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# Spinor-type fields with linear, affine and general coordinate transformations

by

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ABSTRACT. — We demonstrate the existence of bivalued linear (infinite) spinorial representations of the Group of General Coordinate Transformations. We discuss the topology of the G. G. C. T. and its subgroups GA($n$R), GL($n$, R), SL($n$R) for $n = 2, 3, 4$, and the existence of a double covering.

We demonstrate the construction of the half-integer spin representations in terms of Harish-Chandra modules. We give D. W. Joseph's explicit matrices for $j_0 = \frac{1}{2}$, $c = 0$ in SL(3R), which will act as little group in GA(4R).

## 1. INTRODUCTION AND RESULTS

Einstein's Principle of General Covariance imposes two constraints on the equations of Physics in the presence of gravitational fields :

*a*) a smooth transition to the equations of Special Relativity ; note that we require a formulation of the Equivalence Principle in Field Theory [*1*]. Operationally, « locally, the properties of « special-relativistic » matter in a non-inertial frame of reference cannot be distinguished from the properties of the same matter in a corresponding gravitational field [*2*] ».

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*b*) under a general coordinate transformation $x^\mu \rightarrow \bar{x}^\mu$, the equations are general-covariant, i. e. form preserving.

This article relates to (*b*), i. e. it deals with representations of the Covariance Group $\mathscr{C}$, also known as the Group of General Coordinate Transformations (CGT) or in Mathematical language, the Group of Diffeomorphisms. This group has as a subgroup $\mathscr{G}$ the General Linear Group GL(4R); both are defined over a 4-dimensional real manifold $L_4$. We prove that in addition to conventional tensors (namely tensorial representations of $\mathscr{G}$ and $\mathscr{C}$) *there exist bivalued linear spinorial representations* of these groups, reducing to bivalued representations of the Poincaré group $\mathscr{P}$. These « new » representations are *infinite-dimensional and of discrete type;* e. g. for time-like momenta, they reduce to an infinite sum of « ordinary » $\mathscr{P}$ spinors $\left(\text{e. g. } \frac{1}{2} \oplus \frac{5}{2} \oplus \frac{9}{2} \oplus \ldots\right)$ thus somewhat resembling a band of rotational excitations over a half-integer spin deformed nucleus. We shall therefore use the term band-spinor (or « bandor » ) for these infinite spinor representations, so as to distinguish them from (finite) coventional spinors.

Historically, spinors were « fitted » into General Relativity [*3*] [*4*] after their incorporation into Special Relativity through the Dirac equation. It was noted that they behaved like (holonomic, or « world » ) scalars under $\mathscr{C}$, their spinorial behavior corresponding only to the action of a physically distinct *local* Lorentz group $\mathscr{L}_E$, with

$$\mathscr{C} \cap \mathscr{L}_E = 0 \tag{1.1}$$

Both $\mathscr{L}_E$ and conventional spinors thus required the introduction of a Bundle of Cotangent (or Tangent) Frames E, i. e. an orthonormal set of 1-forms ( « vierbeins »; $a = 0, 1, \ldots, 3$ the « anholonomic » indices)

$$\left.\begin{aligned} e^a &= e^a_\mu(x)dx^\mu \\ e_a &= \eta_{ab}e^b \\ \eta_{ab} &= \text{the Minkowski metric} \end{aligned}\right\} \tag{1.2}$$

with the $L_4$ (general affine) or $U_4$ (Riemann-Cartan) manifold metric given by

$$e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu} \tag{1.3}$$

*Band-spinors are « world » spinors, and thus do not require* E *for their definition.* Contrary to what is stated in most texts on General Relativity, the introduction of E should indeed not be construed as resulting just from the world-scalar behavior of spinors. E represents a further geometrical construction corresponding to the physical constraints of a *local gauge group* of the Yang-Mills type, in which the gauged group is the *isotropy group* of the space-time base manifold. We can thus even introduce band-spinors in the vierbein system [*5*]: the isotropy group would then have to be enlarged from $\mathscr{L}_E$ to $\mathscr{G}_E$, i. e. the theory would have to realize a global General Affine ($\mathscr{F}$) symmetry as its starting point,

$$\mathscr{F} = \mathscr{G} \times \mathscr{I}, \qquad \mathscr{F} \supset \mathscr{P} \tag{1.4}$$

where $\mathscr{I}$ represents the translations. The quotient of $\mathscr{F}$ by $\mathscr{I}$ is $\mathscr{G}_E$ (i. e. $\mathscr{G}$ acting on the anholonomic indices).

Note that one source of the prevalent belief that there are no $\mathscr{C}$ spinors ( « world » spinors) stems from an unwarranted extrapolation from a theorem of E. Cartan [6]:

« It is impossible to introduce spinor fields, the term « spinor » being taken in the classical Riemannian connotations; i. e., given an arbitrary coordinate system $x^\mu$, it is impossible to represent a spinor by any finite number N of components $u_\alpha$, so that these should admit covariant derivatives of the form ($\alpha$, $\beta$ are spinor indices, $\mu\nu$ are vector indices)

$$D_\mu u_\alpha = \partial_\mu u_\alpha + \Gamma_{\mu\alpha}^{\beta}(x) u_\beta \tag{1.5}$$

with the $\Gamma_{\mu\alpha}^{\beta}$ as specific functions of $x$ ».

As can be seen, Cartan was aware of the restriction of his proof to a *finite* number of spinor components. Our band-spinors $\Psi^{\bar\alpha}$ indeed do admit covariant derivatives as in (1.5), in the « world » (holonomic) system,

$$D_\mu \Psi^{\bar\alpha} = \partial_\mu \Psi^{\bar\alpha} + \Gamma_{\mu\nu}^{\cdot\tau}(G_{\cdot\tau}^{\nu})_{\bar\beta}^{\bar\alpha} \Psi^{\bar\beta} \tag{1.6}$$

where $\bar\alpha$ runs over the sets $\alpha$, $\alpha \times \lambda$, $\alpha \times \lambda\rho\sigma$, ..., i. e. spins

$$j = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots,$$

$G_\tau^\nu$ is an infinite dimensional representation of $\mathscr{G}$, and $\Gamma_{\mu\nu}^{\cdot\tau}$ is the usual affine connection.

## 2. TOPOLOGICAL CONSIDERATIONS: THE COVERING GROUP OF SL(nR) AND GL(nR)

We are studying the groups,

$$\mathscr{C} \supset \mathscr{F} \supset \mathscr{G} \supset \mathscr{S} \supset \mathscr{O} \tag{2.1}$$

$$\mathscr{C} \supset \mathscr{F} \supset \mathscr{P} \supset \mathscr{O} \tag{2.2}$$

where $\mathscr{S}$ is the Unimodular Linear SL(4R) and $\mathscr{O}$ is the Special Orthogonal SO(4). We do not enter into the further structure induced by the Minkowski metric at this stage. At various stages we shall also deal with the same groups over $n = 3$ and $n = 2$; we shall then use the notation $\mathscr{G}_3$, $\mathscr{S}_3$, etc.

Since our aim is to find unitary representations of $\mathscr{C}$, $\mathscr{F}$, $\mathscr{G}$, $\mathscr{S}$ which reduce to bivalued unitary representations of $\mathscr{P}$ and $\mathscr{O}$, we have *a priori* two candidate solutions:

$$(a) \qquad \mathscr{S} \supset \bar{\mathscr{O}}, \qquad \mathscr{S}_3 \supset \bar{\mathscr{O}}_3, \qquad \mathscr{F} \supset \bar{\mathscr{P}}$$

$$(b) \qquad \bar{\mathscr{C}} \supset \bar{\mathscr{F}} \supset \bar{\mathscr{G}} \supset \bar{\mathscr{S}} \supset \bar{\mathscr{O}}$$

$$\bar{\mathscr{F}} \supset \bar{\mathscr{P}}$$

where the bars denote double-covering of the relevant groups. In the first case, we would be dealing with single-valued representations of $\mathscr{C}$ and its subgroups, and $\mathscr{O}$ would be contained through its covering $\bar{\mathscr{O}}$. In the second case, all groups would display the same bivaluedness as $\mathscr{O}$, and we would have to go to their respective coverings to find a single-valued representation containing $\bar{\mathscr{O}}$. Since

$$\bar{\mathscr{O}}_3 = \mathrm{SU}(2)$$

it is enough that we show that $\mathscr{S}_3 \not\supset \mathrm{SU}(2)$ to cancel solution (*a*).

We introduce an Iwasawa decomposition [7] of $\mathscr{S}$. For a non compact real simple (all invariant subgroups are discrete and in the center) Lie group $\mathscr{B}$, it is always possible to find

$$\mathscr{B} = \mathscr{K} . \mathscr{A} . \mathscr{N} \tag{2.3}$$

where $\mathscr{K}$ is the maximal compact subgroup, $\mathscr{A}$ is a maximal Abelian subgroup homeomorphic to that of a vector space, $\mathscr{N}$ is a nilpotent subgroup isomorphic to a group of triangular matrices with the identity in the diagonal and zeros everywhere below it. The decomposition is unique and holds globally

$$\mathscr{K} \cap \mathscr{A} = \mathscr{A} \cap \mathscr{N} = \mathscr{N} \cap \mathscr{K} = \{ 1 \} . \tag{2.4}$$

Applying (2.3) to $\mathscr{S}_3$, $\mathscr{K}$ is $\mathscr{O}_3$. Since this is maximal and unique,

$$\mathscr{S}_3 \not\supset \bar{\mathscr{O}}_3$$

and we are left with solution $-$ *b*) only. Applying (2.3) to $\mathscr{S}$,

we also have

$$\mathscr{S} = \mathscr{O}\mathscr{A}_s\mathscr{N}_s \tag{2.5}$$

$$\bar{\mathscr{S}} = \bar{\mathscr{O}}\bar{\mathscr{A}}_s\bar{\mathscr{N}}_s \tag{2.6}$$

Now the groups $\mathscr{A}$ and $\mathscr{N}$ in an Iwasawa decomposition are simply connected, and $\bar{\mathscr{A}}\bar{\mathscr{N}} = \mathscr{A}\mathscr{N}$ is contractible to a point. Thus, *the topology of* $\bar{\mathscr{S}}$ *is that of* $\bar{\mathscr{O}}$. The same result has been shown to hold [8] for $\bar{\mathscr{C}}$ when the $L_4$ is Euclidean or Spherical and holds under some weak conditions for any $L_4$.

By the same token, $\bar{\mathscr{G}}$ has the topology of $\overline{\mathrm{O}(n, \mathrm{R})}$, the double covering of the full Orthogonal (which includes the improper orthogonal matrices, with det $= -1$). $\mathscr{G}$ and $\bar{\mathscr{G}}$ thus have two connected components.

For $n \geqslant 3$, $\mathcal{S}$ is thus completely covered by $\bar{\mathcal{S}}$, the double-covering. However, O(2) and SL(2R) are infinitely connected.

$$\bar{\mathcal{S}}_2 < \bar{\bar{\mathcal{S}}}_2 \tag{2.7}$$

where $\bar{\bar{\mathcal{S}}}_2$ is the full covering.

Topologically, solution (*b*) is thus realizable. The single-valued unitary (and thus infinite-dimensional) irreducible representations of $\bar{\mathcal{S}}$ correspond to double-valued representations of $\mathcal{S}$ and reduce to a sum of double valued representations of $\mathcal{O}$.

This being established, it is interesting to check a second source of confusion at the origin of the statements found in the literature of General Relativity and denying the existence of such double-valued representations. This is based upon an error in the statement of a theorem of E. Cartan [9]:

« The three linear unimodular groups of transformations over 2 variables (SL(2C), SU(2), SL(2R)) admit no linear many-valued representation ».

As can be seen from Cartan's proof of this theorem in ref. [9], it holds only for SL(2C) and SU(2). Moreover, Bargmann [10] has constructed the unitary representations of SL(2R), since this is the double covering Spin $(3)_{(+--)}$ of the 3-Lorentz group $(1, -1, -1)$; and even though only single-valued representations of SL(2R) are required for this role, he has also constructed (§ 7 *d*) multivalued linear representations of that group. The representations

$$C^h_q,\ h = \frac{1}{4},\ q = \frac{1}{4} + s^2 \tag{2.8}$$

are bivalued representations of Spin $(3)_{(+--)} =$ SL(2R) as can be derived from Bargmann's formula

$$U(b) = \exp(4ilh\pi)U(a) \tag{2.9}$$

for two elements lying over the same element of SL(2R). We take $l = 1$.

Note that in reducing SL(4R) to SL(2R), the generators are represented on the coordinates (holonomic variables) by,

$$\Sigma_1 = \frac{i}{2}(x_1\partial_1 - x_2\partial_2),\quad \Sigma_2 = \frac{i}{2}(x_1\partial_2 + x_2\partial_1),\quad \Sigma_3 = -\frac{i}{2}(x_1\partial_2 - x_2\partial_1) \tag{2.10}$$

with commutation relations

$$[\Sigma_1, \Sigma_2] = -i\Sigma_3,\quad [\Sigma_3, \Sigma_1] = i\Sigma_2,\quad [\Sigma_2, \Sigma_3] = i\Sigma_1 \tag{2.11}$$

with $\Sigma_3$ generating the compact subalgebra (eigenvalues $m$ in ref. [10]). However, when using the same algebra as the double-covering [10] of SO(1, 2), the identification in terms of the (completely different) $(1, -1, -1)$ space is given by,

$$\Sigma_1 = i(x'_0\partial'_1 + x'_1\partial'_0),\quad \Sigma_2 = -i(x'_2\partial'_0 + x'_0\partial'_2),\quad \Sigma_3 = i(x'_1\partial'_2 - x'_2\partial'_1) \tag{2.12}$$

with the same commutators and the same role for $\Sigma_3$. We stress this correspondance because it has led to some additional confusion and arguments [11] against the existence of bivalued representations of $\mathcal{S}_2$, and with it $\mathcal{S}_4$.

## 3. THE $\overline{\mathrm{SL(3R)}}$ BAND-SPINORS : EXISTENCE

The unitary infinite-dimensional representations of SL(3R) were introduced [12] in the context of an algebraic description of hadron rotational excitations ( « Regge trajectories [13] » ). A construction was provided ( « ladder representations) for the multiplicity-free $|\Delta j| = 2$ bands, where $j$ is the $\mathcal{O}_3$ spin. Such representations are characterized by $j_0$ (the lowest $j$) and $c$, a real number,

$$\mathcal{D}(\mathcal{S}_3 ; j_0, c) \tag{3.1}$$

the ladder representations corresponding to $j_0 = 0$ and $j_0 = 1$.

We shall not dwell here upon the physical context of shear stresses in extended structures, connected with ref. [12], and we refer the reader to the first part of ref. [5], for that purpose. However, it was a result of this physical context that the author noted with D. W. Joseph the possible existence of similar bivalued representations, i. e. band-spinors. Joseph provided [14] a construction for $\mathcal{D}\left(\bar{\mathcal{S}}_3 ; \frac{1}{2}, 0\right)$ and proved that together with the subsets $\mathcal{D}(\mathcal{S}_3 ; 1, c)$, $\mathcal{D}(\mathcal{S}_3 ; 0, c)$, $-\infty < c < \infty$, this formed the entire set of $|\Delta j| = 2$ multiplicity-free representations. The latter result was recently confirmed by Ogievetsky and Sokachev [15], after having been put in question [16].

We shall provide here a different construction, based upon the « subquotient » theorem for Harish-Chandra modules [17]. We return to the Iwasawa decomposition (2.6) for $\bar{\mathcal{S}}_3$

$$\bar{\mathcal{S}}_3 = \bar{\mathcal{O}}_3 \mathcal{A} \mathcal{N} \tag{3.2}$$

and define $\mathcal{M}_3$, the Centralizer of $\mathcal{A}$ in $\mathcal{K}$, i. e. in $\mathcal{O}_3$. This is the set of all $\sigma \in \mathcal{O}_3$ such that

$$(\sigma \in \mathcal{M}_3 \mid \sigma a \sigma^{-1} = a) \tag{3.3}$$

for any $a \in \mathcal{A}$. The elements of $\mathcal{A}$ span a 3-vector space, and $\mathcal{M}_3$ thus has to be in the diagonal. Since det $(\mathcal{M}_3) = 1$, the elements of $\mathcal{O}_3$ belonging to $\mathcal{M}_3$ are the inversions in the 3 planes: $(+1, -1, -1)$, $(-1, +1, -1)$ and $(-1, -1, +1)$. Together with the identity element they form a group of order 4, with a multiplication table $m_1 m_2 = m_3$, $m_2 m_3 = m_1$, $m_3 m_1 = m_2$, $m_n^2 = 1$. It appears Abelian in this representation.

Returning now to $\bar{\mathcal{S}}_3$ and $\bar{\mathcal{O}}_3$, we look for $\bar{\mathcal{M}}_3 \subset \bar{\mathcal{O}}_3$. The inversions are given in SU(2) by exp $(i\pi\sigma_n/2)$, which yields the Non-Abelian group

$$\bar{\mathcal{M}}_3 \ : (\pm i\sigma_n, \pm 1) . \tag{3.4}$$

The subgroup $\bar{\mathcal{Q}}_3 \subset \bar{\mathcal{S}}_3$

$$\bar{\mathcal{Q}}_3 = \bar{\mathcal{M}}_3 \mathcal{A} \mathcal{N}$$

can now be used to induce the representations of $\bar{\mathcal{S}}_3$. Note that

$$\bar{\mathcal{S}}_3 / \bar{\mathcal{Q}}_3 = \mathrm{SU}(2)/\bar{\mathcal{M}}_3 \,. \tag{3.5}$$

The representations $\rho(j_0, c)$ of $\bar{\mathcal{Q}}_3$ are given by $j_0$ for a representation of the $\bar{\mathcal{M}}_3$ group of « plane inversions » in SU(2), and $\lambda$ for the characters of $\mathcal{A}$, since $\mathcal{N}$ is represented trivially. The representations of $\bar{\mathcal{S}}_3$ will thus be labelled accordingly; from (3.5) we see that they will be spin-valued representations of $\bar{\mathcal{Q}}_3$. Since $\bar{\mathcal{A}}\bar{\mathcal{N}} = \mathcal{A}\mathcal{N}$, univalence is guaranteed.

## 4. THE $\overline{\mathrm{SL}(3\mathrm{R})}$ BAND-SPINORS : CONSTRUCTION OF $\mathcal{D}\left(\frac{1}{2}, 0\right)$

Following our original introduction [12] of infinite-dimensional single-valued representations of SL(3R), we now turn to our algebraic point of view. The five non-compact generators of $\mathcal{S}_3$ are isomorphic to a multiplication of the symmetric $\lambda$ matrices of $su(3)$ by $\sqrt{-1}$, and behave like a $j = 2$ representation under the compact $\mathcal{O}_3$ (the antisymmetric $\lambda$ matrices). They can thus mediate transitions between $|\Delta j| = 2, 1, 0$ levels of the compact subalgebra. In the following analysis we shall deal with a highly degenerate subset: the multiplicity-free $|\Delta j| = 2$ representations.

Although several treatments have appeared since [15] [16] we choose to reproduce the results of D. W. Joseph's unpublished 1970 work [14].

In Joseph's notation, the $\mathcal{S}_3$ generators are chosen to be:

$$\hat{h} := 2\begin{vmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{vmatrix}, \quad \hat{e}_{\pm} := \begin{vmatrix} 0 & -i & 0 \\ i & 0 & \pm 1 \\ 0 & \mp 1 & 0 \end{vmatrix}, \quad \hat{f} := \sqrt{\frac{2}{3}}\begin{vmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{vmatrix} \tag{4.1}$$

$$\hat{f}_{\pm} := \begin{vmatrix} 0 & -1 & 0 \\ -1 & 0 & \pm i \\ 0 & \pm i & 0 \end{vmatrix}, \quad \hat{f}_{\pm\pm} := \begin{vmatrix} i & 0 & \pm 1 \\ 0 & 0 & 0 \\ \pm 1 & 0 & -i \end{vmatrix} \tag{4.1}$$

and using Capital letters for their Unitary Representations,

$$\mathrm{H}^{+} = \mathrm{H}, \quad \mathrm{E}_{\mp}^{+} = \mathrm{E}_{-}, \quad \mathrm{F}^{+} = \mathrm{F}, \quad \mathrm{F}_{+}^{+} = -\mathrm{F}_{-}, \quad \mathrm{F}_{++}^{+} = \mathrm{F}_{--} \tag{4.2}$$

the SL(3R) matrices are produced by

$$\left.\begin{aligned} & s = e^{i\alpha} \\ & \alpha = \xi\hat{h} + \xi_{+}e_{+} + \xi_{-}e_{-} + \xi f + \xi_{+}f_{+} + \xi_{-}f_{-} + \xi_{++}f_{++} + \xi_{--}f_{--} \\ & \xi^{*} = \xi, \quad \xi_{+}^{*} = \xi_{-}, \quad \xi^{*} = \xi, \quad \xi_{+}^{*} = -\xi_{-}, \quad \xi_{++}^{*} = \xi_{--} \end{aligned}\right\} \tag{4.3}$$

The commutation relations are,

$$\left.\begin{aligned}
&[\hat{h}, \hat{e}_{\pm}] = \pm 2\hat{e}_{\pm}, \quad [\hat{e}_{+}, \hat{e}_{-}] = \hat{h} \\
&[\hat{h}, \hat{f}_{\pm\pm}] = \pm 4 f_{\pm\pm}, \quad [\hat{h}, \hat{f}_{\pm}] = \pm 2\hat{f}_{\pm}, \quad [\hat{h}, \hat{f}] = 0 \\
&[\hat{e}_{\pm}, \hat{f}_{\pm\pm}] = 0, \quad [\hat{e}_{\pm}, \hat{f}_{\pm}] = \sqrt{4}\,\hat{f}_{\pm\pm}, \quad [\hat{e}_{\pm}, \hat{f}] = \sqrt{6}\, f_{\pm} \\
&[\hat{e}_{\pm}, \hat{f}_{\mp}] = \sqrt{6}\,\hat{f}, \quad [\hat{e}_{\pm}, \hat{f}_{\mp\mp}] = \sqrt{4}\,\hat{f}_{\mp} \\
&[\hat{f}_{++}, \hat{f}_{--}] = -2\hat{h}, \quad [\hat{f}_{+}, \hat{f}_{-}] = \hat{h}, \quad [\hat{f}_{\pm\pm}, \hat{f}_{\mp}] = 2\hat{e}_{\pm} \\
&[\hat{f}_{\pm\pm}, \hat{f}_{\pm}] = 0, \quad [\hat{f}, \hat{f}_{\pm}] = \sqrt{6}\,\hat{e}_{\pm}, \quad [\hat{f}, \hat{f}_{\pm\pm}] = 0.
\end{aligned}\right\} \quad (4.4)$$

By imposing the $|\Delta j| = 2$ requirement upon the generator matrix elements and making use of (4.4), Joseph found a unique half-integer spins solution:

$$\left.\begin{aligned}
&E_{\pm} | j, m \rangle = \sqrt{(j + m)(j \pm m + 1)}\, | j, m \pm 1 \rangle \\
&H | j, m \rangle = 2m | j, m \rangle \\
&\langle j + 2, m + 2 | F_{++} | j, m \rangle \\
&\qquad = \sqrt{(j + m + 4)(j + m + 3)(j + m + 2)(j + m + 1)}\, t \\
&\langle j + 2, m + 1 | F_{+} | j, m \rangle \\
&\qquad = 2\sqrt{(j + m + 3)(j + m + 2)(j + m + 1)(j - m + 1)}\, t \\
&\langle j + 2, m | F | j, m \rangle \\
&\qquad = \sqrt{6(j + m + 2)(j + m + 1)(j - m + 2)(j - m + 1)}\, t \\
&\langle j + 2, m - 1 | F_{-} | j, m \rangle \\
&\qquad = 2\sqrt{(j + m + 1)(j - m + 3)(j - m + 2)(j - m + 1)}\, t \\
&\langle j + 2, m - 2 | F_{--} | j, m \rangle \\
&\qquad = \sqrt{(j - m + 4)(j - m + 3)(j - m + 2)(j - m + 1)}\, t \\
&| t |^2 = \frac{(2j + 3)^2 + c^2}{4(2j + 5)(2j + 1)(2j + 3)^2} \\
&j = \frac{4n + 1}{2}, \qquad n = 0, 1, 2 \ldots, \qquad c^2 \to 0
\end{aligned}\right\} \quad (4.5)$$

All other matrix elements vanish.

This describes $\mathscr{D}\left(\frac{1}{2}, 0\right)$. The same method showed that the only other representations in that set were the previously derived [*12*] band-tensors

$$\begin{matrix} \mathscr{D}(0, c) \\ \mathscr{D}(1, c) \end{matrix} \; -\infty < c < \infty . \quad (4.6)$$

This degenerate set of representations corresponds to the $\overline{\mathrm{SL}(3\mathrm{R})}$ case of a recently discovered class of representations of semi-simple Lie groups in connection with the study of the enveloping algebras and the A. Joseph ideal [*18*] [*19*].

## 5. $\overline{\mathrm{GA}(4\mathrm{R})}$ AND $\overline{\mathrm{GL}(4\mathrm{R})}$

Since the representations of $\mathscr{C}$ are those of $\bar{\mathscr{F}}$, the physical states can be described by induced representations of $\bar{\mathscr{F}}$ over its stability subgroup and the translations. The stability subgroup is $\overline{\mathrm{GL}(3\mathrm{R})}$, and we can thus use the product of our representations of $\mathscr{S}_3$ by the 2-element factor group O(3)/SO(3), since $\mathscr{G}_3$ will have the topology of O(3).

Further complications will arise as a result of the local Minkowskian metric $\eta_{ab}$ of (1.2). The representations we developed fit the case of time-like momenta. We shall study the other possibilities in another publication.

For the construction of fields, we should use $\mathscr{G}_4$. Our analysis in section 3 can be repeated for this group; the $\mathscr{M}_4$ will correspond to a product of two sets $\pm(\sigma_i, 1)$ and $\mathscr{G}_4/\mathscr{Q}_4$ is $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathscr{M}_4$. Band-tensor representations of $\mathscr{S}_4$ were constructed in the second paper refered under [*12*], in connection with the states of a spinning top. For Band-spinors, we shall need $\mathscr{D}\left(j_0^{(1)} = \frac{1}{2}, j_0^{(2)} = 0\right) \oplus \mathscr{D}\left(j_0^{(1)} = 0, j_0^{(2)} = \frac{1}{2}\right)$, with $(\Delta j^{(1)} = 1, \Delta j^{(2)} = 1)$ non-compact action.

Each $(j^{(1)}, j^{(2)})$ level of a band spinor *field* satisfies a Bargmann-Wigner equation [*20*] for $j = |j^{(1)}| + |j^{(2)}|$.

The covariant derivative of a band-spinor field $\Psi^\alpha$ will be given by eq. (1.6). We shall deal with the field formalism in a future publication.

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