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Binding of Schrödinger Particles Through Conspiracy of Potential Wells

by

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ABSTRACT. — We study the ground state energy $E(R)$ for

$$-\Delta + V(x) + W(R - x)$$

when V and W are negative with compact support. In particular, in dimension 3, when $-\Delta + V$ and $-\Delta + W$ both have no bound states but both have zero energy resonances, we prove that $E(R) \sim -\beta R^{-2}$ for R large with $\beta = .321651512\dots$

In this note we want to discuss some properties of the ground state energy, $E(R)$, of the Schrödinger operator on $L^2(\mathbb{R}^v)$

$$-\Delta + V(x) + W(R - x)$$

where V and W have compact support and lie in L^p ($p = \frac{v}{2}$ for $v \geq 3$, $p = 1$ for $v = 1$, $p > 1$ for $v = 2$) and

$$R \equiv |R| > R_0 = \sup \{ |x + y| \mid x \in \text{supp } V, y \in \text{supp } W \}$$

so that $V(x)$ and $W(R - x)$ have disjoint supports. Our first result is (all proofs deferred until later):

THEOREM 1. — Let V, W be negative. In the region $R > R_0$, $|E(R)|$ decreases as R increases, i. e.

$$(R \cdot \nabla_R E) \geq 0. \tag{1}$$

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Remarks. — 1. This is to be compared with the results of Lieb-Simon [2] who prove (1) when V and W are spherically symmetric and increasing but without the restriction of disjoint supports.

2. It is fairly obvious that this will not be true if V and W are sometime positive. For example, if $v = 1$ and V consists of a negative well and W a positive well, then $E(\underline{R}) > E(\infty)$.

Our remaining results are only of interest in $v \geq 3$ dimensions and concern a rather specialized situation. Our interest was stimulated by work of I. Sigal [4] on the Effimov effort who found the results we describe below for $V = W$ spherical potentials. Our proofs in addition to being more general have some degree of greater simplicity and elegance.

DEFINITION. — A potential q on $\mathbb{R}^v$ (in $L^p(\mathbb{R}^v)$ as above) is called *subcritical* if and only if $-\Delta + \lambda q \geq 0$ for $0 \leq \lambda \leq 1 + \varepsilon$. It is called *critical* if and only if $-\Delta + q \geq 0$ but $-\Delta + \lambda q$ has a negative eigenvalue for any $\lambda > 1$. It is called *supercritical* if $-\Delta + q$ has negative eigenvalues.

THEOREM 2. — Let $v \geq 3$. If V and W are both subcritical, then $E(\underline{R}) = 0$ for R sufficiently large.

Remark. — There is an alternative proof [5] of this fact using hitting probabilities for Brownian paths and one that yields fairly explicit lower bounds on how large R needs to be. This proof depends on the fact [5] that q is subcritical if and only if

$$\sup_t \left\| \exp(-t(-\Delta + q)) \right\|_{\infty, \infty} < \infty$$

where $\|\cdot\|_{\infty, \infty}$ is the norm as a map from L^∞ to L^∞ .

THEOREM 3. — Let $v = 3$. If V is subcritical and W is critical, then $E(\underline{R}) = O(R^{-(v-2)})$ at infinity.

THEOREM 4. — Let $v = 3$. If V and W are both negative and critical, then $R^2 E(\underline{R}) \rightarrow -\beta$ as $R \rightarrow \infty$ where $\beta = \alpha^2$ and α is the unique solution of

$$e^{-\alpha} = \alpha \tag{2}$$

Remarks. — 1. The fixed point (2) is easily seen to be stable so that α can be computed by iteration easily on a calculator. 24 iterations on an SR-56 leads to the stable value $\alpha = .5671432904$ and $\beta = .321651512\dots$

2. If $v \geq 3$, $E(\underline{R})R^{2(v-2)}$ has a limit but unlike the case $v = 3$, the limit is V and W dependent and *not* universal.

3. The R^{-2} falloff and the related fact that thus $-(2M)^{-1}\Delta_R + E(R)$ will have an infinity of bound states for suitable M are critical to Sigal's proof of the Effimov effect [4].

THEOREM 5. — If either V or W is supercritical then $E(\infty) = \lim_{R \rightarrow \infty} E(\underline{R})$ exists and $E(\underline{R}) - E(\infty) = o(e^{-aR})$ for suitable $a > 0$.

Remarks. — 1. In fact, $E(\infty) = \min(\inf \sigma(-\Delta + V), \inf \sigma(-\Delta + W))$.
 2. Using the methods of [3], one easily obtains that $E(\underline{R}) - E(\infty) = o(R^n)$ for all n .

We now turn to the method of proof of these results. The same method of proof has been used by one of us [1] to analyze the question of defining self-adjoint Dirac Hamiltonians where one has potentials with several singularities.

For simplicity, we suppose that V and W are non-positive, treating the more general case in remarks following the formal proofs. The basic fact that we exploit is that for $q \leq 0$ in L^p , the ground state energy $E(q)$ of $-\Delta + q$ is determined by the condition that $K_q \equiv |q|^{1/2}(-\Delta - E)^{-1}|q|^{1/2}$ have norm 1; equivalently since K_q is a positive compact operator, 1 is its (simple) largest eigenvalue; equivalently since K_q has a positive integral kernel, it has a pointwise, non-negative eigenvector with eigenvalue 1.

Now if $K_q \eta = \eta$ and $q(\underline{x}) = V(\underline{x}) + W(\underline{R} - \underline{x})$, then $\eta = \tilde{\eta}_1 + \tilde{\eta}_2$ with η_1 having support in $\text{supp } V$ and η_2 in support of $W(\underline{R} - \underline{x})$. If V and $W(\underline{R} - \underline{x})$ has disjoint supports, then this decomposition is unique. Writing $\eta(x) = \eta_1(\underline{x}) + \eta_2(\underline{R} - \underline{x})$ we see that $K_q \eta = \eta$ is equivalent to $L\Phi = \Phi$ where Φ is the two-component vector $\Phi = (\eta_1, \eta_2)$ and L is the two-by-two matrix operator with integral kernel:

$$L = \begin{pmatrix} |V(\underline{x})|^{1/2} G_0(\underline{x} - \underline{y}; E) |V(\underline{y})|^{1/2} & |V(\underline{x})|^{1/2} G_0(\underline{x} + \underline{y} - \underline{R}; E) |W(\underline{y})|^{1/2} \\ |W(\underline{x})|^{1/2} G_0(\underline{x} + \underline{y} - \underline{R}; E) |V(\underline{y})|^{1/2} & |W(\underline{x})|^{1/2} G_0(\underline{x} - \underline{y}; E) |W(\underline{y})|^{1/2} \end{pmatrix}$$

where $G_0(\underline{x} - \underline{y}, E)$ is the kernel of $(-\Delta - E)^{-1}$.

To summarize, $E(\underline{R})$ is determined in the region $E(\underline{R}) < 0$ by the condition $\|L(E, \underline{R})\| = 1$. Since K and hence L is monotone decreasing as E decreases, we see that if $\|L(E_0, \underline{R})\| \leq 1$ (resp ≥ 1), then $E(\underline{R}) \geq E_0$ (resp $\leq E_0$).

Proof of Theorem 1. — Since $\underline{R} \geq \underline{R}_0$, for each $\underline{x}, \underline{y}$ with $\underline{x} \in \text{supp } V$, $\underline{y} \in \text{supp } W$, $G_0(\underline{x} + \underline{y} - \lambda \underline{R}, E) < G_0(\underline{x} + \underline{y} - \underline{R}, E)$ for any $E < 0$ and any $\lambda > 1$. It follows that, for any $\eta \geq 0$, ($\eta \neq 0$),

$$(\eta, L(E, \lambda \underline{R})\eta) < (\eta, L(E, \underline{R})\eta) \quad (3)$$

so, since L has a positive integral kernel, $\|L(E, \lambda \underline{R})\| \leq \|L(E, \underline{R})\|$ proving the result.

Remark. — By the strict inequality in (3) and the compactness of L , we have actually proven that $E(\lambda \underline{R}) > E(\underline{R})$ for $\underline{R} \geq \underline{R}_0$, $\lambda > 1$ and $E(\underline{R}) < 0$.

Proof of Theorem 2. — Write $L = L_D + L_0$ with L_D diagonal and L_0 off diagonal. Since $G(x, 0) = c|x|^{-(v-2)}$ and $V, W \in L^1$,

$$\|L_0(0, \underline{R})\|_{\text{HS}} \leq C|\underline{R} - \underline{R}_0|^{-(v-2)} \quad \text{for } \underline{R} > \underline{R}_0.$$

Since V, W , are subcritical, $\|L_D(0, \underline{R})\| < 1$ ($L_D(0, \underline{R})$ is $\underline{R}$ independent). Thus, for $\underline{R} \geq [C(1 - \|L_D\|)^{-1}]^{1/(v-2)} + \underline{R}_0$ we have that $E(\underline{R}) = 0$.

Remark. — If $\|L\|$ and $\|L_D\|$ (but not $\|L_0\|_{HS}$) are replaced by $\max \sigma(L)$ and $\max \sigma(L_D)$, the proof extends to the case where V and W are not necessarily negative.

Proof of Theorem 3. — Make the decomposition $L = L_D + L_0$ as in the proof of Theorem 2. $L_D(0)$ has 1 as a simple discrete eigenvalue by hypothesis and all other eigenvalues are strictly smaller. Write

$$L(E, R) = L_D(0) + \delta L$$

where $\delta L = [L_D(E) - L_D(0)] + L_0(E, R) \equiv \delta L_1 + \delta L_2$. As above, for $R > R_0$, $\|L_0(E, R)\| \leq CR^{-(v-2)}$ independently of E . Using $E = k^2$:

$$G_0(\tilde{x} - \tilde{y}, E) - G_0(\tilde{x} - \tilde{y}, 0) = c_1 k |\tilde{x} - \tilde{y}|^{-(v-3)} + O(k^2 |\tilde{x} - \tilde{y}|^{-(v-4)})$$

we see that $\|\delta L_1 - kA_1\| \leq Dk^2$ with A_1 the 2×2 matrix operator which is zero off-diagonal and $c_1 V^{1/2} |\tilde{x} - \tilde{y}|^{-(v-3)} V^{1/2}$ and $C_1 W^{1/2} |\tilde{x} - \tilde{y}|^{-(v-3)} W^{1/2}$ on-diagonal.

We now use perturbation theory. The largest eigenvalue $\lambda_0(E, R)$ of $L(E, R)$ is determined by

$$\int_{|\lambda-1|=\varepsilon} (\Phi, (L(E, R) - \lambda)^{-1} \Phi) \lambda d\lambda = \lambda_0 \int (\Phi, (L(E, R) - \lambda)^{-1} d\lambda \quad (4)$$

where $\Phi = (\eta, 0)$ is the normalized vector with $L_D(0)\Phi = \Phi$. Expanding

$$\begin{aligned} (L(E, R) - \lambda)^{-1} &= (L_D(0) - \lambda)^{-1} - (L_D(0) - \lambda)^{-1} \delta L (L_D(0) - \lambda)^{-1} \\ &\quad + (L_D(0) - \lambda)^{-1} \delta L (L_D(0) - \lambda)^{-1} \delta L (L(E, R) - \lambda)^{-1} \end{aligned}$$

(4) becomes:

$$1 + (\eta, \delta L_1^{(11)} \eta) + O(k^2) + O(R^{-2(v-2)}) = \lambda_0(1 + O(k^2) + O(R^{-2(v-2)}))$$

Since $(\eta, \delta L_1^{(11)} \eta) = ck + O(k^2)$ with $c \neq 0$, the condition $\lambda_0 = 1$ becomes $k = O(R^{-2(v-2)})$ or $E = O(R^{-4(v-2)})$.

Remark. — By carrying on the calculations explicitly to second order, one can show that $ER^{4(v-2)}$ converges to an explicit V, W dependent constant as $R \rightarrow \infty$.

Proof of Theorem 4. — For simplicity, consider first the case $V = W$. Then L leaves the subspace $\{\Phi = (\eta, \pm \eta)\}$ invariant. The largest eigenvalue of L is on the (η, η) subspace. On this subspace, 1 is a simple discrete eigenvalue of $L_D(0)$. Using first order as above we obtain the equation:

$$1 + |(\eta, W^{1/2})|^2 (4\pi)^{-1} [-k + e^{-kR}/R] + O(k^2) + O(R^{-2}) + O(k/R) = 1 + O(k^2) + O(R^{-2})$$

Since $\eta > 0$, $(\eta, W^{1/2}) \neq 0$ and thus

$$k = e^{-kR}/R + O(k^2) + O(R^{-2}) \quad (5)$$

so $kR \rightarrow \alpha_0$ and $-k^2 = +E \sim -\alpha_0^2/R^2$.

For the general case, $V \neq W$, $L_D(0)$ has 1 as a degenerate eigenvalue.

So we need to use degenerate perturbation theory. The first order terms then become:

$$(4\pi)^{-1} \begin{pmatrix} -ka^2 & R^{-1}e^{-kR}ab \\ R^{-1}e^{-kR}ab & -kb^2 \end{pmatrix} = F$$

where $a = (\eta, |V|^{1/2})$, $b = (\tilde{\eta}, |W|^{1/2})$ with $\eta(\tilde{\eta})$ the normalized eigenvalue of $|V|^{1/2}G_0|V|^{1/2}$ (resp. $|W|^{1/2}G_0|W|^{1/2}$). The condition that F have a zero eigenvalue is $\det F = 0$ or using $a, b \neq 0$, $k = e^{-kR}/R$. Thus (5) still holds.

Remark. — If $v > 3$, and $V = W$ (for simplicity only), then the first order terms are

$$-kc \int (\eta |V|^{1/2})(\underline{x}) | \underline{x} - \underline{y} |^{-(v-3)} (\eta |V|^{1/2})(\underline{y}) + (\eta, |V|^{1/2})^2 G_0(R, k^2)$$

Since $G_0(R, k^2) \leq dR^{-(v-2)}$, we see that $kR \rightarrow 0$ and thus $G_0(R, k^2) \rightarrow dR^{-(v-2)}$ so that we get $E = -k^2 \sim a^2 R^{-2(v-2)}$ with a explicitly V dependent.

Proof of Theorem 5. — This follows the proof of Theorem 3, except that since one of V, W is supercritical, the off diagonal terms are $O(e^{-aR})$.

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