

ANNALES DE L'I. H. P., SECTION A

P. VANHEUVERZWIJN

Generators for quasi-free completely positive semi-groups

Annales de l'I. H. P., section A, tome 29, n° 1 (1978), p. 123-138

http://www.numdam.org/item?id=AIHPA_1978__29_1_123_0

© Gauthier-Villars, 1978, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Generators for quasi-free completely positive semi-groups

by

P. VANHEUVERZWIJN (*)

Instituut voor Theoretische Fysica.
Universiteit Leuven. B-3030 Leuven, Belgium

ABSTRACT. — We construct quasi-free completely positive (CP) semi-groups on the CCR-C*-algebra, show that they can be extended, in certain representations, to a dynamical semi-group on the associated von Neumann algebra and determine the infinitesimal generator.

RÉSUMÉ. — Nous construisons les semi-groupes complètement positifs et quasi-libres sur la C*-algèbre de relations de commutation.

Ces sémi-groupes pouvant être étendus à l'algèbre de von Neumann associée à certaines représentations, on détermine le générateur infinitésimal.

1. INTRODUCTION

In the algebraic approach to non-equilibrium statistical mechanics, it is generally assumed that the dynamics of an open system, idealized as a C*- or a von Neumann algebra, is given by means of a one parameter semi-group of completely positive maps on the algebra [1] [10]. In case this semi-group extends to a group of *-automorphisms, the system is called conservative, if not the system is called dissipative.

In this paper we study a particular class of dynamical systems, namely quasi-free boson systems. Our algebra will be the CCR-C* algebra $\Delta(H, \sigma)$

(*) Aspirant NFWO, Belgium.

build over a symplectic space (H, σ) . [2] [3], while the CP maps will be of « quasi-free » type.

These CP maps were introduced in [4] [5]. The full class of these maps was characterized in [6]. Some further results, concerning extremality, dilation, implementation and relaxation, were obtained in [7].

It is clear that semi-groups of quasi-free CP maps cannot be strongly continuous, as they map Weyl-operators into Weyl-operators. We show however (theorem 4.4 below) that in certain representations, determined by quasi-free states [8], the semi-group may be extended to a so-called dynamical semi-group [9] [10] on the von Neumann algebra generated by the representation.

As an ultraweakly continuous semi-group of normal maps on a von Neumann algebra, there exists a densely defined and closed generator. We obtain this generator explicitly in theorems 4.7, 9 below. Formally it is of the Lindblad type [10].

The characterization of unbounded generators of dynamical semi-groups being far from complete, the results obtained here should contain some information on the structure of these generators.

The paper is organized as follows :

In § 2 we gather some results on symplectic spaces, operators and semi-groups on it. We briefly recall the definition of the CCR-C* algebra $\overline{\Delta(H, \sigma)}$. In § 3 we construct the class of quasi-free CP semi-groups. § 4 shows, the extension of the semi-group to certain associated von Neumann algebras, is possible. Finally the explicit form of the generator is obtained. If moreover we ask for the existence of a separating vector in the representation space, the domain of the generator is fully determined.

For the general theory of semi-groups and their generators we refer to [13] [14] [15]. For a treatment of quasi-free semi-groups on the CAR algebra, see [21].

2. SYMPLECTIC SPACES AND THE CCR ALGEBRA

The one particle space (H, σ) is a real symplectic space, i. e.

- i) H is a real linear, possibly infinite dimensional, space.
- ii) σ is a real, bilinear antisymmetric and non degenerated form, defined on H .

On H , we define the topology, induced by the family of seminorms $\{p_\phi\}$,

$$p_\phi(\psi) = |\sigma(\phi, \psi)|.$$

The resulting locally convex space is Hausdorff. We call this topology the s -topology.

Given any continuous operator $T: H \rightarrow H$, a unique operator T^+ is

defined through the formula $\sigma(Tx, y) = \sigma(x, T^+y)$. A complex structure is an operator $J: H \rightarrow H$, such that $J^+ = -J$, and $J^2 = -1$.

A symplectic base of H , is a set of vectors $\{f_i, g_i\}_{i=1, \dots}$ such that

- i) $\{f_i, g_i\}$ generate H (we suppose H separable).
- ii) $\sigma(f_i, g_j) = \delta_{ij} \quad \forall i, j$
- iii) $\sigma(f_i, f_j) = \sigma(g_i, g_j) = 0 \quad \forall i, j$

We suppose $\{f_i, g_i\}$ is ordered as follows: $\{f_1, g_1, f_2, g_2, f_3, \dots\}$ and denoted as $\{e_1, e_2, e_3, \dots\}$. Then defining

$$J: H \rightarrow H$$

by

$$\begin{aligned} J e_{2k+1} &= e_{2k+2} & k &= 0, 1, \dots \\ J e_{2k+2} &= -e_{2k+1} & k &= 0, 1, \dots \end{aligned}$$

it is easily checked that J extends to a complex structure.

The following formulas hold

$$\forall \phi \in H: \quad \phi = \sum_k \sigma(\phi, J e_k) e_k \tag{1}$$

$$\forall \psi, \phi \in H: \quad \sigma(\phi, \psi) = \sum_k \sigma(\phi, J e_k) \sigma(e_k, \psi) \tag{2}$$

DEFINITION 2.1. [8]. — \mathcal{Q} is the set of all operators $Q: H \rightarrow H$ s. t.

- i) $s_Q(\psi, \phi) \equiv -\sigma(Q\psi, \phi)$ defines a real scalar product on H .
- ii) $Q^*Q \geq 1$ where $*$ and \geq are taken with respect to s_Q .

It follows then that $Q^* = -Q = Q^+$, hence Q is bounded for the s_Q -norm topology on H ; moreover Q is invertible.

Let \bar{H}^Q denote the completion of H for the s_Q norm topology. The following properties are well known. Suppose H is sequentially s -complete, then $\forall Q \in \mathcal{Q}$, H is s_Q -norm complete. [8] Conversely if H is not sequentially s -complete, then \bar{H}^Q is sequentially s -complete [12], II cor. 29. Henceforth we will suppose H is sequentially s -complete.

DEFINITION 2.2. — A continuous semi-group on H is a 1-parameter family of s -continuous, everywhere defined, operators $A_t, t \in \mathbb{R}^+$ s. t.

- i) $A_0 = 1$
- ii) $A_t A_s = A_{t+s}$
- iii) the map $t \rightarrow A_t$ is s -continuous.

As H is sequentially s -complete, by the closed graph theorem, $\{A_t\}$ is a strongly continuous semi-group on H , equipped with the s_Q -norm topology, whenever $Q \in \mathcal{Q}$.

The infinitesimal generator Z of A_t is defined by

$$Z\psi = \lim_{t \rightarrow 0} \frac{1}{t} (A_t - 1)\psi \quad \text{for any } \psi \in H$$

such that the limit exists (in any of the topologies). When H is sequentially s -complete, Z is s & s_Q -norm densely defined. Moreover Z is s_Q -norm closed.

In the sequel we make use of the following properties $((+), (+ +))$.

i) Let A_t be a continuous semi-group in H , Z its generator, then for all $\psi \in \mathcal{D}(Z)$ the map $t \mapsto ZA_t\psi$ is s -continuous. Hence $t \mapsto \sigma(ZA_t\psi, A_t\psi)$ is continuous for all $\psi \in \mathcal{D}(Z)$ [13] [14].

ii) Let A_t be continuous, then for fixed ψ , and finite s , the set $\{A_t\psi \mid t \in [0, s]\}$ is contained in a finite dimensional subspace of H .

The CCR C^* -algebra $\Delta(H, \sigma)$ ([2] [3]) is the C^* -algebra obtained by completing the $*$ -algebra $\overline{\Delta(H, \sigma)}$ generated by the Weyl elements δ_ψ , $\psi \in H$, satisfying

$$\begin{aligned} \delta_\psi \delta_\phi &= e^{-i\sigma(\psi, \phi)} \delta_{\psi + \phi} \\ (\delta_\psi)^* &= \delta_{-\psi} \end{aligned}$$

We refer to [3] for the exact definition of the norm with respect to which the completion is to be taken.

We recall that ([8]) any $Q \in \mathcal{Q}$ determines a quasi-free state on $\overline{\Delta(H, \sigma)}$ through the formula

$$\omega_Q(\delta_\psi) = e^{1/2\sigma(Q\psi, \psi)}.$$

3. QUASI-FREE CP SEMIGROUPS

Let A be any operator on H .

Denoting $\sigma_A(\psi, \phi) \equiv \sigma(\psi, \phi) - \sigma(A\psi, A\phi)$. It was shown in [6], that the map

$$\begin{aligned} \tau : \Delta(H, \sigma) &\rightarrow \Delta(H, \sigma) \\ \tau(\delta_\psi) &= \delta_{A\psi} f(\psi) \end{aligned} \tag{3}$$

f being a functional on H , such that $f(0) = 1$, extends to a CP map on $\overline{\Delta(H, \sigma)}$ iff ω , defined by

$$\omega(\delta_\psi) = f(\psi)$$

extends to a state on the C algebra $\overline{\Delta(H, \sigma_A)}$.

Imposing some regularity conditions, the general form of semi-groups, consisting of CP maps of type (3), was exhibited in [7]. In the following we suppose $f(\psi)$ to be the generating functional of a quasi-free state on $\overline{\Delta(H, \sigma_A)}$.

THEOREM 3.1. — Let $\tau_t : \overline{\Delta(H, \sigma)} \rightarrow \overline{\Delta(H, \sigma)}$ be a one parameter semi-group of quasi-free CP maps, i. e.

$$\tau_t(\delta_\psi) = \delta_{A_t\psi} f(\psi) \tag{4}$$

such that A_t is continuous and for all ψ the map $t \mapsto f_t(\psi)$ is differentiable.

Suppose $f_t(\psi)$ is the *generating* functional of a quasi-free state on $\overline{\Delta(H, \sigma_{A_t})}$, then it is of the form

$$f_t(\psi) = \exp [\sigma(B_t\psi, \psi)] \tag{5}$$

where

$$B_t = \int_0^t A_x^+ Y A_x dx \tag{6}$$

Here Y satisfies :

- i) Y is uniquely and everywhere defined
- ii) $Y^+ = -Y$ is bounded for all s_Q -norm topologies.
- iii) $\sigma(Y\psi, \psi) \leq 0 \quad \forall \psi$
- iv) $\forall \psi, \phi \in \mathcal{D}(Z)$

$$|\sigma(Z\psi, \phi) + \sigma(\psi, Z\phi)| \leq -\frac{1}{2} [\sigma(Y\psi, \psi) + \sigma(Y\phi, \phi)] \tag{7}$$

Conversely, any continuous semi-group $\{A_t\}$ and operator Y with the above properties, define a CP semi-group through the formulas (4) (5) (6).

Proof. — As $t \rightarrow f_t(\psi)$ is differentiable for all ψ and as $B_t^+ = -B_t$, we obtain that the map $t \rightarrow \sigma(B_t\psi, \phi)$ is differentiable for all ψ, ϕ in H .

On the other hand, as Q is invertible when $Q \in \mathcal{Q}$

$$\forall t \exists B_t^Q \text{ s. t. } B_t = Q B_t^Q$$

then $B_t^Q = B_t^{Q*}$ where the adjoint is taken w. r. t. s_Q . As B_t^Q is everywhere defined, B_t^Q is bounded for the s_Q -norm topology. We have that

$$\lim_{t \rightarrow 0} \frac{1}{t} [\sigma(Q B_t^Q \psi, \phi) - \sigma(Q \psi, \phi)] = - \lim_{t \rightarrow 0} \frac{1}{t} [s_Q(B_t^Q \psi, \phi) - s_Q(\psi, \phi)]$$

exists for all ψ and ϕ in H . Thus there is an s_Q -bounded operator Y^Q such that

$$\left. \frac{d}{dt} s_Q(B_t^Q \psi, \phi) \right|_{t=0} = s_Q(Y^Q \psi, \phi)$$

Defining $Y = Q Y^Q$ we obtain i) and ii). Using Prop. 4.2 in [7] we obtain

$$\begin{aligned} f_t(\psi) &= \exp \left[\int_0^t dx \left(\frac{d}{dt}, f_v(A_x \psi) \Big|_{t'=0} \right) \right] \\ &= \exp \left[\int_0^t dx \sigma(Y A_x \psi, A_x \psi) \right]. \end{aligned}$$

To show iii) we note that $f_t(\psi)$ is the generating functional of a state on a C^* -algebra and that as such $|f_t(\psi)| = |\omega_t(\delta_\psi)| \leq 1$ hence, for all t , $\sigma(B_t\psi, \psi) \leq 0$. On the other hand, $\tau_0 = 1$, and we have $\sigma(B_0\psi, \psi) = 0$ thus

$$\left. \frac{d}{dt} \sigma(B_t\psi, \psi) \right|_{t=0} \leq 0,$$

which is by definition $\sigma(Y\psi, \psi) \leq 0$.

Finally we express that

$$f_t(\psi) = \exp \left[\int_0^t dx \sigma(YA_x\psi, A_x\psi) \right]$$

defines a state on $\overline{\Delta(H, \sigma_{A_t})}$. That $f_t(\psi)$ generates a state, implies ([8])

$$|\sigma_{A_t}(\psi, \phi)| \leq -\frac{1}{2}[\sigma(B_t\psi, \psi) + \sigma(B_t\phi, \phi)];$$

Noting once more that the equality is reached for $t = 0$, we derive for $\psi, \phi \in \mathcal{D}(Z)$

$$|\sigma(Z\psi, \phi) + \sigma(\psi, Z\phi)| \leq -\frac{1}{2}[\sigma(Y\psi, \psi) + \sigma(Y\phi, \phi)]$$

Conversely, suppose $\{A_t\}$ is a continuous semi-group on H , and Y is an operator on H enjoying properties *i*), *ii*), *iii*) and *iv*).

Taking ψ, ϕ in $\mathcal{D}(Z)$, by *iv*) we obtain for all $x \geq 0$

$$\begin{aligned} -\sigma(ZA_x\psi, A_x\phi) - \sigma(A_x\psi, ZA_x\phi) \\ \leq -\frac{1}{2}[\sigma(YA_x\psi, A_x\psi) + \sigma(YA_x\phi, A_x\phi)] \quad (8) \end{aligned}$$

The left and right hand sides of the inequality are integrable on any bounded interval by property (+); integrating (8) yields

$$\begin{aligned} -\int_0^s [\sigma(ZA_x\psi, A_x\phi)dx + \sigma(A_x\psi, ZA_x\phi)dx] \\ \leq -\frac{1}{2} \int_0^s [\sigma(YA_x\psi, A_x\psi) + \sigma(YA_x\phi, A_x\phi)]dx \end{aligned}$$

Hence

$$\sigma(\psi, \phi) - \sigma(A_t\psi, A_t\phi) \leq -\frac{1}{2}[\sigma(B_t\psi, \psi) + \sigma(B_t\phi, \phi)]$$

using the other inequality we arrive at (7). This together with *i*) *ii*) and *iii*) imply that $f_t(\psi)$ defines a state on $\Delta(H, \sigma_{A_t})$. That τ_t defines a semi-group on $\overline{\Delta(H, \sigma)}$ follows now from [7] prop. 4.2. ■

EXAMPLE 3.2. — [4] [16]. — Let $H = \mathbb{R}^2$

$$\sigma((x, y), (x', y')) = \frac{1}{2}(xy' - yx')$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A_t z = e^{-\lambda t} z \quad \lambda > 0 \quad t \geq 0 \quad z \in \mathbb{R}^2$$

$$f_t(z) = \exp \left[-\frac{\theta}{4}(1 - e^{-2\lambda t}) \|z\|^2 \right], \quad \theta \geq 1$$

then $f_t(z)$ is of the form (6) with

$$Y = -\frac{\lambda\theta}{2}J$$

At the risk of being confusing we now introduce

DEFINITION 3.3. — Any semi-group $\{\tau_t\}$ as in theorem 3.1 will be called a quasi-free CP semi-group.

4. GENERATORS OF QUASI-FREE CP SEMIGROUPS

DEFINITION 4.1 [10]. — A one parameter semi-group $\{\tau_t\}$ on a von Neumann algebra \mathcal{M} is called a dynamical semi-group whenever

- i) $\forall t \tau_t$ is a CP map $\mathcal{M} \rightarrow \mathcal{M}$
- ii) $\tau_t(1) = 1$ for any t
- iii) $\tau_0 = 1$
- iv) $\forall t \tau_t$ is an ultraweakly continuous map
- v) $t \rightarrow \tau_t(x)$ is ultraweakly continuous.

It follows that $\{\tau_t\}$ is a contraction semi-group.

If $\{\tau_t\}$ is a dynamical semi-group, there exists an ultraweakly dense set in \mathcal{M} , called $\mathcal{D}(\mathcal{L})$ such that for all $x \in \mathcal{D}(\mathcal{L})$ the

$$\lim_{t \rightarrow 0} \frac{\tau_t(x) - x}{t} \quad \text{exists in ultraweak (u. w.)-sense.}$$

The limit is called $\mathcal{L}(x)$; it is again an element of \mathcal{M} . \mathcal{L} is called the generator of $\{\tau_t\}$. It can be shown that \mathcal{L} is uw-closed [13] [15].

Using the methods of [11] [13] [15] the following can be shown.

THEOREM 4.2. — Let τ_t be a dynamical semi-group on \mathcal{M} . Then

1. $\forall x \in \mathcal{M}, \forall s$, the element $x_s = \int_0^s \tau_t(x)dt$ is well defined and $x_s \in \mathcal{D}(\mathcal{L})$

Moreover $\tau_s(x) - x = \mathcal{L}(x_s)$

2. Let \mathcal{C} be an u. w. dense set, $\mathcal{C} \subseteq \mathcal{D}(\mathcal{L})$, s. t.

$$\tau_t(\mathcal{C}) \subseteq \mathcal{C}, \quad \text{then } \mathcal{C} \text{ is a core for } \mathcal{L}. \quad \blacksquare$$

Let ω_Q be any quasi-free state on $\overline{\Delta(\mathbb{H}, \sigma)}$, and τ_t a quasi-free CP semi-group, then $\forall t, \tau_t^*(\omega_Q)$ is again quasi-free and

$$\tau_t^* \cdot \omega_Q = \omega_{A_t^* Q A_t + B_t}$$

We now introduce our main hypothesis in order to ensure condition *iv*) of definition 4.1 is satisfied when τ_t is considered in a quasi-free representation of the CCR.

DEFINITION 4.3. — A quasi-free state ω_Q is said to be approximately τ_t -invariant, iff $\forall t \geq 0$

$$Q - A_t^+ Q A_t - B_t \tag{9}$$

is an s_Q -trace class operator.

THEOREM 4.4. — Let ω_Q a quasi-free state, τ_t a quasi-free semi-group on $\overline{\Delta(H, \sigma)}$. If H is sequentially s -complete, and ω_Q is approximately τ_t invariant then τ_t extends uniquely to a dynamical semi-group on

$$\Pi_{\omega_Q}(\overline{\Delta(H, \sigma)})'' \equiv \mathcal{M}.$$

Proof (For convenience we denote $\Pi_{\omega_Q}(x)$ as x . — ω_Q and $\tau_t^*(\omega_Q)$ are quasi equivalent. Indeed, as

$$H = \bar{H}^Q = \bar{H}^{A_t^+ Q A_t + B_t}$$

both ω_Q and $\tau_t^*(\omega_Q)$ are factor states [8] and the condition ensures that they are quasi-equivalent [17].

We now show that for any $\omega \in \mathcal{M}_*^+$, the positive part in the predual of \mathcal{M} , $\tau_t^*(\omega)$ is again in \mathcal{M}_*^+ .

Denote by τ_χ the gauge automorphism $\overline{\Delta(H, \sigma)} \rightarrow \overline{\Delta(H, \sigma)}$ defined by $\tau_\chi(x) = \delta_\chi x \delta_{-\chi}$, $\chi \in H$ and, given $\omega \in \mathcal{M}_*^+$, define the state ω^χ

$$\omega^\chi = \omega \circ \tau_\chi$$

then $\tau_t^*(\omega^\chi) = (\tau_t^*(\omega_Q))^{A_t^+ \chi}$.

Hence, $\forall \chi \in H$, the state $\tau_t^*(\omega^\chi)$ is quasi-equivalent to ω_Q .

For a general state $\omega \in \mathcal{M}_*^+$, there is a sequence ρ_n where ρ_n is a finite linear combination of states of the form ω^χ such that $\rho_n \rightarrow \omega$ in norm [18].

On the other hand, $\forall t$, the map $\tau_t^* : \overline{\Delta(H, \sigma)}^* \rightarrow \overline{\Delta(H, \sigma)}^*$ as the dual of a normalized positive map on a C^* -algebra, is norm continuous (in fact $\|\tau_t^*\| = 1$)

Thus we obtain $\tau_t^*(\rho_n) \rightarrow \tau_t^*(\omega)$ in norm and as \mathcal{M}_* is norm closed $\tau_t^*(\omega) \in \mathcal{M}_*^+$. This is nothing but saying that for any t , the map $\tau_t : \Pi_\omega(\overline{\Delta(H, \sigma)}) \rightarrow \Pi_{\omega}(\overline{\Delta(H, \sigma)})$ is ultraweakly continuous.

This implies that $\forall t$, τ_t can be extended to an uw-continuous map $\bar{\tau}_t : \mathcal{M} \rightarrow \mathcal{M}$. Using a Kaplansky-type approximation, we show that $\bar{\tau}_t$ is positive.

In the same way we have $\bar{\tau}_t \otimes 1_n : \mathcal{M} \otimes \mathcal{M}_n \rightarrow \mathcal{M} \otimes \mathcal{M}_n$ are positive, such that $\bar{\tau}_t$ is CP.

Remains to show *v*) of definition 4.1, i. e. for all ω in \mathcal{M}_* and for all x

in \mathcal{M} the map $t \mapsto \omega(\tau_t(x))$ is continuous. This is clearly true for $\omega = \omega_Q \cdot \tau_x$ and

$$x = \sum_{i=1}^n \lambda_i \delta \psi_i$$

Using once more the norm density of the linear combinations of the states $\omega_Q \cdot \tau_x$ in \mathcal{M}_*^+ , we obtain the continuity of

$$t \mapsto \omega(\tau_t(x))$$

for all $\omega \in \mathcal{M}_*$, and $x = \sum_{i=1}^n \lambda_i \delta \psi_i$

Finally, for general $x \in \overline{\Pi(\Delta(H, \sigma))}$, there is a sequence $x_n, x_n \in \overline{\Pi(\Delta(H, \sigma))}$

hence $x_n \rightarrow x$ in norm

$$|\omega(\tau_t(x)) - \omega(\tau_t(x_n))| = |(\omega_Q \tau_t)(x - x_n)| \leq \|x - x_n\| < \varepsilon \quad \text{for } n \text{ large enough.}$$

By the uniform convergence $t \mapsto \omega(\tau_t(x))$ is continuous.

Then, using the method of [19], we obtain the continuity for all x in \mathcal{M} . \blacksquare

Let ω_Q be any quasi-free state on $\overline{\Delta(H, \sigma)}$; for all ψ and ϕ in H , the map

$$\lambda \in \mathbb{R} \mapsto \omega_Q(\delta_{\lambda\psi + \phi})$$

is infinitely differentiable. Hence, $\forall \psi$, there exists a selfadjoint operator $B_Q(\psi)$ on \mathcal{H}_{ω_Q} , the GNS space for ω_Q such that

$$\Pi_{\omega_Q}(\delta_{\lambda\psi}) = e^{i\lambda B_Q(\psi)},$$

(Remark that \mathcal{H}_{ω_Q} is separable).

Moreover $\forall \psi, \phi, \eta \in H$ we have

$$i) \quad \Pi_{\omega_Q}(\delta_\psi) \Omega_{\omega_Q} \in \mathcal{D}(B_Q(\phi))$$

here Ω_{ω_Q} denotes the cyclic vector in the GNS space.

$$ii) \quad B_Q(\psi) \Pi_{\omega_Q}(\delta_\phi) \Omega_{\omega_Q} \in (B_Q(\eta)).$$

If $\{e_k\}$ is a symplectic base for H , we will denote $B_Q(e_k)$ by B_k . From now on we drop all indices referring to ω_Q .

The following equalities are easily verified:

$$\begin{aligned} &\langle \Pi(\delta_\phi) \Omega, B_k [B_l, \Pi(\delta_\psi)] \Pi(\delta_\eta) \Omega \rangle \\ &= 2[\sigma(e_k, \psi + \phi + \eta) + i\sigma(Qe_k, -\phi + \psi + \eta)] \\ &\quad \cdot \sigma(e_l, \psi) \langle \pi(\delta_\phi) \Omega, \pi(\delta_\psi) \pi(\delta_\eta) \Omega \rangle \end{aligned} \tag{10}$$

$$\begin{aligned} &\langle \Pi(\delta_\phi) \Omega, [B_k, \Pi(\delta_\psi)] B_l \Pi(\delta_\eta) \Omega \rangle \\ &= 2[-\sigma(e_l, \psi + \phi + \eta) + i\sigma(Qe_l, -\phi + \psi + \eta)] \\ &\quad \cdot \sigma(e_k, \psi) \langle \pi(\delta_\phi) \Omega, \pi(\delta_\psi) \pi(\delta_\eta) \Omega \rangle \end{aligned} \tag{11}$$

LEMMA 4.5. — Let τ_t and ω_Q as in theorem 4.4 Π as above. Then $\forall k, l \in \mathbb{N}$ and $\forall \psi, \phi \in H$, and for all finite s the elements

$$\int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \tag{12}$$

$$\int_0^s \Pi(\tau_t(\delta_\psi))B_l\Pi(\delta_\phi)\Omega dt \tag{13}$$

are well defined and belong to respectively $\mathcal{D}(B_k B_l)$ and $\mathcal{D}(B_k)$.

Proof (for (12)). — It is easily checked that the maps

$$t \rightarrow \|\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega\|$$

and

$$t \rightarrow \|B_l\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega\|$$

are continuous; thus the elements

$$\int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt$$

and

$$\int_0^s B_l\Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt$$

exist in Bochner sense.

By the continuity of $t \mapsto \tau_t$, $\int_0^s \Pi(\tau_t(\delta_\psi))dt$ exists as an operator (in ultraweak sense).

Let $\xi \in \mathcal{D}(B_l)$, then

$$\begin{aligned} \langle B_l \xi, \int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \rangle &= \int_0^s \langle B_l \xi, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega \rangle dt \\ &= \int_0^s \langle \xi, B_l \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega \rangle dt \\ &= \langle \xi, \int_0^s B_l \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \rangle \end{aligned}$$

Thus, since $B_l = B_l^*$, $\int_0^s \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \in \mathcal{D}(B_l)$. In the same way we prove that

$$\int_0^s B_l \Pi(\tau_t(\delta_\psi))\Pi(\delta_\phi)\Omega dt \text{ is in } \mathcal{D}(B_k). \quad \blacksquare$$

PROPOSITION 4.6. — Let H be finite dimensional, τ_t a CP quasi-free semi-group; ω_Q a quasi-free state on $\overline{\Delta(H, \sigma)}$.

Then there exist real sequences

$$\begin{aligned} \{ a_{lk} \} & \quad k, l = 1, \dots, 2N \\ \{ b_{lk} \} & \quad k, l = 1, \dots, 2N \end{aligned}$$

such that $\forall x \in \mathcal{D}(\mathcal{L})$ and all ξ, η in a dense set \mathcal{D} , one has

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle = & -\frac{i}{4} \sum_{k,l} a_{lk} [\langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle] \\ & + a_{kl} [\langle B_k \xi, x B_l \eta \rangle - \langle \xi, x B_k B_l \eta \rangle] \\ & - \frac{1}{4} \sum_{k,l} b_{lk} [\langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle] \\ & - b_{kl} [\langle B_k \xi, x B_l \eta \rangle - \langle \xi, x B_k B_l \eta \rangle] \end{aligned} \tag{12}$$

Moreover there is a core \mathcal{C} for \mathcal{L} , such that $\forall x \in \mathcal{C}$ and $\forall n \in \mathcal{D}$

$$\begin{aligned} \mathcal{L}(x)\eta = & -\frac{i}{4} \sum_{k,l} (a_{kl} B_k [B_l, x]\eta + a_{kl} [B_k, x] B_l \eta) \\ & - \frac{1}{4} \sum_{k,l} (b_{kl} B_k [B_l, x]\eta - b_{kl} [B_k, x] B_l \eta). \end{aligned} \tag{13}$$

Proof. — Let \mathcal{D} be the linear span of $\{ \Pi(\delta_\psi)\Omega \mid \psi \in H \}$ in \mathcal{H} . Denote by \mathcal{C} the linear span of $\left\{ \int_0^s \Pi(\tau_t(\delta_\psi)) dt \mid s < \infty, \psi \in H \right\}$ in \mathcal{M} .

Then, by theorem 4.2, \mathcal{C} is a core for \mathcal{L} .
Then define (e_k being a symplectic base)

$$\begin{aligned} a_{kl} &= \sigma(Je_k, Z^+ J e_l) \\ b_{kl} &= \sigma(Je_k, Y J e_l) \end{aligned}$$

By theorem 4.2 we know

$$\mathcal{L} \left(\int_0^s \Pi(\tau_t(\delta_\psi)) dt \right) = \Pi(\tau_s(\delta_\psi)) - \Pi(\delta_\psi)$$

on the other hand, the function

$$t \mapsto \frac{d}{dt} \langle \Pi(\delta_\phi)\Omega, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\eta)\Omega \rangle$$

being continuous,

$$\langle \Pi(\delta_\phi)\Omega, \Pi(\tau_s(\delta_\psi) - \delta_\psi)\Pi(\delta_\eta)\Omega \rangle = \int_0^s \frac{d}{dt} [\langle \Pi(\delta_\phi)\Omega, \Pi(\tau_t(\delta_\psi))\Pi(\delta_\eta)\Omega \rangle] dt.$$

For notational convenience, we put $\phi = \eta = 0$.

$$\left\langle \Omega, \mathcal{L} \left[\int_0^s \Pi(\tau_t(\delta_\psi)) dt \right] \Omega \right\rangle = \int_0^s dt \omega_Q(\tau_t(\delta_\psi)) [-\sigma(ZA_t\psi, QA_t\psi) + \sigma(YA_t\psi, A_t\psi)]$$

which by (1) equals

$$= \int_0^s dt \omega_Q(\tau_t(\delta_\psi)) \sum_{k,l} [\sigma(A_t\psi, Je_l)\sigma(Ze_l, Je_k)\sigma(Qe_k, A_t\psi) - \sigma(A_t\psi, Je_l)\sigma(e_l, YJe_k)\sigma(e_k, A_t\psi)]$$

substituting (Je_l) for (e_l) and $(-e_l)$ for (Je_l)

$$\begin{aligned} &= - \int_0^s dt \omega_Q(\tau_t(\delta_\psi)) \sum_{k,l} [\sigma(A_t\psi, e_l)\sigma(Ze_l, Je_k)\sigma(Qe_k, A_t\psi) - \sigma(A_t\psi, e_l)\sigma(Je_l, YJe_k)\sigma(e_k, A_t\psi)] \\ &= - \frac{i}{2} \int_0^s dt \omega_Q(\tau_t(\delta_\psi)) \sum_{k,l} a_{lk} [\sigma(e_l, A_t\psi)\sigma(e_k, A_t\psi) + i\sigma(Qe_k, A_t\psi)\sigma(e_l, A_t\psi) + a_{kl}[-\sigma(e_k, A_t\psi)\sigma(e_l, A_t\psi) + i\sigma(e_k, A_t\psi)\sigma(Qe_l, A_t\psi)]] \\ &\quad - \frac{1}{2} \int_0^s dt \omega_Q(\tau_t(\delta_\psi)) \sum_{k,l} b_{kl} [\sigma(e_l, A_t\psi)\sigma(e_k, A_t\psi) + i\sigma(Qe_k, A_t\psi)\sigma(e_l, A_t\psi) + b_{lk}[\sigma(e_k, A_t\psi)\sigma(e_l, A_t\psi) - i\sigma(e_k, A_t\psi)\sigma(Qe_l, A_t\psi)]] \end{aligned}$$

by (10) (11) this equals

$$\begin{aligned} &= - \int_0^s dt \left[\frac{i}{4} \sum_{k,l} [a_{lk} \langle \Omega, \mathbf{B}_k [\mathbf{B}_l, \tau_t(\delta_\psi)] \Omega \rangle + a_{kl} \langle \Omega, [\mathbf{B}_k, \tau_t(\delta_\psi)] \mathbf{B}_l \Omega \rangle] \right. \\ &\quad \left. - \frac{1}{4} \sum_{kl} [b_{lk} \langle \Omega, \mathbf{B}_k [\mathbf{B}_l, \tau_t(\delta_\psi)] \Omega \rangle - b_{kl} \langle \Omega, [\mathbf{B}_k, \tau_t(\delta_\psi)] \mathbf{B}_l \Omega \rangle] \right] \quad (14) \end{aligned}$$

as all terms in the sum are integrable, we obtain by making use of lemma 4.5.

$$\begin{aligned} &= - \frac{i}{4} \sum_{kl} \left[a_{lk} \left\langle \Omega, \mathbf{B}_k \left[\mathbf{B}_l, \int_0^s \tau_t(\delta_\psi) dt \right] \Omega \right\rangle + a_{kl} \left\langle \Omega, \left[\mathbf{B}_k, \int_0^s \tau_t(\delta_\psi) dt \right] \mathbf{B}_l \Omega \right\rangle \right] \\ &\quad - \frac{1}{4} \sum_{kl} \left[b_{lk} \left\langle \Omega, \mathbf{B}_k \left[\mathbf{B}_l, \int_0^s \tau_t(\delta_\psi) dt \right] \Omega \right\rangle - b_{kl} \left\langle \Omega, \left[\mathbf{B}_k, \int_0^s \tau_t(\delta_\psi) dt \right] \mathbf{B}_l \Omega \right\rangle \right] \quad (15) \end{aligned}$$

which is (13).

For general x in $\mathcal{D}(\mathcal{L})$, we proceed as follows $\exists x_\alpha \in \mathcal{C}$ such that $x_\alpha \rightarrow x$ u. w. and $\mathcal{L}(x_\alpha) \rightarrow \mathcal{L}(x)$ u. w. thus, for $(\xi, \eta) \in \mathcal{D}$:

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \lim_\alpha [\langle \xi; \mathcal{L}(x_\alpha)\eta \rangle] \\ &= \lim_\alpha \left[\frac{-i}{4} \sum_{kl} a_{lk} [\langle \mathbf{B}_l \mathbf{B}_k \xi, x_\alpha \eta \rangle - \langle \mathbf{B}_k \xi, x_\alpha \mathbf{B}_l \eta \rangle + \dots] \right] \end{aligned}$$

since there isn't but a finite number of terms, this equals

$$= -\frac{i}{4} \sum_{kl} a_{lk} \langle \mathbf{B}_l \mathbf{B}_k \xi, x \eta \rangle - \langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle + \dots$$

which is (12). ■

NOTATION 4.7. — For fixed x in $\mathcal{D}(\mathcal{L})$, the right hand side in (12) defines a bilinear form on \mathcal{D} . We will denote it as

$$\phi_x(\xi, \eta)$$

Let $(H_n)_{n \in \mathbb{N}}$ be an increasing and absorbing net of finite dimensional regular symplectic subspaces of H . Then we define $\mathcal{M}_F \subseteq \mathcal{M}$ as

$$\mathcal{M}_F \equiv \bigcup_{n \in \mathbb{N}} \overline{\Pi(\Delta(H_n, \sigma))}''$$

PROPOSITION 4.8. — Let H be infinite dimensional and sequentially s -complete, ω_Q and τ_t as in theorem 4.4.

Then there exist real infinite sequences $\{a_{lk}\}$, $\{b_{ek}\}$ such that $\forall x \in \mathcal{D}(\mathcal{L}) \cap \mathcal{M}_F$ and all ξ , in a dense set \mathcal{D} one has

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \frac{i}{4} \sum_{k,l} [a_{lk} [\langle \mathbf{B}_l \mathbf{B}_k \xi, x \eta \rangle - \langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle] \\ &\quad + a_{kl} [\langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle - \langle \xi, x \mathbf{B}_k \mathbf{B}_l \eta \rangle]] \\ &\quad - \frac{1}{4} \sum_{k,l} [b_{lk} [\langle \mathbf{B}_l \mathbf{B}_k \xi, x \eta \rangle - \langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle] \\ &\quad - b_{lk} [\langle \mathbf{B}_k \xi, x \mathbf{B}_l \eta \rangle - \langle \xi, x \mathbf{B}_k \mathbf{B}_l \eta \rangle]] \quad (16) \\ &\equiv \phi_x(\xi, \eta) \end{aligned}$$

Moreover there is a core \mathcal{C} for \mathcal{L} , such that $\forall x \in \mathcal{C}$, a formula similar to (13) holds.

Proof. — Define \mathcal{D} as in proposition 4.7, and \mathcal{C} as the linear span of $\left\{ \int_0^s \Pi(\tau_t(\delta_\psi)) dt \mid s < \infty, \psi \in \mathcal{D}(z) \right\}$. Again by theorem 4.2 \mathcal{C} is a core for \mathcal{L} .

Choose a symplectic base in $\{e_k\}$ in $\mathcal{D}(Z^+)$ and define

$$\begin{aligned} a_{kl} &= \sigma(Je_k, Z^+Je_l) \\ b_{kl} &= \sigma(Je_k, YJe_l) \end{aligned}$$

If $\psi \in \mathcal{D}(Z)$, then by property (+) stated in § 2, the map

$$t \mapsto \frac{d}{dt} \omega_Q(\tau_t(\delta_\psi)) = \omega_Q(\tau_t(\delta_\psi))[-\sigma(ZA_t\psi, QA_t\psi) + \sigma(YA_t\psi, A_t\psi)]$$

is continuous.

The proof is then a mere extension of the method in 4. 7. Indeed, by the continuity, we arrive at a formula similar to (14). The elements $\int_0^s \Pi(\tau_t(\delta_\psi))dt$ belong to \mathcal{M}_F , by property (+ +) stated in § 2.

Hence we obtain (15), and thus (13).

For general $x \in \mathcal{D}(\mathcal{L})$, there is a net $\{x_\alpha\} \in \mathcal{C}$ such that $x_\alpha \rightarrow x$ and $\mathcal{L}(x_\alpha) \rightarrow \mathcal{L}(x)$ u. w.

Thus for $\xi, \eta \in \mathcal{D}$

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \lim_\alpha \langle \xi, \mathcal{L}(x_\alpha)\eta \rangle \\ &= \lim_\alpha \phi_{x_\alpha}(\xi, \eta) \end{aligned} \tag{17}$$

If moreover $x \in \mathcal{M}_F$, then for l sufficiently large we obtain e. g.,

$$\langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle = 0.$$

Since any term in (17) is convergent to a term which eventually vanishes, we obtain (16). ■

Formula (16) is of the form

$$\begin{aligned} \langle \xi, \mathcal{L}(x)\eta \rangle &= \lim_{N \rightarrow \infty} \sum_{k,l=1}^N -\frac{i}{4} [a_{lk} \langle B_l B_k \xi, x\eta \rangle - \langle B_k \xi, x B_l \eta \rangle] \\ &\quad + a_{kl} \dots + \dots \\ &\equiv \lim_{N \rightarrow \infty} \phi_x^N(\xi, \eta) \end{aligned}$$

We remark however that in general the bilinear forms ϕ_x^N are not associated to a generator of a quasi-free CP semi-group on $\Pi(\overline{\Delta(H_N, \sigma)})'$, the reason being that when Z generates a semi-group in H , and P_N is the projection onto H_N , $P_N Z P_N$ need not generate a semi-group on H_N , except when H_N is a reducing subspace.

Remark also that if $\sum_{k,l} (a_{lk} + a_{kl}) B_k B_l$ can be given a sense as a self-

adjoint operator, then by re-summing the series, we recover the Lindblad form of the generator [10] [22].

If we impose the condition that \mathcal{M} has a separating vector, then we

can determine $\mathcal{D}(\mathcal{L})$, in the same way as was done in [11] for spatial derivations.

THEOREM 4.9. — Let $H, \omega_Q, \tau, \mathcal{D}$ as above. Suppose moreover that ω_Q extends faithfully to \mathcal{M} .

An element x in \mathcal{M}_F belongs to $\mathcal{D}(\mathcal{L})$ iff the bilinear form

$$\phi_x(\cdot, \cdot) : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{C}$$

has a bounded extension to $\mathcal{H} \times \mathcal{H}$.

Proof. — If $x \in \mathcal{D}(\mathcal{L}) \cap \mathcal{M}_F$, then by proposition 4.7 and 4.9, there is an element $\mathcal{L}(x)$ in \mathcal{M} such that $\phi_x(\xi, \eta) = \langle \xi, \mathcal{L}(x)\eta \rangle$. Hence ϕ_x has a bounded extension.

Let $\phi_x(\cdot, \cdot)$ have a bounded extension, then there is a bounded operator B_x such that for all $\xi, \eta \in \mathcal{D}$

$$\langle \xi, B_x \eta \rangle = \phi_x(\xi, \eta)$$

As $\mathcal{D}(\mathcal{L}) \cap \mathcal{M}_F$ is u. w. dense in \mathcal{M} , there is a net x_α in this set such that

$$x_\alpha \rightarrow x \text{ u. w.}$$

then $\phi_{x_\alpha}(\xi, \eta) \rightarrow \phi_x(\xi, \eta)$ for all $\xi, \eta \in \mathcal{D}$ or $\langle \xi, \mathcal{L}(x_\alpha)\eta \rangle \rightarrow \langle \xi, B_x \eta \rangle$.

As both $\mathcal{L}(x_\alpha)$ and B_x are bounded we have that $\mathcal{L}(x_\alpha) \rightarrow B_x$ weakly, hence $B_x \in \mathcal{M}$.

In case \mathcal{M} has a separating vector, then the weak and ultraweak topologies coincide [20]. Thus we constructed a net x_α in $\mathcal{D}(\mathcal{L})$

$$x_\alpha \rightarrow x \text{ u. w.}$$

and

$$\mathcal{L}(x_\alpha) \rightarrow B_x \text{ u. w.}$$

as \mathcal{L} is u. w. closed, we obtain $x \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}(x) = B_x$. ■

ACKNOWLEDGMENT

It is a pleasure to thank Prof. A. Verbeure for helpful discussions and reading the manuscript.

REFERENCES

[1] G. EMCH, *Comm. Math. Phys.*, t. **49**, 1976, p. 191-215.
 D. EVANS, *Comm. Math. Phys.*, t. **48**, 1976, p. 15-22.
 [2] J. MANUCEAU, *Ann. Inst. Henri Poincaré*, **VIII**, 1968, p. 139-161.
 [3] J. MANUCEAU, M. SIGUGUE, D. TESTARD, A. VERBEURE, *Comm. Math. Phys.*, t. **32**, 1973, p. 231-243.
 [4] E. B. DAVIES, *Comm. Math. Phys.*, t. **27**, 1972, p. 309.
 [5] D. E. EVANS, J. T. LEWIS, *Journ. Funct. Anal.*, t. **26**, 1977, p. 369-377.
 [6] B. DEMOEN, P. VANHEUVERZWIJN, A. VERBEURE, *Lett. Math. Phys.*, t. **2**, 1977, p. 161-166.

- [7] B. DEMOEN, P. VANHEUVERZWIJN, A. VERBEURE, *Preprint KUL-TH-77/008*, to appear in *Rep. Math. Phys.*
- [8] J. MANUCEAU, A. VERBEURE, *Comm. Math. Phys.*, t. **9**, 1968, p. 293.
- [9] A. KOSSAKOWSKI, *Rep. Math. Phys.*, t. **3**, 1972, p. 247-274.
- [10] G. LINDBLAD, *Comm. Math. Phys.*, t. **48**, 1976, p. 119.
- [11] G. BRATTELI, D. ROBINSON, *Ann. Inst. Henri Poincaré*, **XXV**, 1976, p. 139-164.
- [12] N. DUNFORD and J. SCHWARTZ, *Linear Operators*, part 1, Interscience Publishers, N. Y., 1958.
- [13] K. YOSIDA, *Functional Analysis*, Springer, 1968.
- [14] T. KOMURA, *J. Funct. Anal.*, t. **2**, 1968, p. 258-296.
- [15] M. REED, B. SIMON, *Fourier Analysis, Self-Adjointness*, Academic Press, N. Y., 1975.
- [16] G. EMCH, J. ALBEVERIO, J. P. ECKMANN, *preprint Genève*, 1977, p. 1-126.
- [17] A. VAN DAELE, *Comm. Math. Phys.*, t. **21**, 1971, p. 171-191.
- [18] M. FANNES, A. VERBEURE, *Journ. Math. Phys.*, t. **16**, 1975, p. 2086-2088.
- [19] R. KALLMAN, *Comm. Math. Phys.*, t. **14**, 1969, p. 13-14.
- [20] R. KADISON, *Proc. of the 1973 Varenna Summer-School North-Holland*, Amsterdam, 1976.
- [21] E. B. DAVIES, *Comm. Math. Phys.*, t. **55**, 1977, p. 231.
- [22] V. GORINI, A. KOSSAKOWSKI, E. SUDARSHAN, *Journ. Math. Phys.*, t. **17**, 1976, p. 821-825.

(Manuscrit reçu le 12 octobre 1977)