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Induction from a normal nilpotent subgroup

by

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ABSTRACT. — The problem of the factor systems involved in the induction procedure (in the sense of Mackey) for the unitary representations of group extensions, is considered for the case where the normal subgroup is a nilpotent Lie group. An explicit expression is given for these factor systems, which becomes especially simple for an interestingly large class of such extensions.

RÉSUMÉ. — Le problème des systèmes de facteurs intervenant dans la procédure d'induction (dans le sens de Mackey) pour les représentations unitaires d'extensions de groupes est considéré dans le cas où le sous-groupe normal est un groupe de Lie nilpotent. Une expression explicite est donnée pour ces systèmes de facteurs, expression qui devient spécialement simple pour une classe de telles extensions suffisamment large pour être intéressante.

INTRODUCTION

The theory of induction of representations for separable locally compact topological groups, as begun in the last years of the 19th century by Frobenius [1] for the finite groups, is now quite complete, especially since the well known work of Wigner [2] and Mackey [3]. In the simplest and commonest case the theory provides an explicit procedure for the construction of all irreducible unitary representations of a (regular) inessential extension $G \cong N \rtimes_{\phi} H$ (semidirect product) with N normal abelian, from the repre-

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sentations of N and of certain subgroups of the factor group H . It has also been shown by Mackey [4] that if the semidirectness or the abelian condition is left out, it becomes necessary to consider projective representations of these subgroups of H as well.

While the occurring factor systems are easy to find when N is still abelian, the problem becomes much more difficult in the general case. In this paper we consider this problem when N is a nilpotent Lie group, making profit of the work of Kirillov on the structure of the irreducible unitary representations for this case [5-6]. In the first part we review briefly the theory of Kirillov on the structure of the dual $\hat{N}$ of N . For a special kind of element of $\hat{N}$ we prove in addition some interesting and useful extra properties. We show with examples that for a large class (with respect to practical purposes) of nilpotent Lie groups, but not for all, the whole $\hat{N}$ satisfies these properties. In the second part we use Mackey's theory of induction and derive there an explicit expression for the factor sets involved. For the special kind mentioned above, and if the extension is central, the result then becomes especially simple.

1. The dual of N

We first review briefly the structure of the set $\hat{N}$ of classes of irreducible unitary representations of a connected nilpotent Lie group N , as described by Kirillov [6]. For simplicity, we assume that N is simply connected, the generalization giving no difficulty, as is well known [5-6]. Let thus $\underline{n}$ be the Lie algebra of N , $\underline{n}'$ the dual space of $\underline{n}$ and $\text{coAd}_{\underline{n}}(N)$ the co-adjoint representation of N on $\underline{n}'$, defined for $v \in \underline{n}'$ by

$$(\text{coAd}_{\underline{n}}(n_1)v)(x) \stackrel{\text{def}}{=} v(\text{Ad}_{\underline{n}}(n_1)^{-1}x) \quad (1.1)$$

with $n_1 \in N$, $x \in \underline{n}$. Let then $\mathcal{O}_v$ be the orbit of an element $v \in \underline{n}'$, i. e. the set of all images of v under the action (1.1) of N . Since (1.1) defines a representation of N , the set of all orbits is a partition of $\underline{n}'$. Consider then in each such orbit one arbitrary (but fixed) element v and consider a subalgebra $\underline{l} \subseteq \underline{n}$ such that

$$[\underline{l}, \underline{l}] \subseteq \text{Ker } v \quad (1.2)$$

Such a subalgebra $\underline{l}$ is called *subordinate* to v . The following map T_v on the Lie subgroup L of N generated by $\underline{l}$ ($L = \exp \underline{l}$)

$$T_v(\exp x) \stackrel{\text{def}}{=} \exp \{ i \cdot v(x) \}, \quad x \in \underline{l} \quad (1.3)$$

is then a one-dimensional irreducible unitary representation of L . This representation can then be induced to a representation $V_{v,\underline{l}}$ of N by

$$(V_{v,\underline{l}})(n)f(L\lambda) \stackrel{\text{def}}{=} T_v(\lambda \cdot n \cdot (\lambda')^{-1})f(L\lambda') \quad (1.4)$$

where λ, λ' are members of an (arbitrary but fixed) set of representatives

of the (right) coset decomposition of N with respect to L , with λ' fixed by the condition $\lambda \cdot n \cdot (\lambda')^{-1} \in L$, and $f(L\lambda)$ is a measurable and quadratic integrable function on the coset space with respect to the (quasi-) invariant (under N) measure μ and with values in the carrier space of the representation T_v .

The following then holds

THEOREM 1.1 (Kirillov). — (i) Any irreducible unitary representation of N is obtained in this way, up to equivalence.

(ii) $V_{v,\underline{l}}$ is irreducible if and only if $\dim \underline{l}$ is maximal

(iii) $V_{v,\underline{l}} \sim V_{v',\underline{l}'}$, both irreducible, if and only if $\mathcal{O}_v = \mathcal{O}_{v'}$.

(iv) $\hat{N}$, the dual of N , is always of Type I

From now on, $\underline{l}$ will always be assumed to be a subalgebra subordinate to v and of maximal dimension. Such a subalgebra is also called a *real polarization at v* . Let us now distinguish the following two types in $\hat{N}$.

DEFINITION. — The class $[\hat{n}] \in \hat{N}$ to which $V_{v,\underline{l}}$ belongs is said to be of *type a* if and only if it is possible to choose $\underline{l}$ ideal. Else it will be called of *type b* .

It is easy to verify that the implied condition does not depend on the choice of a particular element on the orbit $\mathcal{O}_v$ so that, together with theorem 1.1, this condition depends effectively only on the class $[\hat{n}]$ of $V_{v,\underline{l}}$.

Clearly any $\hat{N}$ contains elements of type a . We shall call N to be of type a if $\hat{N}$ contains only elements of type a . Let us show with examples that the class of nilpotent Lie groups which are of type a is large enough to be interesting for practical purposes, but that it does not contain every N .

Examples. — 1) any nilpotent Lie group whose lower central series has a length less or equal to 2 is of type a . Indeed any $\underline{l}$ contains then the centre (else it is not maximal) and any subalgebra containing the centre is in this case ideal.

2) any nilpotent Lie group of dimension less or equal to 5 is of type a . This follows from direct checking, using the classification by Dixmier [7] of the corresponding Lie algebras. This check shows in addition that a choice may be necessary: for example for $\mathcal{G}_{5,3}$ in the notation of [7] and for $y_4 = 0 \neq y_5$ (with $v(\sum \lambda_i X_i) \equiv \sum \lambda_i y_i$). The subalgebra generated by X_1, X_3, X_5 is subordinate to v (with maximal dimension) but is not an ideal whereas the subalgebra generated by X_3, X_4, X_5 satisfies both conditions.

3) any direct product of nilpotent Lie groups of type a is clearly of type a . This enlarges slightly the examples just mentioned.

4) not any nilpotent Lie group is of type a , contrary to what one might perhaps expect. An example of a class of type b can be given for the 10-dimensional Lie algebra of 5×5 matrices A with $A_{\mu\nu} = 0$ for $\mu \geq \nu$. It is a quite easy and useful exercise to verify that for $v(A) \stackrel{\text{def}}{=} A_{15}$ there can be no subordinate $\underline{l}$ (of maximal dimension) which is an ideal.

We do not know however if there exists a *simple immediate* characterization of the class of groups of type a . Let us now mention, for nilpotent Lie groups of type a , some useful additional properties.

PROPOSITION 1.2. — Let N be of type a , $v \in \mathcal{O}_v$ in $\underline{n}'$, $[\underline{l}, \underline{n}] \subseteq \underline{l}$ a corresponding ideal real polarization at v , then

(i) $\underline{l}$ is his own centralizator with respect to v , i. e. $\forall x \in \underline{n}$

$$v([\underline{l}, x]) = 0 \Leftrightarrow x \in \underline{l}$$

(ii) The coset space N/L is (Borel) isomorphic with the orbit of T_v under N and is one-to-one characterized by the classes $\bar{v}$ of elements of $\mathcal{O}_v$ which coincide with eachother when restricted to $\underline{l}$.

Proof. — (i) $v([\underline{l}, x]) = 0$ implies $v([\underline{l} + x, \underline{l} + x]) = 0$ and $\underline{l}$ ideal implies $\underline{l} + \mathbb{R}x$ is a subalgebra hence $\underline{l} + \mathbb{R}x$ is subordinate to v . This is a contradiction to the maximality of $\underline{l}$ unless $x \in \underline{l}$. The converse is trivial.

(ii) Since $\underline{l}$ is ideal one may consider the following exact sequence of groups

$$1 \rightarrow L \rightarrow N \rightarrow N/L \rightarrow 1, \quad \rho, \Psi \quad (1.5)$$

with factor set $\rho \in Z_{\Psi}^2(N/L, L)$ and $\Psi: N/L \rightarrow \text{Aut } L$ defined canonically. One may then construct (all) irreducible representations of N by induction from the dual $\hat{L}$ of L , via the stability subgroups of N defined for a representative $\hat{l}$ of each class $[\hat{l}] \in \hat{L}$ (see section 2). For $\hat{l} = T_{v, \underline{l}}$ it follows from (i) that the stabiliser is L itself (and the induced representation $V_{v, \underline{l}}$). The result is then evident.

Since the orbit of T_v , the coset space N/L and a set of coset representatives are in 1-to-1 Borel correspondance with eachother, we will use, or better said abuse, as is usual, the parameter λ to describe all these spaces.

We now turn to the general induction procedure from a normal nilpotent subgroup N .

2. General inducing from a nilpotent normal subgroup

Mackey's theory of induced representations is a well known procedure, when at least applied to a (regular) semidirect product $G = N \rtimes_{\phi} H$, with G separable locally compact and N abelian (ϕ denoting some given homomorphism from H to $\text{Aut}(N)$) [3]. Less known perhaps is the more general case where G , separable locally compact, is any (regular) extension of a group N , not necessarily abelian, by a group H , i. e. appears in the following exact sequence of groups

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1, \quad m, \phi \quad (2.1)$$

characterized by a factor set $m: H \times H \rightarrow N$, with

$$m(h_1, h_2 h_3) + \phi(h_1) m(h_2, h_3) = m(h_1, h_2) + m(h_1 h_2, h_3),$$

and a map $\phi: H \rightarrow \text{Aut}(N)$ satisfying $\phi(h_1)\phi(h_2) = \mu(m(h_1, h_2)) \cdot \phi(h_1 h_2)$, μ being the canonical epimorphism from N to $\text{In}(N)$, the group of inner automorphisms of N . The essential difference, in this more general case is that, as shown by Mackey [4], if N is no more abelian or the extension (2.1) does not split, no longer ordinary representations of the adequate subgroups of H have to be considered, but certain projective ones. It is the purpose of this section to calculate explicitly the factor sets which are then involved, for the case where N is a nilpotent Lie group. Since the dependence of m (as in (2.1)) in these factor sets is easy to find, we assume first, in order to lighten the notation, that $m = 0$.

Let us first indicate briefly, in a way which is not usual in this context but which is, at our opinion, the best adapted one, the essential steps of the explicit generalized Wigner-Mackey construction of induced representations. Let therefore $[\hat{n}] \in \hat{N}$, the dual of N , with $\hat{n}$ a representant of this class of irreducible representations. One defines from ϕ a map $\hat{\phi}$ on H with

$$\hat{\phi}(h) : \hat{N} \rightarrow \hat{N}, \quad \hat{\phi}(h)[\hat{n}] \equiv [\hat{n}_h]$$

as follows

$$\hat{n}_h(n) = \hat{n}(\phi(h)^{-1}n) \quad (2.2)$$

The set of all classes $\{[\hat{n}_h]\}$ generated by $\hat{\phi}$ from $[\hat{n}]$ is called the orbit of $[\hat{n}]$ under H and is denoted $\mathcal{O}_{[\hat{n}]} \subseteq \hat{N}$. The action (2.2) defines directly a subgroup $H_{\hat{n}}$ of H , which we shall call the homogeneous little group, by

$$h \in H_{\hat{n}} \Leftrightarrow [\hat{n}_h] = [\hat{n}]$$

i. e. $h \in H_{\hat{n}}$ if and only if there exists a unitary operator $S(h) \in \mathcal{U}(\mathcal{H}(\hat{n}))$, $\mathcal{H}(\hat{n})$ being the representation space of $\hat{n}$, such that

$$\hat{n}_h(n) = S(h)^{-1} \hat{n}(n) S(h), \quad \forall n \in N \quad (2.3)$$

The map $S: H_{\hat{n}} \rightarrow \mathcal{U}(\mathcal{H}(\hat{n}))$ is in general not an homomorphism, but, from (2.3), using that ϕ is an homomorphism in (2.2) and, that $\hat{n}$ is irreducible, it follows from Schur's Lemma that it is necessarily a projective one, satisfying thus

$$S(h_1)S(h_2) = \tau(h_1, h_2)S(h_1 h_2) \quad (2.4)$$

$h_1, h_2 \in H_{\hat{n}}$ and $\tau(h_1, h_2)$ some factor set in $\mathcal{U}(1)$, the unit circle of the complex plane.

We note here that if G is not semidirect then the elements $h \in H_{\hat{n}}$ can no longer be identified with a subgroup of $G_{\hat{n}}$. On the other side, as we saw, ϕ is no longer necessarily an homomorphism. The generalization of (2.4) is then easy to compute and is given by

$$S(h_1)S(h_2) = \tau(h_1, h_2)\hat{n}(m(h_1, h_2))S(h_1 h_2) \quad (2.4)'$$

It is quite clear from the above formulas that it is in general not easy to find explicitly operators $S(h)$ satisfying (2.3) and thus the factor systems τ we shall need in the sequel. It is just the purpose of this paper to solve this problem for the more special case we are interested in.

The isotropy group $G_{\hat{n}} \subseteq G$ (defined as the subgroup of G leaving $[\hat{n}]$ invariant under the action $g: \hat{n}(n) \rightarrow \hat{n}(g^{-1}ng)$, N being identified with its image as subgroup of G) appears then as an extension (inessential if so is G) of N by $H_{\hat{n}}$, as shown in the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & N & \xrightarrow{(I)} & G_{\hat{n}} & \xrightarrow{\pi} & H_{\hat{n}} \longrightarrow 1, & 0, \phi \\ & & \parallel & & \downarrow (I) & & \downarrow (I) & \\ 1 & \longrightarrow & N & \xrightarrow{(I)} & G & \xrightarrow{\pi} & H \longrightarrow 1, & 0, \phi \end{array} \quad (2.5)$$

(I) denoting the injection monomorphisms. We construct then in a first step a (projective) representation of $G_{\hat{n}}$ as follows. Let L be a projective representation of $H_{\hat{n}}$ with factor set ω and carrier space $\mathcal{H}(L)$ then it is easy to see that by defining for all $(n, h_{\hat{n}}) \in G_{\hat{n}}$

$$(\hat{n}_S \cdot L)(n, h_{\hat{n}}) \stackrel{\text{def}}{=} \hat{n}(n)S(h_{\hat{n}}) \otimes L(h_{\hat{n}}) \quad (2.6)$$

on the Hilbert space $\mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$, $\otimes$ denoting the external Kronecker product, we get a (projective) representation of $G_{\hat{n}}$ with factor set σ , where

$$\sigma(g_1, g_2) = \tau(\pi g_1, \pi g_2) \cdot \omega(\pi g_1, \pi g_2) \quad (2.7)$$

so that, choosing $\omega = \tau^{-1}$ on $H_{\hat{n}}$, we get an ordinary representation of $G_{\hat{n}}$ with carrier space $\mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$. The last step is now the following: we decompose H in right cosets with respect to $H_{\hat{n}}$ with coset representatives $\{h_i \mid i \in I, I \text{ some index set (Borel) isomorphic to } H/H_{\hat{n}}\}$ and similarly G with respect to $G_{\hat{n}}$, choosing now as coset representatives the image of the $\{h_i\}$ under a fixed section $r: H \rightarrow G$, the set of representatives being then given by $\{(0, h_i) \mid i \in I\}$. Let now $\mu_{\hat{n}}$ be a quasi-invariant ergodic measure on $\hat{N}$, not identically zero but vanishing on all orbits outside $\mathcal{O}_{[\hat{n}]}$. We assume again, in order to simplify the notation, that this measure is right and left invariant, the generalization being straightforward. Since we have assumed also that the action of H on $\hat{N}$ was regular (in the sense of Mackey [4]), this measure is unique (as a class) and also transitive, i. e. concentrated on the orbit [4]. Further, the coset space $G/G_{\hat{n}} \cong H/H_{\hat{n}}$ can be identified with the orbit $\mathcal{O}_{[\hat{n}]}$ by the 1-to-1 Borel isomorphism $[\hat{n}_i] \leftrightarrow h_i$, so that we may use, similarly as in section 1 the parametrization $\{h_i\}$ to describe both spaces. We now consider on this space a vector valued function f

$$f: h_i \rightarrow \mathcal{H}(\hat{n}) \otimes \mathcal{H}(L)$$

satisfying the two conditions

$$\begin{aligned} (i) & \quad (f(h_i), \Phi) \text{ is } \mu_{\hat{n}}\text{-measurable, } \forall \Phi \in \mathcal{H}(\hat{n}) \otimes \mathcal{H}(L) \\ (ii) & \quad \|f\|^2 \stackrel{\text{def}}{=} \int_{\mathcal{O}[\hat{n}]} \|f(h_i)\|^2 d\mu_{\hat{n}}(h_i) < \infty \end{aligned} \quad (2.8)$$

where scalar product and norm under the integrals are taken in $\mathcal{H}(n) \otimes \mathcal{H}(L)$.

Identifying functions equal almost everywhere, the set of functions satisfying the above conditions can be shown [3] to form a separable Hilbert space, $\mathcal{H}$, with scalar product

$$(f, g) = \int_{\mathcal{O}[\hat{n}]} (f(h_i), g(h_i)) d\mu_{\hat{n}}(h_i)$$

The induced representation is then defined on $\mathcal{H}$ as follows: let $(n, h) \in G$, then

$$(\hat{n} \uparrow G)^L(n, h)f(h_i) \stackrel{\text{def}}{=} (\hat{n}_s \cdot L)((0, h_i)(n, h)(0, h_j)^{-1}f(h_j)) \quad (2.9)$$

where h_j is the (unique) coset representative satisfying

$$h_i h h_j^{-1} \in H_{\hat{n}}$$

The following then holds, and follows from [4] for this special case:

THEOREM 2.1 (Mackey). — Consider an orbit $\mathcal{O}[\hat{n}] \subseteq \hat{N}$ and a transitive ergodic measure $\mu_{\hat{n}}$ concentrated on this orbit as described above. Then the representation (2.9) is unitary and irreducible if and only if L is unitary and irreducible. Two such representations are equivalent only if they are based on the same orbit. Moreover all irreducible unitary representations of G are obtained, up to equivalence, once and only once, when one induces once per orbit, and for each orbit one considers all projective, inequivalent, irreducible unitary ω -representations L of the corresponding $H_{\hat{n}}$ with ω satisfying the equation (2.7) with σ taken equal to 1.

Up to now we have made no use of the fact that N is nilpotent and the results described above are valid for any regular (split) extension (2.1). Our purpose is now to use the results of section 1 to construct explicitly the factor sets ω .

Let thus $[\hat{n}] \in \hat{N}$, and N nilpotent. It follows from theorem 1.1 (i) that for each class $[\hat{n}]$, one can choose a representative $V_{v, \underline{L}}$ as defined in (1.4) with carrier space

$$\mathcal{H}(V) = \int_{N/L}^{\oplus} \mathcal{H}(T) d\mu(\lambda) \quad (2.10)$$

with T and λ as in section 1. The canonical action $\hat{\phi}$ of $h \in H$ is now given by (2.2)

$$\hat{\phi}(h)V_{v, \underline{L}}(n) = V_{v, \underline{L}}(\phi(h)^{-1}n) \quad (2.11)$$

defining the little group $G_{\hat{n}}$ and its factor group $H_{\hat{n}}$ as in (2.5). It follows then from (2.3), and by the definition of $H_{\hat{n}}$, that for all $h \in H_{\hat{n}}$ there exists a unitary operator $S(h)$ such that

$$S(h)(\hat{\phi}(h)V_{v,\underline{l}})(n)S(h)^{-1} = V_{v,\underline{l}}(n)$$

and, since G is split, $S(h)$ satisfies the condition

$$S(h_1)S(h_2) = \omega^{-1}(h_1, h_2)S(h_1h_2) \quad (2.12)$$

$\forall h_1, h_2 \in H_{\hat{n}}$ with the factor set $\omega^{-1}(h_1, h_2)$ we have to determine. We now construct this operator explicitly. In a first step we consider on $\mathcal{H}(V)$ an operator $U(h)$ with

$$U(h) : \mathcal{H}(V) \rightarrow \mathcal{H}(V')$$

where $V' \equiv V_{v',\underline{l}'}$, $v' = \text{coAd}(h)v$, $\underline{l}' = \text{Ad}(h)\underline{l}$. That $\underline{l}'$ is indeed subordinate to v' whenever $\underline{l}$ is subordinate to v follows from the fact that by (1.1) we have

$$[\underline{l}, \underline{l}] \subseteq \text{Ker } v \Leftrightarrow [\text{Ad}(h)\underline{l}, \text{Ad}(h)\underline{l}] \subseteq \text{Ker } \text{coAd}(h)v$$

Because N is normal in G we may choose now the set $\{h\lambda h^{-1}\}$, which is clearly Borel isomorphic to the set $\{\lambda\}$, to describe the coset space $N/hLh^{-1} \equiv N/L'$ on which the representation $V_{v',\underline{l}'}$ is based and we may now define the action of $U(h)$ by

$$(U(h)f)(h\lambda h^{-1}) \stackrel{\text{def}}{=} f(\lambda) \quad (2.13)$$

Using now the invariance of the measure μ on the coset space N/L and the definition of the scalar product in $\mathcal{H}(V)$

$$(f, g) = \int_{N/L} (f(\lambda), g(\lambda)) d\mu(\lambda) \quad (2.14)$$

f, g as in (1.4), it follows from N normal in G that $\mu'(\lambda) \equiv c_h \mu(h^{-1}\lambda h)$ is invariant under N , and because this measure is unique (as a class), one may choose c_h such that $U(h)$ becomes unitary. This implies in turn that $(Uf)(h\lambda h^{-1})$ is measurable and squared integrable on N/hLh^{-1} if and only if $f(\lambda)$ is such on N/L . We now use this map to bring the representation (2.11) on a representation explicitly of the Kirillov type. We have by (1.4) and (2.13), with $g \in \mathcal{H}(V')$,

$$\begin{aligned} (U(h)V_{v,\underline{l}}(h^{-1}nh)U(h)^{-1}g)(h\lambda h^{-1}) \\ = (V_{v,\underline{l}}(h^{-1}nh)U(h)^{-1}g)(\lambda) = (T_{v,\underline{l}}(l_1)U(h)^{-1}g)(\lambda_1) \end{aligned} \quad (2.15)$$

with $l_1 = \lambda(h^{-1}nh)\lambda_1^{-1} \in L$, i. e. $hl_1h^{-1} = (h\lambda h^{-1})n(h\lambda_1h^{-1})^{-1} \in hLh^{-1}$, so that we get for (2.15), using $T_{v,\underline{l}}(l_1) = T_{v',\underline{l}'}(l'_1)$, with $v' = \text{coAd}(h)v$, $l'_1 = hl_1h^{-1}$, $\underline{l}' = \text{Ad}(h)\underline{l}$, and the fact that T is one-dimensional:

$$\begin{aligned} &= (T_{v',\underline{l}'}(l'_1)U(h)^{-1}g)(\lambda_1) = T_{v',\underline{l}'}(l'_1)g(h\lambda_1h^{-1}) \\ &= (V_{v',\underline{l}'}(n)g)(h\lambda h^{-1}) \end{aligned}$$

so that we have indeed

$$U(h)V_{v,\underline{l}}(h^{-1}nh)U(h)^{-1} = V_{v',\underline{l}'}(n) \quad (2.16)$$

with $v' = \text{coAd}(h)v$ and $\underline{l}' = \text{Ad}(h)\underline{l}$.

This result is actually true for all $h \in H$, since we have made no use of the isotropy condition.

Let now h be in $H_{\hat{n}}$. This implies per hypothesis, and from theorem 1.1 (iii), that v' is in the same orbit $\mathcal{O}_v$ in $\underline{n}'$ as v (under the action of N). Thus there exists a $n \in N$, depending on h (and denoted $n(h)$) such that $v' = \text{coAd}(n(h))v$ and $U(n(h)) = U(h)$. In other words we know there exists a $n(h)$ such that, from (2.16)

$$V_{v',\underline{l}'}(n) = U(h)V_{v,\underline{l}}(n(h)^{-1} \cdot n \cdot n(h))U(h)^{-1}, \quad \forall n \in N$$

and, since $V_{v,\underline{l}}$ is a representation of N

$$V_{v',\underline{l}'}(n) = U(h)V_{v,\underline{l}}^{-1}(n(h))V_{v,\underline{l}}(n)V_{v,\underline{l}}(n(h))U(h)^{-1} \quad (2.17)$$

We note here that the choice of $n(h)$ is in general not unique: one may indeed still add to it any element of the centre of the representation.

Combining now (2.17) with (2.16), and with the definition (2.3) of $S(h)$, we have found however an explicit expression for the latter:

$$S(h) = V_{v,\underline{l}}(n(h)) \quad (2.18)$$

which is unitary

If the extension is not split the same intertwining operator may still be used, corresponding then to the action of the element $(0, h)$ of $G_{\hat{n}}$, in the extension notation. Since H can then no longer be identified with a subgroup of G , the action of H on $\underline{n}'$ is no longer a representation: we have then

$$\text{coAd}(h_1)\text{coAd}(h_2) = \text{coAd}(m(h_1, h_2))\text{coAd}(h_1h_2) \quad (2.19)$$

corresponding to the analogous change in (2.4)'. Taking these modifications into account we may now formulate a first general result for our factor systems:

PROPOSITION 2.2. — Let G , separable locally compact, be any regular extension of a nilpotent Lie group N by a group H . Then given $[\hat{n}] \in \hat{N}$, $V_{v,\underline{l}} \in [\hat{n}]$, the intertwining operator $S(h) \equiv S((0, h))$ of (2.3) and the factor system ω of (2.7) on $H_{\hat{n}}$ needed for the Wigner-Mackey generalized induction procedure are respectively given by

$$S((0, h)) = V_{v,\underline{l}}(n(h))$$

with $n(h) \in N$ such that $\text{coAd}(n(h))v = \text{coAd}(h)v$, and

$$\begin{aligned} \omega(h_1, h_2)\mathbb{1} &= [S((0, h_1)) \cdot S((0, h_2)) \cdot S((0, h_1)(0, h_2))^{-1}]^{-1} \\ &= V_{v,\underline{l}}(m(h_1, h_2)) \cdot V_{v,\underline{l}}(n(h_1h_2))V_{v,\underline{l}}^{-1}(n(h_1)n(h_2)) \end{aligned} \quad (2.20)$$

This factor set can thus now be computed straightforwardly.

However, when the representation class of $V_{v,\underline{l}}$ is of type a with $\underline{l}$ ideal, and for central extensions, the result becomes rather simpler.

Indeed we note that then the choice of $n(h)$ is unique up to the subgroup of N leaving the orbit of $T_{v,\underline{l}}$ point per point invariant, i. e. by Prop. 1.2, up to an element of L . In other words for a given $h \in H_{\hat{n}}$ the set of $n(h)$ satisfying (2.17) corresponds uniquely to an element of the factor group N/L .

Let us therefore consider again the exact sequence of groups (1.5):

$$1 \rightarrow L \rightarrow N \rightarrow N/L \rightarrow 1, \quad \rho, \Psi$$

and suppose first G is semidirect. We have then (identifying for simplicity L with its image in N under (ι) , the canonical injection, and the elements of N/L with their image under a fixed section $s : N/L \rightarrow N$)

$$\begin{aligned} S(h_1)S(h_2)S(h_1h_2)^{-1} &= V_{v,\underline{l}}(n(h_1))V_{v,\underline{l}}(n(h_2))V_{v,\underline{l}}^{-1}(n(h_1h_2)) \\ &= V_{v,\underline{l}}(\rho(n(h_1), n(h_2))) \end{aligned}$$

Hence, with (2.20) and the definition of $V_{v,\underline{l}}$

$$\omega(h_1, h_2) = T_{v,\underline{l}}^{-1}(\rho(n(h_1), n(h_2)))$$

For example if N is abelian, $L = N$, so that the extension (1.5) is trivially split. Thus ρ and hence ω are indeed trivial. This result also generalizes straightforwardly for the case where the extension (2.1) is not inessential but central, i. e. characterized by a factor set m in the centre of N . Indeed $V_{v,\underline{l}}(m(h_1, h_2))$ in (2.20) is then necessarily a phase, this corresponds also to the fact that m in the centre of N implies $m \subseteq L$, for any orbit. On the other side, $m \subseteq L$ implies also that $n(h_1) \cdot n(h_2)$ is in the same coset of N with respect of L as $n(h_1h_2)$, $\forall h_1, h_2 \in H_{\hat{n}}$. Hence we have the following

PROPOSITION 2.3. — Let G , separable locally compact be a central (regular) extension of a nilpotent Lie group N of type a by a group H . Then given $[\hat{n}] \in \hat{N}$, $V_{v,\underline{l}} \in [\hat{n}]$ (with $\underline{l}$ ideal) the factor system ω described above is given, up to equivalence, by

$$\omega(h_1, h_2) = T_{v,\underline{l}}(m(h_1, h_2))T_{v,\underline{l}}^{-1}(\rho(n(h_1), n(h_2))) \quad (2.21)$$

Again, for N abelian this gives the known result back, ρ being trivial.

The proposition 2.3 clearly remains true if N is not of type a and/or the extension is non central, but only for the representations $\hat{n}$ of N which satisfy the two conditions

- (i) $[\hat{n}]$ is of type a
- (ii) $m(H_{\hat{n}}, H_{\hat{n}}) \subseteq L$

This problem has been developed for the purpose of a specific physical situation [8]. We refer to this paper for a practical application of these results. Let us just give here a short example.

Example. — Let $G = N \Lambda_{\phi} H$ (semidirect) where N is the 9-dimensional

nilpotent Lie group with infinitesimal generators P_μ , X_μ and I ($\mu = 0, 1, 2, 3$) satisfying the following commutation relations

$$[P_\mu, X_\nu] = g_{\mu\nu} \cdot I \quad \mu, \nu = 0, 1, 2, 3$$

where $g_{\mu\nu}$ is some invertible « metric tensor ». All other commutators vanish. For example one may have $H = SL(2, C)$ with $g_{\mu\nu}$ the Minkowski metric and $\phi(SL(2, C))$ the usual action on momentum and position operators.

Since the lower central series has length 2, N is of type a ; let now $v \in \mathfrak{h}'$. There are obviously two cases:

(i) $I \in \text{Ker } v$. Then $\underline{l} = \underline{n}$ and ρ is trivial hence ω is trivial too (by (2.21), m being zero).

(ii) $I \notin \text{Ker } v$. The orbits $\mathcal{O}_v$ are then 8-dimensional and characterized by the value of v on I . The maximal dimension of $\underline{l}$ is thus 5, since $[\underline{l}, \underline{l}] \subseteq \text{Ker } v$

$$\text{Max}(\dim \underline{l}) = \dim \underline{n} - \frac{1}{2} \dim \mathcal{O}_v$$

$\underline{l}$ can obviously always be chosen as the (abelian) subalgebra generated by I, P_0, P_1, P_2 and P_3 , which is an ideal. The extension (1.5) is then clearly split hence ρ and thus ω are trivial.

This nilpotent Lie group behaves thus in the induction procedure of a semidirect product always like an abelian one (*), i. e. the procedure reduces to the usual case.

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(*) For the special case of $H = SL(2, C)$ mentioned above this result has already been obtained by ANGELOPOULOS [9], in a paper considering the same problem, but from an opposite point of view: this author explored namely conditions that could be imposed on H (in the semi-direct product case) so to keep ω inessential and proved that for H semi-simple, ω was not in general inessential but that it always could be chosen real (hence equal to ± 1).

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