - \eta^{is}L^{r\alpha}_{\beta}\,,$$

which, in its turn reduces the (11) to $\partial_{w}L^{i\alpha}_{\beta} = 0$.

It follows $L^{i\alpha}_{\beta} = \text{cost}$, provided that there are, on the manifold $\mathscr{V}_4$, four indipendent Killing's fields $w_h (h = 1, 2, 3, 4)$.

Concluding the conditions of invariance of equation (3) with respect to the isometry group produced by the infinitesimal generators w_h are:

$$\begin{cases} L^{i\alpha}_{\beta} = \text{cost} \\ [L^{i\varepsilon}_{\beta},\ A^{[rs]\alpha}_{\varepsilon}] = \eta^{ir}L^{s\alpha}_{\beta} - \eta^{is}L^{r\alpha}_{\beta}\,. \end{cases} \tag{13}$$

4. DE SITTER SPACE AND INVARIANT FIELD EQUATION IN POLAR COORDINATES

De Sitter space can be regarded as a pseudo-sphere S embedded an a flat 5-dimensional space R^5 with signature $+\,3$. If $(x^i) = (x^1, x^2, x^3, x^4, x^5)$

are global cartesian coordinates in $\mathbf{R}^5$, it can be represented by the equation

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 - (x^5)^2 - r^2 = 0 .$$

In the coordinates $(\theta^a) = (\theta^1, \theta^2, \theta^3, \theta^4)$ of polar type, defined by the relations

$$\begin{cases} x^1 = r \operatorname{ch} \theta^4/r \cos \theta^3/r \cos \theta^2/r \cos \theta^1/r \\ x^2 = r \operatorname{ch} \theta^4/r \cos \theta^3/r \cos \theta^2/r \operatorname{sen} \theta^1/r \\ x^3 = r \operatorname{ch} \theta^4/r \cos \theta^3/r \operatorname{sen} \theta^2/r \\ x^4 = r \operatorname{ch} \theta^4/r \operatorname{sen} \theta^3/r \\ x^5 = r \operatorname{sh} \theta^4/r \end{cases}$$

and choosing the following tetrad field

$$\begin{cases} \boldsymbol{e}_1 = (\operatorname{ch} \theta^4/r \cos \theta^3/r \cos \theta^2/r)^{-1} \dfrac{\partial}{\partial \theta^1}, & \boldsymbol{e}_2 = (\operatorname{ch} \theta^4/r \cos \theta^3/r)^{-1} \dfrac{\partial}{\partial \theta^2}, \\ \boldsymbol{e}_3 = (\operatorname{ch} \theta^4/r)^{-1} \dfrac{\partial}{\partial \theta^3}, & \boldsymbol{e}_4 = \dfrac{\partial}{\partial \theta^4}, \end{cases}$$

the corresponding equation for a Lorentz field ψ of arbitrary type writes

$$\mathrm{L}^{i\alpha}_{\beta} \nabla_{\boldsymbol{e}_i} \psi^\beta + i\chi\psi^\alpha = 0 . \tag{14}$$

Since in De Sitter space there are ten Killing's vector fields, (which is the maximum possible) the conditions (13) ensure the invariance of equation (14) with respect to the whole De Sitter group. Therefore the most general De Sitter-invariant equation associated to the field ψ of type under consideration, obtained by expliciting the equation (14), is given by

$$\begin{aligned} & r\left(\mathrm{L}^{1\alpha}_{\beta} \frac{\partial \psi^\beta}{\partial \theta^1} + \mathrm{L}^{2\alpha}_{\beta} \cos \frac{\theta^2}{r} \frac{\partial \psi^\beta}{\partial \theta^2} + \mathrm{L}^{3\alpha}_{\beta} \cos \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \frac{\partial \psi^\beta}{\partial \theta^3} \right. \\ & \qquad\qquad \left. + \mathrm{L}^{4\alpha}_{\beta} \operatorname{ch} \frac{\theta^4}{r} \cos \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \frac{\partial \psi^\beta}{\partial \theta^4} \right) \\ & + \mathrm{L}^{1\alpha}_{\beta} \left(- \operatorname{sen} \frac{\theta^2}{r} \mathrm{A}^{[12]\beta}_{\gamma} \psi^\gamma - \operatorname{sen} \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \mathrm{A}^{[13]\beta}_{\gamma} \psi^\gamma \right. \\ & \qquad\qquad \left. + \operatorname{sh} \frac{\theta^4}{r} \cos \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \mathrm{A}^{[14]\beta}_{\gamma} \psi^\gamma \right) \\ & + \mathrm{L}^{2\alpha}_{\beta} \left(- \operatorname{sen} \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \mathrm{A}^{[23]\beta}_{\gamma} \psi^\gamma + \operatorname{sh} \frac{\theta^4}{r} \cos \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \mathrm{A}^{[24]\beta}_{\gamma} \psi^\gamma \right) \\ & + \mathrm{L}^{3\alpha}_{\beta} \operatorname{sh} \frac{\theta^4}{r} \cos \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \mathrm{A}^{[34]\beta}_{\gamma} \psi^\gamma + i\chi r \operatorname{ch} \frac{\theta^4}{r} \cos \frac{\theta^3}{r} \cos \frac{\theta^2}{r} \psi^\alpha = 0, \end{aligned} \tag{15}$$

with the condition that the operators $\mathrm{L}^{i\alpha}_{\beta}$ satisfy the relations (13). Finally, we conclude by observing that, in the limit $r \to \infty$ the equation (15) tends to its correspondent in the flat space for fields of the same type. Since the conditions (13) are indipendent of the coordinate system, the techniques and the results of [*1*] and [*2*] concerning the determination of the matrix elements can be transported unchanged to the De Sitter case.

REFERENCES

[*1*] M. A. NAIMARK, *Les représentations linéaires du groupe de Lorentz*, Dunod, Paris, 1963.

[*2*] I. M. GEL'FAND, R. A. MINLOS, Z. Ya SHAPIRO, *Representations of the Rotation and Lorentz Groups*, Pergamon Press, London, 1963.

[*3*] T. BRUGARINO, Derivata covariante e derivata di Lie dei campi di Lorentz. *Rend. del Circ. Matem. di Palermo*, Serie II, Tomo **XXIII**, 1974, p. 145-150.

[*4*] T. BRUGARINO, Nozioni e condizioni d'invarianza per equazioni di campo in una varietà curva. *Rend. del. Circ. Matem. di Palermo*, Serie II, Tomo **XXIV**, 1975, p. 151-156.

[*5*] P. A. DIRAC, The electron wave equation in De Sitter Space, *Ann. Math.*, t. **36**, 1935.

[*6*] G. BORNER, H. P. DURR, Classical and Quantum Fields in De Sitter Space. *Nuovo Cimento*, t. **64** A, N, 3, 1969, p. 669-713.

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