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G. PRASAD
B. B. SINHA

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Relativistic magnetofluids and space-like conformal mapping

by

G. PRASAD and B. B. SINHA
Department of Mathematics, Faculty of Science,
B. H. U., Varanasi 221005, India

ABSTRACT. — In this paper we have obtained a set of necessary and sufficient conditions for the existence of a space-like conformal Killing vector collinear to the magnetic field vector. Under these conditions, it has been shown that the square of the magnitude of « magnetic vorticity » vector increases fastly for contracting « fluid shell ».

INTRODUCTION

The search for geometrical symmetries like motions and collineations is essentially motivated by the necessity of discovering the conservation laws in the theory of relativity. Oliver and Davis [*1*] have pointed out that certain space-time symmetry properties play fundamental role in the investigation of local conservation expressions. In addition to conservation expressions, the kinematical and dynamical properties ([*2*]-[*4*]) associated with the time-like congruence are also shared by the symmetry properties like affine motions, conformal motions and motions. Oliver and Davis [*5*] have shown that the vanishing of shear is itself a symmetry condition that partially underlies conformal motion. Such types of interesting consequences of symmetries admitted by the perfect fluid space-times are known to us. But it seems that very little attention has been paid to the space-like symmetry mappings admitted by the magnetofluid space-times whereas the study of relativistic magnetohydrodynamics (RMHD) initiated and developed by

Lichnerowicz [6] is important in several contexts ([7]-[9]) viz. cosmology, the late stages of stellar collapse and the pulsar theory.

Ciubotariu [10] has obtained a relativistic generalization of Ferraro's law under the assumption that the magnetofluid space-time admits a time-like killing vector collinear to the fluid flow vector. A relativistic analogue of this law has been obtained by Mason [11] using the assumption that the magnetofluid space-time admits a space-like killing vector collinear to the magnetic field vector. The existence of such a space-like killing vector essentially motivates for further study of more general symmetry mappings and their physical implications. The present authors [12] have recently studied the space-like motions admitted by magnetofluid space-times.

The purpose of this paper is to study the space-like conformal motions admitted by the magnetofluid space-times. In particular, we shall obtain a set of necessary and sufficient conditions for the existence of a space-like conformal killing vector collinear to the magnetic field vector. Further, we shall establish a theorem which shed light on the behaviour of « magnetic vorticity » (as defined in [16]).

1. KINEMATICAL PARAMETERS

In this section we mention briefly the theory of kinematical parameters associated with the congruences of fluid flow lines and magnetic field lines. We assume that the signature of space-time is $(+ - - -)$. The kinematical properties of the fluid flow lines are characterized by the usual decomposition for the rate of change of the fluid velocity vector u^i [13]

$$u_{i;j} = \sigma_{ij} + \omega_{ij} + \theta\gamma_{ij} + \mathrm{D}u_i, u_j, \tag{1.1}$$

where σ_{ij}, ω_{ij} and θ denote the shear, the rotation and the scalar expansion of the congruence of fluid flow lines respectively. D stands for the directional derivative along the fluid flow line and γ_{ij} is the projection operator onto the 3-space orthogonal to u^i.

The covariant derivative of the 4-vector n^i tangential to the congruence of magnetic field lines is decomposed according to Greenberg [14] as follows:

$$n_{i;j} = \overset{*}{\sigma}_{ij} + \overset{*}{\omega}_{ij} + \overset{*}{\theta}\overset{*}{\gamma}_{ij} + \mathrm{D}^*n_in_j - \mathrm{D}n_iu_j - (\mathrm{D}n_ku^k)u_iu_j + (\mathrm{D}^*n_ku^k)u_in_j + n_{k;j}u^ku_i \tag{1.2}$$

where n_i denotes unit magnetic field vector. The shear, rotation and expansion of the congruence of magnetic field lines are denoted by $\overset{*}{\sigma}_{ij}$, $\overset{*}{\omega}_{ij}$ and $\overset{*}{\theta}$ respectively. D* represents the directional derivative along the magnetic field line. $\overset{*}{\gamma}_{ij}$ is Greenberg's projection tensor.

2. SPACE-LIKE CONFORMAL KILLING VECTOR

A space-like conformal motion is defined by the relation [*15*]

$$\mathfrak{L}_\xi g_{ij} = \psi g_{ij}, \tag{2.1}$$

where ξ^i is a space-like vector and ψ a non-zero scalar function. The vector ξ^i satisfying (2.1) is called conformal Killing vector. In this section we shall state and prove two theorems giving an important information related to the congruence of the magnetic field lines when the magnetofluid space-time exhibits a space-like conformal motion.

THEOREM (2.1). — A magnetofluid space-time admits a conformal Killing vector collinear to the magnetic field vector iff *i)* the congruence of the magnetic field lines is shear- free, *ii)* $\overset{*}{\theta} = u^k \mathrm{D} n_k$, *iii)* $(\ln \varphi), i = \mathrm{D}^* n_i - \overset{*}{\theta} n_i$ and *iv)* $\mathrm{D} n_i = 2(u^k \mathrm{D} n_k) u_i - (u^k \mathrm{D}^* n_k) n_i - n_{k;i} u^k$.

Proof. — Let us assume that the magnetofluid space-time admits a conformal Killing vector collinear to the unit magnetic field vector n^i, i. e. (2.1) holds for $\xi^i = \varphi n^i$. The condition (2.1) may equivalently be written as

$$\varphi_{,i} n_j + \varphi_{,j} n_i + \varphi(n_{i;j} + n_{j;i}) = \psi g_{ij} \tag{2.2}$$

Contracting (2.2) with $n^i n^j$, $u^i u^j$, $u^i n^j$ and g^{ij}, we get

$$\psi = 2\mathrm{D}^* \varphi, \tag{2.3}$$

$$\psi = 2\varphi u^i \mathrm{D} u_i, \tag{2.4}$$

$$\mathrm{D}(\ln \varphi) = u^k \mathrm{D}^* n_k \tag{2.5}$$

and

$$\mathrm{D}^* \varphi + \varphi n^i_{;i} = 2\psi \tag{2.6}$$

respectively. Combining (2.4), (2.6) and the definition [*14*],

$$2\overset{*}{\theta} = n^i_{;i} - u^i \mathrm{D} n_i, \tag{2.7}$$

we get

$$\psi = 2\overset{*}{\theta}\varphi \tag{2.8}$$

Splitting up (2.2) orthogonal to u^i and n^i, and making use of (1.2), we have

$$2\overset{*}{\sigma}_{ij} = (\psi/\varphi - 2\overset{*}{\theta})\overset{*}{\gamma}_{ij}. \tag{2.9}$$

By means of (2.8) and (2.9), we obtain the first necessary condition stated in the theorem. On account of (2.4) and (2.8) we get second condition. Contracting (2.2) with $\overset{*}{\gamma}{}^i_k n^j$ and using (2.5), we get

$$(\ln \varphi), k = \mathrm{D}^* n_k - \overset{*}{\theta} n_k \tag{2.10}$$

which is the third required condition. Again contracting (2.2) with $\gamma^i_k u^j$ and using (2.5), we get

$$Dn_k + (u^i D^* n_i) n_k - 2(u^i D n_i) u_k + n_{i;k} u^i = 0 \qquad (2.11)$$

which is the fourth required condition. Conversely, we shall obtain (2.1) for $\xi^i = \varphi n^i$ assuming that the conditions *i*), *ii*), *iii*) and *iv*) hold. For this purpose, we will write

$$\mathfrak{L}_\xi g_{ij} = \varphi_{,i} n_j + \varphi_{,j} n_i + \varphi(n_{i;j} + n_{j-i}) \qquad (2.12)$$

Using condition *iii*) in (2.12), we obtain

$$\varphi^{-1} \mathfrak{L}_\xi g_{ij} = (D^* n_i n_j + D^* n_j n_i) - 2\overset{*}{\theta} n_i n_j + (n_{i;j} + n_{j;i}) \qquad (2.13)$$

Using (1.2) in (2.13), we get

$$\varphi^{-1} \mathscr{L}_\xi g_{ij} = 2\overset{*}{\sigma}_{ij} + 2\overset{**}{\theta} \gamma_{ij} + (Dn_i u_j + Dn_j n_i) - 2(u^k D n_k) u_i u_j + (u^k D^* n_k)(u_i n_j + u_j n_i) + n_{k;i} u^k u_j + n_{k;j} u^k u_i. \qquad (2.14)$$

Setting $\psi = 2\overset{*}{\theta}\varphi$ and making use of the conditions *i*), *ii*), *iii*) and *iv*), we observe that the right hand side of (2.14) is equivalent to $\varphi^{-1}\psi g_{ij}$. This completes the proof of the theorem (2.1).

THEOREM (2.2). — If a magnetofluid space-time admits a space-like conformal Killing vector collinear to the magnetic field vector then $D^*(\overset{*}{\omega}{}^2 A^*) = 0$, where $\overset{*}{\omega}{}^2$ is the square of the magnitude of the « magnetic vorticity » vector and A^* the proper area subtended by the magnetic field lines as they pass through the screen in the two surface dual to the « Maxwellian surface » [*16*] formed by the magnetic field lines and the fluid flow lines.

Proof. — Let us write

$$\mathfrak{L}_n \overset{*}{\omega}_{ij} = \overset{*}{\omega}_{ij;k} n^k + \overset{*}{\omega}_{kj} n^k_{;i} + \overset{*}{\omega}_{ik} n^k_{;j} \qquad (2.15)$$

Using (1.2) in (2.15), we get

$$\mathfrak{L}_n \overset{*}{\omega}_{ij} = D^* \overset{*}{\omega}_{ij} - 2\overset{*}{\omega}_{k[i} \overset{*}{\sigma}_{j]}{}^k + 2\overset{**}{\theta} \omega_{ij} + 2(D^* n^k) \overset{*}{\omega}_{k[i} n_{j]} - 2(Dn^k) \overset{*}{\omega}_{k[i} u_{j]}, \qquad (2.16)$$

where the square bracket around the indices denotes antisymmetrizationl. Now we write

$$\overset{*}{\perp} D^* \overset{*}{\omega}_{ij} \equiv \overset{*}{\gamma}{}^k_i \overset{*}{\gamma}{}^l_j D^* \overset{*}{\omega}_{kl} = D^* \overset{*}{\omega}_{ij} + 2(D^* n^k) \overset{*}{\omega}_{k[i} n_{j]} - 2(Dn^k) \overset{*}{\omega}_{k[i} u_{j]} \qquad (2.17)$$

Combining (2.16) and (2.17), we obtain

$$\mathfrak{L}_n \overset{*}{\omega}_{ij} = \overset{*}{\perp} \mathrm{D}^* \overset{*}{\omega}_{ij} + 2\overset{**}{\theta}\omega_{ij} - 2\overset{*}{\omega}_{k[i}\overset{*}{\sigma}_{j]}{}^k \tag{2.18}$$

In accordance with our assumption that the vector $\xi^i = \varphi n^i$ is a conformal. Killing vector, the conditions *i*), *ii*), *iii*) and *iv*) of the theorem (2.1) are true. Thus using *iii*) and an identity giving the propagation equation for « magnetic rotation » tensor $\overset{*}{\omega}_{ij}$ [16].

$$\begin{aligned}\overset{**}{\mathrm{D}\omega}_{ij} = {} & (\mathrm{D}^* n_{[i})_{;j]} - 2\overset{**}{\theta\omega}_{ij} + 2\overset{*}{\omega}_{k[i}\overset{*}{\sigma}_{j]}{}^k - u_{[i}\overset{*}{\mathrm{A}}_{j]k}\overset{*}{\mathrm{D}}n^k + n^l \mathrm{A}_{l[i}\mathrm{D}n_{j]} \\ & + n^l \mathrm{A}_{lk}\mathrm{A}^{*k}{}_{.[i}u_{j]} - n_{[i}\mathrm{A}^*{}_{j]k}\mathrm{D}^* n^k \\ & + \mathrm{A}_{lk}n^l \mathrm{D}^* n^k u_{[i}n_{j]} + (n_l \mathrm{D}u^l)\mathrm{D}n_{[i}u_{j]} \\ & - (\mathrm{D}^*\mathrm{D}n_{[i})u_{j]} + \mathrm{D}n_{[i}\mathrm{D}^* u_{j]} + (\mathrm{D}^{*2}n_{[i})n_{j]} \\ & + n^k \mathrm{D}^* \mathrm{A}_{k[i}u_{j]} - \mathrm{D}^* n^k \mathrm{A}_{k[i}u_{j]} + n^k \mathrm{A}_{k[i}\mathrm{D}^* u_{j]} \\ & + n_{[i}u_{j]}\mathrm{D}^*(n_k \mathrm{D}^* n^k) + (u_k \mathrm{D}^* n^k)\mathrm{D}^* n_{[i}u_{j]} \\ & + (u_k \mathrm{D}^* n^k)n_{[i}\mathrm{D}^* u_{j]},\end{aligned} \tag{2.19}$$

where

$$\overset{*}{\mathrm{A}}_{ik} \equiv \overset{*}{\sigma}_{ik} + \overset{*}{\omega}_{ik} + \overset{**}{\theta}\gamma_{ik} \,; \qquad \mathrm{A}_{ik} \equiv \sigma_{ik} + \omega_{ik} + \theta\gamma_{ik},$$

and the Greenberg transport law [14], we finally obtain

$$\overset{*}{\perp}\mathrm{D}^*\overset{*}{\omega}_{ij} \equiv \overset{*}{\perp}(\mathrm{D}^* n_{[i})_{;j]} - 2\overset{**}{\theta\omega}_{ij} + 2\overset{*}{\omega}_{k[i}\overset{*}{\sigma}_{j]}{}^k \tag{2.20}$$

after some manipulation. The integrability condition for *iii*) of theorem (3.1) leads to

$$\overset{*}{\perp}(\mathrm{D}n_{[i})_{;j]} = \overset{**}{\theta\omega}_{ij}. \tag{2.21}$$

By means of (2.18), (2.20) and (2.21), we get

$$\mathfrak{L}_n \overset{*}{\omega}_{ij} = \overset{**}{\theta\omega}_{ij}. \tag{2.22}$$

Combining (2.18), (2.22), the condition *i*) of the theorem (2.1) and the definition [14], $2\overset{*}{\theta} = \frac{1}{\mathrm{A}^*}\mathrm{D}^*\mathrm{A}^*$, we get

$$\mathrm{D}^*(\mathrm{A}^*\overset{*}{\omega}{}^2) = 0 \tag{2.23}$$

which proves the statement made in the theorem (2.2).

For physical interpretation of this theorem, we define « fluid shell » as a 2-dimensional surface constituted by fluid particles through which the magnetic flux is passing normally. This definition ensures that the « fluid shell » is orthogonal to the « Maxwellian surface » and A* represents its area element. The « fluid shell » contracts or expands according as the area element A* decreases or increases. Thus from (2.23), we conclude that the square of the magnitude of « magnetic vorticity » vector increases fastly for contracting « fluid shell ».

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