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Section A :
*Physique théorique.*

# On the absolutely and singularly continuous subspaces in scattering theory

by

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ABSTRACT. — A physical criterion is described to distinguish between absolutely continuous and singularly continuous subspaces of a Hamiltonian. Some models are discussed in this connection.

## 1. INTRODUCTION

This paper can be considered as a natural sequel to that of Amrein and Georgescu [*1*] on the evanescence of scattering states. The authors in [*1*] essentially extended Ruelle's [*2*] treatment to a larger class of potentials. In both [*1*] and [*2*], the criterion used to identify $\mathcal{H}_c$, the subspace of $\mathcal{H}$ corresponding to the continuous part of the evolution operator $V_t$, is the following:

$$\psi \in \mathcal{H}_c \quad \text{iff} \quad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \| P_r V_t \psi \|^2 dt = 0 ,$$

where one needs some assumption on $V_t$ for the implication « only if », and $P_r$ is the projection onto a sphere of radius $r$. But this criterion firstly has no direct physical interpretation and secondly, the conclusion is not sufficiently discerning about the absolutely continuous and the singular continuous parts of the continuous spectrum.

In this paper, we give a criterion which is directly physically inter-

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pretable and attempt to relate it with the absolutely continuous part of the Hilbert space with respect to the evolution. In this connection, the reader is also referred to the article of Gustafson [3].

## 2. THE MAIN THEOREM AND ITS INTERPRETATION

Let $V_t$ be the unitary group generated by the self-adjoint Hamiltonian H, i. e. $V_t = e^{-iHt}$ and let $P_S$ be the operator defined as

$$\begin{aligned}(P_S f)(\vec{x}) &= f(\vec{x}), \quad \text{if} \quad \vec{x} \in S \\ &= 0 \quad , \quad \text{if} \quad \vec{x} \notin S\end{aligned} \qquad \text{where} \quad |S| = \text{Lebesgue measure of } S < \infty$$

In quantum mechanics, we define the probability that a particle with an initial state $\psi(\|\psi\| = 1)$ be found in a space region S after time $t$ as

$$p(S, \psi ; t) \equiv \| P_S V_t \psi \|^2 .$$

Then, it is reasonable to talk about « time of Sojourn » [4] and define

$$J(S ; \psi) \equiv \int_{-\infty}^{\infty} p(S, \psi ; t)dt = \int_{-\infty}^{\infty} \| P_S V_t \psi \|^2 dt$$

as the total time a particle with state $\psi$ at $t = 0$ spends in a space region S. In principle, it seems possible to measure $J(S ; \psi)$, or at least decide whether such a quantity is finite or not. Such an expression was used for the description of time-delay in scattering [5]. It is the purpose of this paper to associate this quantity J with absolute continuity of H.

The principal theorem in this direction is

THEOREM 1. — If there exists a sequence of regions $\{ S_n \}$ such that $s.\lim_{n\to\infty} P_{S_n} = I$ and $J(S_n, \psi) < \infty$ for all $n$, then $\psi \in \mathscr{H}_{a.c.}(V_t)$.

The interpretation is two-fold and quite straight-forward. If a particle with state $\psi$ spends finite time in every finite region in space, then the state $\psi$ is absolutely continuous. Since if $\psi \in \mathscr{H}_p$, subspace of point spectra of $V_t$, then $J(S, \psi) = \infty$ for every S, one can give an equivalent statement for the singular continuous states. If $\psi$ is a singular continuous state, then there exists at least one finite (however large) region in space $\tilde{S}$ such that $J(\tilde{S}, \psi) = \infty$.

This means that if $\psi \in \mathscr{H}_{s.c.}(V_t)$, then the probability $p(\tilde{S}, \psi ; t)$, though not independent of time as is the case for states belonging to the point spectrum, decays sufficiently slowly for large times not to be integrable. If we set up observers on spheres of increasing radii around the scattering center, the state $\psi \in \mathscr{H}_p$ will lead to a conclusion that it is trapped in every spherical region. On the other hand a state $\psi \in \mathscr{H}_{s.c.}$ may appear to be moving out inside smaller spheres only to be trapped inside a larger sphere later.

For the proof of the Theorem we need the following

LEMMA 1. — Let $(\phi, V_t\psi) \in L_2(\mathbb{R}^1, dt)$ where $\phi$, $\psi$ are any two vectors in $\mathcal{H}$. Then $(\phi, E_\lambda\psi)$ is an absolutely continuous function of $\lambda$, where $\{E_\lambda\}$ is the spectral family of H.

*Proof of Lemma.* — From [6] and [7; appendix], it is known that

$$(\phi, E_{\lambda+h}\psi) - (\phi, E_\lambda\psi) = \lim_{\omega\to\infty} \frac{1}{2\pi} \int_{-\omega}^{\omega} (\phi, V_t\psi)\left(\frac{e^{ith}-1}{it}\right)e^{it\lambda}dt \tag{1}$$

For every $h \neq 0$, setting $f_h(t) = \dfrac{e^{ith}-1}{ith}$, we observe that

$$(\phi, V_t\psi)f_h(t) \in L_2 \cap L_1, \qquad \text{since} \qquad |f_h(t)| \leqslant 1.$$

Therefore

$$\frac{(\phi, E_{\lambda+h}\psi) - (\phi, E_\lambda\psi)}{h} = \mathcal{F}\{(\phi, V_t\psi)f_h(t)\}(\lambda), \tag{2}$$

where $\mathcal{F}$ is the unitary operator of Fourier transformation in $L_2(\mathbb{R}^1)$.

On the other hand $f_h(t) \underset{h\to 0}{\longrightarrow} 1$ pointwise in $t$ and hence by Lebesgue dominated convergence [8, cor. 16, p. 151] $(\phi, V_t\psi)f_h(t) \underset{h\to 0}{\longrightarrow} (\phi, V_t\psi)$ in $L_2$-topology. However, since $\mathcal{F}$ is unitary in $L_2$, we obtain

$$\frac{d(\phi, E_\lambda\psi)}{d\lambda} = \mathcal{F}\{(\phi, V_t\psi)\}(\lambda) \tag{3}$$

Then clearly $\dfrac{d(\phi, E_\lambda\psi)}{d\lambda} \in L_2$ and

$$\frac{d(\phi, E_\lambda\psi)}{d\lambda} = L_2 - \lim_{\omega\to\infty} \frac{1}{2\pi} \int_{-\omega}^{\omega} (\phi, V_t\psi)e^{it\lambda}dt \tag{4}$$

Since an $L_2$-function is necessarily locally $L_1$, it follows from (4) and an application of Fubini's theorem that

$$\int_\mu^\rho \frac{d(\phi, E_\lambda\psi)}{d\lambda}d\lambda = (\phi, E_\rho\psi) - (\phi, E_\mu\psi)$$

Hence, $(\phi, E_\lambda\psi)$ is an absolutely continuous function of $\lambda$.

From the lemma, two corollaries follow easily.

COROLLARY 1. — If $(\phi, V_t\psi) = 0(|t|^{-1/2-\varepsilon})$; $\varepsilon > 0$, then $(\phi, E_\lambda\psi)$ is an absolutely continuous function of $\lambda$.

Since $(\phi, V_t\psi)$ is a bounded continuous function of $t$ and it is square-integrable at $\infty$, it is square-integrable and hence the result.

COROLLARY 2. — Under the same hypothesis as in the Lemma, it follows that $(\phi, E_s\psi) = 0$, where $E_s$ is the projection onto $\mathcal{H}_s(V_t)$, the singular subspace.

*Proof.* — Let $\phi = \phi_{\text{a.c.}} + \phi_s$, $\psi = \psi_{\text{a.c.}} + \psi_s$. Then

$$(\phi, \mathrm{E}_\lambda\psi) = (\phi_{\text{a.c.}}, \mathrm{E}_\lambda\psi_{\text{a.c.}}) + (\phi_s, \mathrm{E}_\lambda\psi_s)$$

Since $(\phi, \mathrm{E}_\lambda\psi)$ is absolutely continuous and $(\phi_{\text{a.c.}}, \mathrm{E}_\lambda\psi_{\text{a.c.}})$ absolutely continuous by the definition of $\mathscr{H}_{\text{a.c.}}$ [9], it follows that

$$(\phi_s, \mathrm{E}_\lambda\psi_s) = 0\,, \qquad \forall\lambda \tag{5}$$

Therefore, on taking limit as $\lambda \to +\infty$ in equation (5), we obtain

$$(\phi_s, \psi_s) = (\phi, \mathrm{E}_s\psi) = 0\,.$$

*Proof of theorem 1.* —

$$\int |(\mathrm{P}_{\mathrm{S}_n}\psi, \mathrm{V}_t\psi)|^2 dt < \|\psi\|^2 \int \|\mathrm{P}_{\mathrm{S}_n}\mathrm{V}_t\psi\|^2 dt = \|\psi\|^2 \mathrm{J}(\mathrm{S}_n, \psi) < \infty$$

by hypothesis.

Hence by the corollary 2 of the Lemma, it follows that

$$(\mathrm{P}_{\mathrm{S}_n}\psi, \mathrm{E}_s\psi) = 0 \qquad \text{for all } \mathrm{S}_n\,.$$

Taking limit as $n \to \infty$, we conclude

$$(\psi, \mathrm{E}_s\psi) = 0 \qquad \text{or} \qquad \mathrm{E}_s\psi = 0 \qquad \text{i. e.}$$

$$\psi \in \mathscr{H}_{\text{a.c.}}(\mathrm{V}_t)$$

Q. E. D.

Some useful properties of $\mathrm{J}(\mathrm{S}, \psi)$ are established in

THEOREM 2. — *a*) If $\mathrm{S}_1 \subset \mathrm{S}_2$ then $\mathrm{J}(\mathrm{S}_1, \psi) < \mathrm{J}(\mathrm{S}_2, \psi)$.
*b*) $\mathrm{J}(\mathrm{S}, \psi) < \infty \Rightarrow \lim_{t\to\pm\infty} p(\mathrm{S}, \psi\,; t) = 0$

$$\Rightarrow \lim_{\mathrm{T}\to+\infty} \frac{1}{2\mathrm{T}} \int_{-\mathrm{T}}^{\mathrm{T}} p(\mathrm{S}, \psi\,; t)dt = 0\,.$$

*Proof of theorem 2.* — *a*) The result follows from the observation that

$$\|\mathrm{P}_{\mathrm{S}_1}\mathrm{V}_t\psi\|^2 < \|\mathrm{P}_{\mathrm{S}_2}\mathrm{V}_t\psi\|^2 \qquad \text{whenever } \mathrm{S}_1 \subset \mathrm{S}_2\,.$$

*b*) $\mathrm{J}(\mathrm{S}, \psi) < \infty$ means that the positive function $t \to \|\mathrm{P}_\mathrm{S}\mathrm{V}_t\psi\|$ is in $\mathrm{L}_2(\mathbb{R}^1)$.

On the other hand, the unitary group property of $\mathrm{V}_t$ ensures that the same function is *uniformly continuous* in $t$ for $-\infty < t < \infty$. In fact,

$$|\,\|\mathrm{P}_\mathrm{S}\mathrm{V}_t\psi\|^2 - \|\mathrm{P}_\mathrm{S}\mathrm{V}_{t'}\psi\|^2\,| < 2\|\psi\|\,\|(\mathrm{V}_{t-t'} - \mathrm{I})\psi\| \to 0$$

uniformly in $t$, since $\mathrm{V}_t$ is strongly continuous.

Therefore $t \to \|\mathrm{P}_\mathrm{S}\mathrm{V}_t\psi\|^2$ is a *uniformly continuous* $\mathrm{L}_2$-function on $\mathbb{R}^1$ and by the proposition 1 in the appendix, we conclude that

$$\lim_{t\to\pm\infty} \|\mathrm{P}_\mathrm{S}\mathrm{V}_t\psi\|^2 = 0\,.$$

The last implication of (*b*) is wellknown and we omit the proof.

*Remarks.* — (1) The result (*a*) of the above theorem tells us that if $J(S, \psi) = \infty$ for some S, then it remains infinite for all regions containing S. Hence if $\psi \in \mathscr{H}_{s.c.}(V_t)$, then there exists a minimal region $\tilde{S}$ such that $J(\tilde{S}, \psi) = \infty$.

(2) It is not difficult to construct a counterexample to converse of the Theorem 1, i. e. to construct a vector $\psi \in \mathscr{H}_{a.c.}(V_t)$ such that

$$\int \| P_S V_t \psi \|^2 = \infty$$

for all finite S. This can be easily done using Pearson's example [*10*] and Theorem 2. Pearson's example constructs a Hamiltonian such that the wave operators $\Omega_\pm$ exists, but Ran $(\Omega_+) \neq$ Ran $(\Omega_-)$. In fact, it is shown that

$$\lim_{t\to\infty} \| P_S V_t \psi \|^2 \neq 0 \qquad \text{if} \qquad \psi \in \mathscr{H}_{a.c.} - \text{Ran}\,(\Omega_+)$$

and

$$\lim_{t\to-\infty} \| P_S V_t \psi \|^2 \neq 0 \qquad \text{if} \qquad \psi \in \mathscr{H}_{a.c.} - \text{Ran}\,(\Omega_-)$$

Therefore by virtue of the first implication of Theorem 2 *b*, $J(S, \psi) = \infty$ for all finite S and all $\psi$ belonging to either of the two subspaces of $\mathscr{H}_{a.c.}(V_t)$.

## 3. SOME RESULTS ON THE CONVERSE PROBLEM

For this we define

$$M(H) = \left\{ f \in \mathscr{H} \mid \int \| P_S e^{iHt} f \|^2 dt < \infty \,;\, |S| < \infty \right\}$$

The implication of Theorem 1 can be restated as $M(H) \subseteq \mathscr{H}_{a.c.}(H)$. M(H) is easily seen to be a linear manifold, not necessarily closed.

THEOREM 3. — Let S be any set in $\mathbb{R}^3$ with $|S| < \infty$ and let there be a number $\kappa > 0$ such that $P_S(H + i)^{-\kappa}$ is Hilbert-Schmidt. Then

$$\overline{M(H)} = \mathscr{H}_{a.c.}(H)$$

*Proof.* — Let $D_0 = \{ f \in \mathscr{H}_{a.c.}(H) \mid \| f(\lambda) \|_\lambda \in \mathscr{C}_0^\infty(\mathbb{R}^1) \}$, where $f(\lambda)$ is the representative of $f$ in the H-spectral representation.

$$\begin{aligned} \int \| P_S e^{-iHt} f \|^2 dt &= \int (f, e^{iHt} P_S e^{-iHt} f) dt \\ &= \int (g, e^{iHt}(H - i)^{-\kappa} P_S (H + i)^{-\kappa} e^{-iHt} g) dt, \end{aligned}$$

where $g = (H + i)^\kappa f \in \mathscr{H}$ because $D_0 \subset D(H)$.

Since by hypothesis $(H - i)^\kappa P_S (H + i)^{-\kappa}$ is Trace-Class and

$$\| g(\lambda) \|_\lambda = (\lambda^2 + 1)^{\kappa/2} \| f(\lambda) \|_\lambda$$

is essentially bounded, we can apply Birman-Kato-Rosenblum Lemma [5], to deduce the finiteness of the above integral. Therefore $D_0 \subseteq M(H)$ and since the domain $D_0$ is clearly dense in $\mathscr{H}_{a.c.}$, we have the result, viz.,

$$\overline{M(H)} = \mathscr{H}_{a.c.}(H). \qquad \text{Q. E. D.}$$

Let $H_0 = -\Delta$, the free Schrödinger-Hamiltonian in $L^2(\mathbb{R}^3)$ and let $H = H_0 + V$ formally, where V is the potential operator, given by the operator of multiplication $V(\vec{x})$. Now we state various conditions under which the hypothesis of the above theorem is satisfied.

THEOREM 4. — If either

$$(\alpha) \qquad D(H) \supseteq D(H_0)$$

or

$$(\beta) \qquad V \in \mathscr{R} + L_\infty,$$

where $\mathscr{R}$ is the Rollnik class defined in Simon [*11*], then the hypothesis of theorem 3 is satisfied and therefore, $\overline{M(H)} = \mathscr{H}_{a.c.}(H)$.

*Proof.* — $\alpha$) Since $D(H) \supseteq D(H_0)$, it follows that $(H_0 + i)(H + i)^{-1}$ is an everywhere defined bounded operator. Also, it is easy to verify that $P_S(H_0 + i)^{-1}$ is a Hilbert-Schmidt operator. Therefore,

$$P_S(H + i)^{-1} = P_S(H_0 + i)^{-1}(H_0 + i)(H + i)^{-1} \in \mathscr{B}_2$$

the class of Hilbert-Schmidt operators.

$\beta$) We know from Simon [*11*; Theorem II.34, p. 73] that when $V \in \mathscr{R} + L_\infty$, one can establish the « second resolvent equation »,

$$(H - z)^{-1} = (H_0 - z)^{-1} - \left[(H_0 - z)^{-1}A_2^*\right](1 + Q(z))^{-1}[A_1(H_0 - z)^{-1}],$$

where $z$ is a nonreal complex number, $A_1 = |V|^{1/2}$, $A_2^* = \operatorname{sgn} V . |V|^{1/2}$, and $Q(z) =$ closure of $A_1(H_0 - z)^{-1}A_2^* \in \mathscr{B}_2$. We also observe from [*11*] that $\left[(H_0 - z)^{1/2}A_2^*\right]$ and $[A_1(H_0 - z)^{1/2}]$ are both bounded operators and that $\left[P_S(H_0 - z)^{-1}A_2^*\right] \in \mathscr{B}_2$. Since $\chi_S \in L_{3/2} \subset \mathscr{R}$, by [*11*; Theorem I.17, p. 14] we know that the kernel of the closure of $P_S(H_0 - z)^{-1}A_2^*$, is a Hilbert-Schmidt kernel. $P_S(H_0 - z)^{-1}$ is already known to be Hilbert-Schmidt, therefore $P_S(H - z)^{-1} \in \mathscr{B}_2$ for every non real $z$. Q. E. D.

Next we shall consider some cases where we can prove that M(H) is closed or equivalently $M(H) = \mathscr{H}_{a.c.}(H)$. In all these examples, however, it is already known that $\mathscr{H}_{a.c.}(H) = \mathscr{H}$. To prove these results, we depend on the theory of H-smooth operator developed by Kato [*12*].

THEOREM 5. — ($\alpha$) In $L_2(\mathbb{R}^n)$, $n \geqslant 3$ let $H_0 = -\Delta$. Then

$$M(H_0) = \mathscr{H}_{a.c.} = \mathscr{H} = L_2(\mathbb{R}^n).$$

($\beta$) In $L_2(\mathbb{R}^n)$, $n \geqslant 3$ let the perturbation be small and $H_0$-smooth as described in [*12*]. Then $M(H) = \mathscr{H}_{a.c.} = \mathscr{H} = L_2(\mathbb{R}^n)$.

($\gamma$) In $L_2(\mathbb{R}^1)$, let $L(p) = -\Delta + p \equiv -\Delta + p_2 + p_1$, where $p_2$ is Stark-like and $p_1$ is short-range, as described in [*13*]. Then

$$M(H) = \mathscr{H}_{\text{a.c.}} = \mathscr{H} = L_2(\mathbb{R}^1).$$

*Proof.* — We recall from Kato [*12*] that

$$\sup_{z \in \mathbb{C}} \| P_S(H - z)^{-1} P_S \| < \infty \ldots \tag{6}$$

is a sufficient condition for the operator $P_S$ to be H-smooth, i. e.

$$\int_{-\infty}^{\infty} \| P_S e^{-iHt} \psi \|^2 dt < C \| \psi \|^2, \qquad \forall \psi \in \mathscr{H} \ldots \tag{7}$$

($\alpha$) Clearly $\chi_S \in L_1(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$; $n > 3$. Then by virtue of Theorem 6.4 of [*12*], we obtain

$$\sup_z \| P_S(H_0 - z)^{-1} P_S \| < \infty,$$

on choosing $p = 1$ and $q = \infty$. Therefore

$$M(H_0) = \mathscr{H} = \mathscr{H}_{\text{a.c.}}(H_0).$$

($\beta$) We start with the « second resolvent equation » established in Kato [*12*; p. 263], viz.

$$(H - z)^{-1} = (H_0 - z)^{-1} - g[(H_0 - z)^{-1} A_2^*] (I + gQ(z))^{-1} [A_1(H_0 - z)^{-1}]$$

where

$$A_1 = | V |^{1/2}, \qquad A_2^* = \operatorname{sgn} V | V |^{1/2},$$

$V \in L_p(\mathbb{R}^n) \cap L_q(\mathbb{R}^n)$; $1 \leqslant p < n/2 < q \leqslant \infty$ and $Q(z) = [A_1(H_0 - z)^{-1} A_2^*]$.

Then $\| Q(z) \| < N$, independent of $z$ and for all

$$| g | < \frac{1}{N}, \qquad \| (I + gQ(z))^{-1} \| < (1 - | g | N)^{-1}.$$

Also, since $\chi_S \in L_p$ for all $p$; $1 \leqslant p \leqslant \infty$, the proof of Theorem 6.4 in page 277 of [*12*] goes through with the necessary minor modifications to yield

$$\sup_z \| P_S(H_0 - z)^{-1} A_2^* \| < \infty$$

and

$$\sup_z \| A_1(H_0 - z)^{-1} P_S \| < \infty.$$

Hence $\sup_z \| P_S(H - z)^{-1} P_S \| < \infty$, where $H = H_0 + gV$ and $| g | < \dfrac{1}{N}$, leading to the desired conclusion.

($\gamma$) In this case, we shall need to assume that S is not only a region in $\mathbb{R}^1$ with Lebesgue measure $| S | < \infty$, but also that it is a bounded region.

The « resolvent equation » established in [13] is

$$\begin{aligned} \mathrm{R}(z, \mathrm{L}(p)) = \mathrm{R}(z, \mathrm{L}(q(z))) + [\mathrm{R}(z, \mathrm{L}(q(z)))\mathrm{V}^{1/2}]\,. \\ (\mathrm{I} - \mathrm{V}^{1/2}\mathrm{R}(z, \mathrm{L}(q(z)))\mathrm{V}^{1/2})^{-1}[\mathrm{V}^{1/2}\mathrm{R}(z, \mathrm{L}(q(z)))]\,, \end{aligned} \tag{8}$$

where $\mathrm{V}^{1/2}$ = Multiplication operator by $(p - q(z))^{1/2}$, and

$$q(z)(\xi) = \begin{cases} z & \xi \in [a, b] \\ p_2(\xi) + \dfrac{5}{16}\left(\dfrac{p_2'(\xi)}{p_2(\xi) - z}\right)^2 - \dfrac{1}{4}\,\dfrac{p_2''(\xi)}{p_2(\xi) - z}, & \xi \notin [a, b]\,. \end{cases}$$

We also recall that for every compact $\mathrm{I} \subset \mathbb{R}^1$, one defines

$$\mathrm{R}_\pm(\mathrm{I}) = \{ z \in \mathbb{C} \mid \mathrm{Re}\, z \in \text{Interior I}, 0 < \pm \arg z < \bar{\theta} \}$$

and

$$n(\xi) = \sup_{z \in \mathrm{R}_\pm(\mathrm{I})} |\, p(\xi) - q(z)(\xi)\,| + \exp\left( - x(\xi) \int_0^{|\xi|} s^2(\sigma) d\sigma \right),$$

where

$$s(\xi) = \begin{cases} 1 & , \quad \xi \in [a, b] \\ \sup\limits_{z \in \mathrm{R}_\pm(\mathrm{I})} |\, p_2(\xi) - z\,|^{-1/4}, & \xi \notin [a, b]\,, \end{cases}$$

$x(\xi)$ is a positive function such that

*i)* $$x(\xi) = x(-\xi)$$

*ii)* $$\int_{-\infty}^{\infty} s^2(\xi) \exp\left( - x(\xi) \int_0^{|\xi|} s^2(\sigma) d\sigma \right) d\xi < \infty$$

*iii)* $$\lim_{|\xi| \to \infty} x(\xi) = 0$$

Now we restate the main result of [13], viz.

$$\sup_{z \in \mathrm{R}_\pm(\mathrm{I})} \|\, \mathrm{M}(n^{1/2})\mathrm{R}(z, \mathrm{L}(q(z)))\mathrm{M}(n^{1/2})\,\| < \infty \tag{9}$$

Since

$$\|\, \mathrm{P_S M}(1/n)^{1/2}\,\| < \sup_{\xi \in \mathrm{S}} \exp\left( \frac{x(\xi)}{2} \int_0^{|\xi|} s^2(\sigma) d\sigma \right) < \infty\,,$$

it follows from equation (9) that

$$\begin{aligned} &\sup_{z \in \mathrm{R}_\pm(\mathrm{I})} \|\, \mathrm{P_S R}(z, \mathrm{L}(q(z)))\mathrm{P_S}\,\| \\ &\qquad < \text{Constant}\,.\, \sup_{z \in \mathrm{R}_\pm(\mathrm{I})} \|\, \mathrm{M}(n^{1/2})\mathrm{R}(z, \mathrm{L}(q(z))\mathrm{M}(n^{1/2})\,\| < \infty \end{aligned} \tag{10}$$

Similarly, the fact that $\left\| \mathrm{M}\left(\dfrac{1}{n}\right)^{1/2} \mathrm{V}^{1/2} \right\| < 1$ leads to the result

$$\sup_{z \in \mathrm{R}_\pm(\mathrm{I})} \|\, \mathrm{P_S R}(z, \mathrm{L}(q(z)))\mathrm{V}^{1/2}\,\| < \infty \tag{11}$$

We also know from section 5 and 6 of [*14*] that $I - V^{1/2}R(z, L(q(z)))V^{1/2}$ is bounded invertible for all $z \in \overline{R_\pm(I)}$. Combining this with equations (8), (10) and (11), we conclude that

$$\sup_{z \in R_\pm(I)} \| P_S R(z, L(p)) P_S \| < \infty \tag{12}$$

for every compact $I \subset \mathbb{R}^1$ and bounded $S \subset \mathbb{R}^1$.

In order to complete the proof, we need only to show that

$$\| [P_S R_\pm(\omega, L(p)) P_S] \| \quad \text{is bounded, when } \omega \to \pm\infty . \tag{13}$$

where $\omega = \mathrm{Re}\, z$ and the above operator is to be understood as the limit of $[P_S R(z, L(p)) P_S]$ as $\mathrm{Im}\, z \to \pm 0$, which exists by virtue of results in [*13*]. This we do for an exact Stark potential, viz. $p_2(\xi) = -eE\xi$ $(E > 0)$.

In this case, the turning point $\xi_0$ is given by the equation

$$p_2(\xi_0) = \omega \qquad \text{or} \qquad \xi_0 = -\frac{\omega}{eE} \tag{14}$$

Therefore, the limit $\omega \to \mp\infty$ is equivalent to the limit $\xi_0 \to \pm\infty$. We choose the turning interval $[a, b]$ to be given by

$$a = \xi_0 - v, \qquad b = \xi_0 + v \tag{15}$$

All through the following, we hold $v$ fixed and let $\xi_0 \to \pm\infty$.

Define the unitary translation operator $T_{\bar\xi}$ in $L_2(\mathbb{R}^1)$, as

$$T_{\bar\xi} f(\xi) = f(\xi - \bar\xi) \tag{16}$$

It is clear from the Weyl construction of the kernel $R(z, L(q(z)))(\xi, \eta)$ in section 6 of [*13*] that the kernel $R_\pm(\omega, L(q(\omega)))(\xi, \eta)$ is actually a function of $(\xi - \xi_0, \eta - \xi_0)$ and of $v$ only. In fact, by letting

$$B_\pm(\omega) = V(\omega)^{1/2} R_\pm(\omega, L(q(\omega))) V(\omega)^{1/2} \tag{17}$$

one obtains

$$T_{\xi_0}^{-1} B_\pm(\omega) T_{\xi_0} = \tilde{B}_\pm(\xi_0), \tag{18}$$

where

$$\tilde{B}_\pm(\xi_0)(\xi, \eta) = (p_1(\xi + \xi_0) + u(\xi))^{1/2} R_\pm(0, L(q(0)))(\xi, \eta) \cdot (p_1(\eta + \xi_0) + u(\eta))^{1/2} \tag{19}$$

with

$$u(\xi) = \begin{cases} -eE\xi, & \xi \in [-v, v] \\ -\dfrac{5}{16}\xi^{-2}, & \xi \notin [-v, v] \end{cases}$$

Let $\tilde{B}_\pm$ be a Hilbert-Schmidt operator defined by the kernel

$$\tilde{B}_\pm(\xi, \eta) = u(\xi)^{1/2} R_\pm(0, L(q(0)))(\xi, \eta) u(\eta)^{1/2} \tag{20}$$

Then we claim that

$$\widetilde{\mathrm{B}}_{\pm}(\xi_0) \xrightarrow[\xi_0 \to \pm\infty]{} \widetilde{\mathrm{B}}_{\pm} \qquad \text{in Hilbert-Schmidt norm} \tag{21}$$

In order to prove (21), we need the following estimate from [*13*], viz.

$$|\,\mathrm{R}_{\pm}(\omega, \mathrm{L}(q(\omega)))(\xi, \eta)\,| \leqslant \mathrm{A}t(\xi - \xi_0)t(\eta - \xi_0) \tag{22}$$

where A is a function of $v$ only and

$$t(\xi) = \begin{cases} 1 & , \quad \xi \in [-v, v] \\ (e\mathrm{E})^{-1/4}\,|\,\xi\,|^{-1/4}, & \quad \xi \notin [-v, v] \end{cases}$$

Then,

$$\begin{aligned} |\,\widetilde{\mathrm{B}}_{\pm}(\xi_0)(\xi, \eta) - \widetilde{\mathrm{B}}_{\pm}(\xi, \eta)\,|^2 & \\ \leqslant 2\mathrm{A}^2 \{ (\,|\,p_1(\xi + \xi_0)\,|\,t^2(\xi))(\,|\,p_1(\eta + \xi_0) + u(\eta)\,|\,t^2(\eta)) & \\ + (\,|\,p_1(\xi + \xi_0) + u(\xi)\,|\,t^2(\xi))(\,|\,p(\eta + \xi_0)\,|\,t^2(\eta) \} & \end{aligned}$$

where we have used the estimate (22) and the inequality

$$|\,|\,a\,|^{1/2} - |\,b\,|^{1/2}\,| \leqslant |\,|\,a\,| - |\,b\,|\,|^{1/2} \leqslant |\,a - b\,|^{1/2}.$$

On the other hand,

$$\begin{aligned} &\int |\,p_1(\xi + \xi_0) + u(\xi)\,|\,t^2(\xi)d\xi \\ &\quad \leqslant \int |\,p_1(\xi + \xi_0)\,|\,t^2(\xi)d\xi + \int |\,u(\xi)\,|\,t^2(\xi)d\xi \\ &\quad \leqslant \max\{1, (e\mathrm{E}v)^{-1/2}\} \,.\, \|\,p_1\,\|_1 + \left(e\mathrm{E}v^2 + \frac{5}{12}v^{-3/2}\right) \equiv \Sigma(v) < \infty \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^2} |\,\widetilde{\mathrm{B}}_{\pm}(\xi_0)(\xi, \eta) - \widetilde{\mathrm{B}}_{\pm}(\xi, \eta)\,|^2 d\xi d\eta \leqslant 4\mathrm{A}^2\Sigma \int_{\mathbb{R}^1} |\,p_1(\xi + \xi_0)\,|\,t^2(\xi)d\xi$$

$\to 0$ as $\xi_0 \to \pm\infty$ by virtue of proposition 2 in the appendix with the choice $p = 4$ and we have proven (21). It is also clear from definitions (17) and (20) that

$$\widetilde{\mathrm{B}}_{\pm} = \mathrm{M}(p_2 - q(0))^{1/2}\mathrm{R}(0, \mathrm{L}(q(0)))\mathrm{M}(p_2 - q(0))^{1/2} = \mathrm{B}_{\pm}(0)\,|_{p_1 = 0}$$

and hence by virtue of results in [*13*] $(\mathrm{I} - \widetilde{\mathrm{B}}_{\pm})^{-1}$ is a bounded operator. Then,

$$(\mathrm{I} - \widetilde{\mathrm{B}}_{\pm}(\xi_0))^{-1} \xrightarrow[\xi_0 \to \pm\infty]{} (\mathrm{I} - \widetilde{\mathrm{B}}_{\pm})^{-1} \qquad \text{in norm.} \tag{23}$$

From relations (18), (21) and (23) it follows that

$$\begin{aligned} \|\,(\mathrm{I} - \mathrm{B}_{\pm}(\omega)^{-1}\,\| &= \|\,\mathrm{T}_{\xi_0}(\mathrm{I} - \widetilde{\mathrm{B}}_{\pm}(\xi_0))^{-1}\mathrm{T}_{\xi_0}^{-1}\,\| \\ &= \|\,(\mathrm{I} - \widetilde{\mathrm{B}}_{\pm}(\xi_0))^{-1}\,\| < \infty \qquad \text{as} \qquad \xi_0 \to \pm\infty. \end{aligned} \tag{24}$$

Also

$$\iint |[P_S R(z, L(q(z)))P_S](\xi, \eta)|^2 d\xi d\eta$$
$$< A^2 \left( \int \chi_S(\xi) t^2(\xi - \xi_0) d\xi \right)^2$$
$$< A^2 \max \{ 1, (eEv)^{-1/2} \} |S|^2, \qquad \text{independent of } \xi_0 . \quad (25)$$

Similarly, it can be shown that

$$\iint |[P_S R(z, L(q(z)))V^{1/2}](\xi, \eta)|^2 d\xi d\eta < \infty, \quad \text{as} \quad \xi_0 \to \pm\infty \quad (26)$$

Finally, the resolvent equation (18) and relations (12), (24), (25) and (26) together help us conclude that $\sup_z \| P_S R(z, L(p))P_S \| < \infty$ and equivalently,

$$M(L(p)) = \mathscr{H} = \mathscr{H}_{\text{a.c.}}(L(p)) .$$

*Remarks*

(4) With respect to ($\beta$) of theorem 5, it can be added that this result extends easily to the $n$-body problem with a small perturbation as treated by Iorio and O'Carroll [*15*]. In this case, the physically relevant projection operator is the relative one, denoted $P_{iS}$, defined in $L_2(\mathbb{R}^3)$ as follows:

$$(P_{iS} f)(\vec{x}_i) = f(\vec{x}_i), \qquad \text{if} \qquad \vec{x}_i = \vec{r}_j - \vec{r}_k \in S$$
$$= 0, \qquad \text{otherwise.}$$

Then the necessary estimates follow from [*15*] with minor adjustments as was the case with ($\beta$).

(5) In order to show that $M(H) = \mathscr{H}_{\text{a.c.}}$, it suffices to establish that $\sup_z \| P_S(H_{\text{a.c.}} - z)^{-1} P_S \| < \infty$. Since there is no simple way of doing this, we have only attempted to arrive at a stronger estimate, viz.

$$\sup_z \| P_S(H - z)^{-1} P_S \| < \infty$$

which implies that $H = H_{a.c.}$.

(6) From the method of proof of ($\gamma$) of theorem 5, it is evident that one can do relative scattering theory between a pure Stark-Hamiltonian $L(p_2)$ and one with a shortrange part thrown in, viz. $L(p_2 + p_1)$ where $p_1 \in L_1(\mathbb{R}^1)$. In other words, one can prove that the multiplication operator $|p_1|^{1/2}$ is $L(p_2)$-smooth.

(7) It is worth mentioning that in ($\gamma$), the operator $V^{1/2} R_{\pm}(\omega, L(q(\omega)))V^{1/2}$ does not converge to zero as $\omega \to \pm\infty$ in contrast to the case of short-range perturbations [*11*, theorem I.23].

## 4. EXAMPLE OF A HAMILTONIAN WITH SINGULAR CONTINUOUS SPECTRUM

Finally, we end with an example of a Hamiltonian such that the spectrum has a nontrivial singular continuous part and $(\psi, e^{-iHt}\psi) \to 0$ as $t \to \pm\infty$, where $\psi$ is some vector in $\mathscr{H}_{s.c.}(H)$. In this context, reader is also referred to Lemma 2 of page 626 in [*7*].

For this, we need an example of a singularly continuous stieltjes measure constructed by Schaeffer [*16*], which is stated as follows:

Given $r : [0, \infty) \to [0, \infty)$ any increasing function, there exists a real non-decreasing singularly continuous function

$$F(\lambda) \quad (\text{i, e. } F'(\lambda) = 0 \text{ a. e.}); \qquad -\pi < \lambda < \pi$$

such that

$$\int_{-\pi}^{\pi} e^{-it\lambda} dF(\lambda) = 0\left(\frac{r(|t|)\ln t}{|t|^{1/2}}\right) \qquad \text{for} \qquad t \to \pm\infty \tag{27}$$

Since addition of a constant does not change anything, we normalize F so that $F(-\pi) = 0$ and define

$$\sigma(\lambda) = \begin{cases} 0 & , \quad \lambda \leqslant 0 \\ \dfrac{2}{F(\pi)}\sqrt{\left(\dfrac{2}{\pi}\right)}F(\lambda - \pi), & \quad \lambda \in [0, 2\pi] \\ \dfrac{2}{\pi}\sqrt{\lambda} & , \quad \lambda \geqslant 2\pi \end{cases} \tag{28}$$

Then clearly $\sigma$ is a non-decreasing continuous function, singularly continuous in $[0, 2\pi]$. Now we use the main theorem of Naimark [*17*, p. 282] to arrive at the unique second order linear differential operator associated with the spectral measure $\sigma(\lambda)$. Conditions (A) and (B) of page 270 in [*17*] are easily verified as also the fact that the set of points at which $\sigma(\lambda)$ increases has at least one finite limit point. Therefore, one has a unique differential operator $L(p)$ given as

$$L(p)f(x) = -f''(x) + p(x)f(x); \qquad 0 \leqslant x < \infty \tag{29}$$

with a potential function $p$ continuous in $[0, \infty)$ and boundary condition of the type

$$f'(0) - \theta f(0) = 0.$$

Let $L_0$ be the self-adjoint operator given by (29) and let $\bar{L}_l$ be any self-adjoint extension of the real symmetric operator $L_l$ in $L_2[0, \infty)$ given as

$$L_l f(x) = -f''(x) + \left\{\frac{l(l+1)}{x^2} + p(x)\right\} f(x), \tag{30}$$

where $f \in \mathscr{C}_0^\infty(0, \infty)$ and $l = 1, 2, \ldots$. Then, we construct the direct sum

$$= \bigoplus_{l=0}^{\infty} \mathrm{L}_2([0, \infty)) \approx \mathrm{L}_2(\mathbb{R}^3) \tag{31}$$

and

$$\mathrm{H} = \mathrm{L}_0 \oplus \left( \bigoplus_{l=1}^{\infty} \bar{\mathrm{L}}_l \right).$$

It is clear that for any $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^3 - \{0\})$,

$$(\mathrm{H}\psi)(\vec{x}) = -\Delta\psi(\vec{x}) + p(\vec{x})\psi(\vec{x}) \tag{32}$$

In other words, we have embedded the one-dimensional $[0, \infty)$ problem in three dimensions.

Now let $\psi = \{\psi_l\}_{l=0,1,2\ldots}$ such that $\tilde{\psi}_l = 0$ for all $l \geqslant 1$ and $\tilde{\psi}_0 =$ the characteristic function of $[0, 2\pi]$, where $\tilde{\psi}_0$, $\tilde{\psi}_l$ denote the representatives of $\psi_0$, $\psi_l$ in the spectral representation of $\mathrm{L}_0$, $\bar{\mathrm{L}}_l$ respectively. Then

$$\begin{aligned}(\psi, e^{-i\mathrm{H}t}\psi) &= \int_0^{2\pi} e^{-i\lambda t} d\sigma(\lambda) \\ &= 0\left(\frac{r(|t|) \ln |t|}{|t|^{1/2}}\right)\end{aligned}$$

and with the choice $r(|t|) = \ln |t|$, one has

$$(\psi, e^{-i\mathrm{H}t}\psi) = 0\left(\frac{1}{|t|^{1/2-\varepsilon}}\right)$$

for every $\varepsilon > 0$ and $|t| \to \infty$.

Also, since the potential function $p$ is continuous in $[0, \infty)$ it is locally square-integrable in $\mathbb{R}^3$ and therefore by [*18*, p. 106] one observes that $\mathrm{P_S}(\mathrm{H} - z)^{-1}$ is compact. This leads to the conclusion that

$$\| \mathrm{P_S} e^{-i\mathrm{H}t}\psi \|^2 \to 0 \qquad \text{as} \qquad t \to \pm\infty$$

where $\psi$, the same vector as given above, clearly belongs to $\mathscr{H}_{\mathrm{s.c.}}(\mathrm{H})$.

# APPENDIX

Here, we prove two propositions which have been used in the text.

**Proposition** 1. — Let $f$ be a function belonging to $L_p(\mathbb{R}^1)$ and uniformly continuous. Then

$$\lim_{|x|\to\infty} f(x) = 0$$

*Proof.* — Since $f(x)$ is continuous, the points where $|f(x)| > \delta > 0$ form a set of intervals. The length of such an interval $(x_1, x_2)$ tends to 0 as $x_1 \to \infty$, since

$$(x_2 - x_1)\delta^p < \int_{x_1}^{x_2} |f(x)|^p dx \to 0$$

Now since $|f(x_1)| = \delta$,

$$|f(x)| < |f(x_1)| + |f(x) - f(x_1)| = \delta + |f(x) - f(x_1)|$$

which tends to 0, by choosing first $\delta$, then $x_1$ and observing that $|f(x) - f(x_1)| \to 0$ uniformly in $x_1$.

**Proposition** 2. — Let $f \in L_1(\mathbb{R}^1)$ and $g \in L_p(\mathbb{R}^1) \cap L_\infty(\mathbb{R}^1)$ where $1 < p < \infty$. Then

$$\int f(\xi + \xi_0)g(\xi)d\xi \xrightarrow[|\xi_0|\to\infty]{} 0$$

*Proof.* — The proof is done in three steps. First, we claim that $\int f(\xi + \xi_0)g(\xi)d\xi \xrightarrow[|\xi_0|\to\infty]{} 0$ whenever $f, g \in \mathcal{S}(\mathbb{R}^1)$, the class of Schwartz functions. This follows from the fact that $f(\xi + \xi_0) \to 0$ for all $\xi$ as $|\xi| \to \infty$ and an application of Lebesgue dominated convergence theorem. Next, we approximate $g$ in $L_p$ by a sequence $g_n \in \mathcal{S}$ and observe that

$$\left| \int f(\xi + \xi_0)g(\xi)d\xi \right| < \| f \|_q \, \| g - g_n \|_p + \left| \int f(\xi + \xi_0)g_n(\xi)d\xi \right| \xrightarrow[|\xi_0|\to\infty]{} 0,$$

where $f \in \mathcal{S}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, we approximate $f \in L_1$ by a sequence $f_m \in \mathcal{S}$ and conclude that

$$\left| \int f(\xi + \xi_0)g(\xi)d\xi \right| < \| f - f_m \|_1 \, \| g \|_\infty + \left| \int f_m(\xi + \xi_0)g(\xi)d\xi \right| \to 0 \quad \text{as} \quad |\xi_0| \to \infty,$$

where we have used the fact that $g \in L_\infty$.

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