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<http://www.numdam.org/item?id=AIHPA_1977__26_1_87_0>
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by

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I. INTRODUCTION

In this paper we investigate the analytic dependence of Feynman amplitudes on the mass variable associated with a single line. We work in the context of the *generic* amplitude defined in [7], i. e., the usual amplitude regularized with both the $\hat{\lambda}$ parameters of analytic renormalization and a complex dimension $\nu$; this avoids divergence difficulties and in addition...
leads to singular behavior which has a clear relation to the structure of
the underlying Feynman graph. Our main result, described in Section II,
gives a complete description of the singularities of the physical sheet of
the amplitude at a point where the mass variable \( z \) vanishes. This is a cor-
rected version of an erroneous result of [1] (Theorem 3.1), which in fact
is correct only when all lines of the graph have non-zero masses.

In [2] a partial desingularization of the integration space for the Feyn-
man amplitude was obtained, permitting a discussion of the meromorphic
structure of the amplitude in \( \lambda, \nu \). In Section III we modify this desingula-
risation slightly, to reach a geometry in the integration space for which
the \( z \to 0 \) pinch is always of one fairly simple type. The singularity
generated by this local pinch configuration is analyzed in the Appendix:
in Section IV we apply this analysis to the Feynman integral to prove
the main theorem.

We will follow the notation of [2], but have tried to define most terms as
they arise. In Section III we have omitted proofs, since they involve only
slight modifications of those given in [2].

II. TERMINOLOGY AND STATEMENT OF RESULTS

For any Feynman graph \( G \) we let \( \Omega_G \) denote the set of lines of \( G \), \( \Omega_G^M \subset \Omega_G \)
the set of massive lines, \( \Theta_G \) the set of vertices, and \( \Theta_G^E \) the set of external
vertices; further, \( N(G) = |\Omega_G|, n(G) = |\Theta_G|, c(G) = \) the number of connected components of \( G \), and \( h(G) = N(G) - n(G) + c(G) \), the number
of loops. The generic Feynman amplitude \( F_G(\delta, \lambda; \nu, \eta) \) is a function of regularizing parameters \( \nu \in \mathbb{C} \) and \( \lambda, \eta \in \mathbb{C} \), \( l \in \Omega_G \), of external invariants \( s(\chi), \chi \in \Theta_G^E \) (which satisfy certain linear relations [1]), and of squared mass
variables \( z_l, l \in \Omega_G^M \), defined for \( G \) connected by

\[
F_G(\delta, \lambda; \nu, \eta) = \gamma(-\pi_G) \int_{\Omega_G} \prod_{l \in \Omega_G} z_l^{1/2} d_G(z) \gamma^{\nu/2} D_G^*(z, \delta, \eta). \tag{2.1}
\]

Here

\[
\pi_G = h(G)v/2 - \sum_{l \in \Omega_G} (\lambda_l + 1) \tag{2.2}
\]

\[
d_G(z) = \sum \prod_{l \in \Omega_G} z_l. \tag{2.3}
\]

\[
D_G(z, \delta, \eta) = \frac{1}{2} \sum_{\chi \in \Theta_G^E} s(\chi) \left( \sum_{l \in \Omega_G^M} z_l \right) - \left( \sum_{l \in \Omega_G^M} z_l \right) D_G(z). \tag{2.4}
\]

\[
D_G^*(z, \delta, \eta) = D_G(z, \delta, \eta)/d_G(z), \tag{2.5}
\]
with sums running over all trees $T$ in $G$ and all 2-trees $T_2$ which separate $\chi$ from $\Theta^E - \chi$; $\mathcal{D}_G = \{ \alpha \in \mathbb{P}^{N(G)} - \{ \alpha_i \geq 0 \} \}$ and $\eta$ is the fundamental projective differential form. The integral (1) is convergent for $(s, z)$ in the Symanzik region $R_G = \{ (s, z) | s(\chi) > 0, z_l < 0 \}$ and $(\omega, \nu)$ in a suitable convergence region [2]; it is understood that a complex power of a positive quantity is defined using the principle branch of the logarithm. Analytic continuation of $F_G$ in the $(\omega, \nu)$ parameters is discussed in [2].

We now consider a fixed, 2-connected graph $G_0$ with a distinguished massive line $\omega$; we will drop the subscript $G_0$ from the line and vertex sets of this graph. Our goal is to describe the behavior of $F_{G_0}$ when $z_\omega$ varies in a neighborhood of zero (and all other $s, z$ variables have the signs of the Symanzik region).

**Remark 2.1.** The singularity is simple to discuss if $G_0$ has no (or one) external vertices and $\omega$ is the only massive line. Then since

$$D(s, z, \omega) = - z_\omega z_\omega d(z),$$

$$F_{G_0}(s, z ; \omega, \nu) = [- z_\omega]^\eta \Phi_{G_0}(\omega, \nu)$$

with

$$\Phi_{G_0} = \Gamma(- \pi \frac{\nu}{2}) \left( \prod_{l \neq \omega} \left( \prod_{l \neq \omega} \right) \right)^{\omega_l} \times \left( \frac{\nu}{2} \right)^{\eta}$$

a meromorphic function of $\omega, \nu$, i.e., $F_{G_0}$ behaves like $z_\omega^{\pi \nu}$ at $z_\omega = 0$. We will refer to this as the trivial case.

To discuss the general case we need the concepts of a saturated graph and a link (previously defined in [2]) and one additional definition, that of a mass singularity graph. In what follows, $G^\omega_0$ denotes the graph $G_0$ modified so that $\omega$ is a massless rather than massive line; $G^\omega_0$ the graph obtained from $G_0$ by adding one vertex, $\infty$, and joining it by one line to each vertex of $\Theta^E$. For any graphs $G, H$, with $H$ a subgraph of $G$, $G/H$ is the quotient graph obtained from $G$ by contracting all lines in $H$, and $p_{G/H} : G \to G/H$ is the associated mapping.

**Definition 2.2.**

a) Suppose that $H \subset G_0$, and that $G^\omega_0 / H$ has pieces $Q_1, \ldots, Q_k$ numbered so that $\infty \notin \theta_{Q_i}$, and $\Omega_{Q_i} = \emptyset$, for $i > i_0$. Then $H = H \cup p_{G^\omega_0/H}(Q_{i_0+1} \cup \ldots \cup Q_k)$ is called the saturation of $H$; $H$ is saturated if $H = H$. b) A subgraph $S \subset G_0$ is called a link (in $G_0$) if (i) $S = G_0$, and (ii) the removal of any piece of $S$ destroys property (i). c) A subgraph $B \subset G_0$ is a mass singularity (MS) graph for $\omega$ if (i) $\omega \notin B$, (ii) $B$ is a link in $G^\omega_0$, and (iii) $B$ is saturated.

We can now state the main result of this paper, to be proved in Section IV.

**Theorem 2.3.** Let $G_0$ be a non-trivial graph with $\omega \notin \Omega_{G_0}^\omega$. Then for $s(\chi)$ and $s_l (l \neq \omega)$ restricted to a compact subset of the Symanzik region
for $G_0^\omega$, $F_{G_0}$ may be analytically continued in $z_\omega$ to a fixed punctured neighborhood $\{0 < |z_\omega| < \varepsilon\}$ of $z_\omega = 0$. In this neighborhood,

$$F_{G_0}(s, z; \hat{s}, \hat{z}; v) = H(s, z; \hat{s}, \hat{z}; v) + \sum_B (-z_\omega)^{r_{G_0/B}} K_B(s, z; \hat{s}, \hat{z}; v),$$

where the sum is over all MS graphs $B$. $H$ and $K_B$ are analytic at $z_\omega = 0$, in fact,

$$H|_{z_\omega=0} = F_{G_0^\omega},$$

$$K_B|_{z_\omega=0} = \int_{G_0/B} \prod_{i=1}^r F_{B_i},$$

if $B$ has connected components $B_1, \ldots, B_r$.

Thus near $z_\omega = 0$, $F_{G_0}$ decomposes into a regular piece together with pieces which behave like a power of $z$; one piece for each of a certain class of subgraphs of $G_0$. Both the power and the «residue» $K_B|_{z_\omega=0}$ are simply characterized in terms of the subgraph $B$ and quotient graph $G_0/B$.

**Example 2.4.** — a) If $G_0$ is massive (i.e., $Q^M = \Omega$) there is a unique MS graph $B$ containing all lines except $\omega$. In this case the singularity structure

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*Fig. 1. — Feynman graph $G_0$.***
given by Theorem 1.3 is the same as that described in [1]. b) In the graph of figure 1, in which dotted lines are massless, solid lines massive, and wavy lines denote external vertices, there are two MS graphs, shown in figure 2. Thus the corresponding sum in (2.8) has two terms.

Fig. 2. — MS subgraphs of $G_0$.

**Remark 2.5.** — If $G^0$ is trivial (Remark 2.1), the singular behavior described by (2.6) may in fact be regarded as a special case of Theorem 1.3; we must take the empty graph to be the unique MS graph in $G_0$ and make the convention that $F_G = 0$ whenever $\Theta^E_G = \Omega^M_G = \emptyset$ unless $G$ is the empty graph, in which case $F_G = 1$.

### III. DESINGULARIZATION

In order to discuss the singular behavior at $z_0 = 0$ of the integral (2.1), it is necessary to analyze the pinch which occurs in the integration space. For this purpose we introduce here a desingularization of the integration space which reduces the pinch geometry to a relativity simple form, analyzed in the appendix. The desingularization is described by *s-families* of sub and quotient graphs of $G_0$, which label necessary blow-ups and blow-downs of the boundary of the integration region in (2.1). Our procedure is a modification of that used in [2]; the *s-families* used here differ from those of [2] primarily by the inclusion of mass singularity graphs. The important properties of these *s-families* are summarized in Lemma 3.2; the proof of this Lemma is quite similar to the proofs given in [2] and is therefore omitted.

Let $\mathcal{H} = \{ H \subseteq G_0 \mid H \text{ is saturated and irreducible, or } H \text{ is an MS graph} \}$. $\mathcal{Q} = \{ Q = G_0/Z \mid Z \text{ is a link, and } Q \text{ is irreducible} \}$. If $H \in \mathcal{H}$, a *link in $H$* is a subgraph $H_1 \subseteq H$ such that $H_1 = H$ and $H_1$ is either irreducible or a link in $G^0$. If $Q \in \mathcal{Q}$ with $Q = G_0/Z$, a subgraph $S_1 \subseteq Q$ is a *link in $Q$* if $Q \cup S_1$ is a link. For any $\mathcal{E} \subseteq \mathcal{Q} \cup \mathcal{H}$ we write $\mathcal{E}_q = \mathcal{E} \cap \mathcal{Q}$, $\mathcal{E}_h = \mathcal{E} \cap \mathcal{H}$, $\mathcal{E}^0 = \{ K \in \mathcal{E} \mid \Omega_K \text{ is maximal} \}$, and, for $K \in \mathcal{E}$, $\mathcal{E}(K) = \{ K' \in \mathcal{E} \mid \Omega_K \subseteq \Omega_K \}$. We write $\mathcal{E}^0_q(K) \equiv (\mathcal{E}(K)^0)_q$, etc.
DEFINITION 3.1. — An s-family $\mathcal{S} \subseteq \mathcal{E}$ is a maximal family satisfying

1) $G_0 \in \mathcal{S}$;
2) The sets $\Omega_K$, $K \in \mathcal{S}$, are non-overlapping;
3) If $K \in \mathcal{S}$, and $\mathcal{S}'_q(K) = \{ K/S_1, \ldots, K/S_r \}$, then $S = \bigcup_{i=1}^r S_i$ is a link in $K$, and the pieces of $K/S$ are precisely the elements of $\mathcal{S}'_q(K)$;
4) If $K \in \mathcal{S}$, $\mathcal{F} \subseteq \mathcal{S}'_q(K)$ with $|\mathcal{F}| \geq 2$, and $H_1 = \bigcup_{H \in \mathcal{F}} H$, then (a) $H_1$ is not irreducible and is not a link in $G_0$, and (b) if $K = G_0/S \in \mathcal{S}$, the pieces of $H_1 \cup S$ are precisely the pieces of $H_1$ together with the pieces of $S$.

LEMMA 3.2. — a) If $\mathcal{S}$ is an s-family and $K \in \mathcal{S}$, there is precisely one line, denoted $\sigma(K)$, in $\Omega_K - \bigcup_{K \in \mathcal{S}(K)} \Omega_K$; in particular, this implies that $|\mathcal{S}| = |\Omega| = N$.

b) For any s-family $\mathcal{S}$, define
$$\mathcal{D}(\mathcal{S}) = \{ \alpha \in \mathbb{P}N^{-1} | \alpha_l \geq 0, \, \alpha_{\sigma(H)} \leq \alpha_{\sigma(K)} \leq \alpha_{\sigma(Q)}, \text{ whenever } H \in \mathcal{S}'_q(K), \, Q \in \mathcal{S}'_q(K) \}.$$

Then $\mathcal{D} = \cup \mathcal{D}(\mathcal{S})$, and if $\alpha \in \mathcal{D}(\mathcal{S}) \cap \mathcal{D}(\mathcal{S}')$, for $\mathcal{S} \neq \mathcal{S}'$, then $\alpha_l = \alpha_{l'}$ for some $l \neq l'$.

c) For any s-family $\mathcal{S}$ there is a distinguished tree $T$ in $G_0$; $T$ consists of all lines $\sigma(H)$, where $H \in \mathcal{S}_H$ and $\sigma(H)$ is a piece of the graph formed by adjoining $\sigma(H)$ to $\bigcup_{H \in \mathcal{S}'_q(H)} H'$. $T \cap \Omega_K$ is a (spanning) tree in $K$ for each $K \in \mathcal{S}$, moreover, either (i) $\sigma(G_0) \in \Omega^M$, (ii) $\sigma(G_0) \in T$, and the 2-tree $T - \sigma(G_0)$ separates $\Theta^E$ into two non-empty subsets $\psi, \Theta^E - \psi$, or (iii) both.

Proof. — Omitted; see [2].

We may now use this lemma to rewrite the integral (2.1) defining $F = F_{G_0}$.

For by Lemma 3.2 (b),
$$F(\xi, \varphi, \lambda, v) = \Sigma F_{\mathcal{S}}(\xi, \varphi, \lambda, v),$$
the sum running over all s-families, with
$$F_{\mathcal{S}}(\xi, \varphi, \lambda, v) = \Gamma(-\pi_{G_0}) \int_{\mathcal{D}(\mathcal{S})} \prod_{H \in \mathcal{S}} t_H^{-1} d^{v/2}(\mathcal{D}^*)^{-\pi_{G_0}} \eta$$

In (3.2) we make the variable change
$$\alpha_l = \prod_{H \in \mathcal{S}_H} t_H \prod_{Q \in \mathcal{S}_Q} t_Q^{-1}.$$
and normalize by setting \( z_{\sigma(G_0)} = t_{G_0} = 1 \); the integration region \( \mathcal{D}(\mathcal{E}) \) then becomes the cube \( \{ t \mid 0 \leq t_K \leq 1, K \in \mathcal{E}(G_0) \} \), and

\[
\eta = \prod_{\mathcal{E}_n(G_0)} t_H^{N(H)-1} dt_H \prod_{\mathcal{E}_q} t_Q^{-N(Q)-1} dt_Q. \tag{3.4}
\]

Any tree in \( G_0 \) must intersect each \( H \in \mathcal{E}_n(G_0) \) in at most \( n(H) - c(H) \) lines, and each \( Q \in \mathcal{E}_q \) in at least \( n(Q) - c(Q) = n(Q) - 1 \) lines; from Lemma 3.2 (c), these numbers are exact for the tree \( T \). Thus the definition (2.3) of \( d_{G_0}(\mathcal{E}) \) becomes

\[
d(\mathcal{E}) = \prod_{\mathcal{E}_n(G_0)} t_H^{n(H)} \prod_{\mathcal{E}_q} t_Q^{-n(Q)}(1 + e_\mathcal{E}(t)), \tag{3.5}
\]

with \( e_\mathcal{E} \) a polynomial having positive coefficients. Similarly,

\[
D^*(\mathcal{E}, s, z) = \zeta_\mathcal{E} + g_\mathcal{E}(t, s, z),
\]

where \( \zeta_\mathcal{E} \) depends on the different cases of Lemma 3.2 (c):

\[
\zeta_\mathcal{E} = \begin{cases} 
-z_{\sigma(G_0)}, & \text{case (i)}, \\
sl_\mathcal{E}(1 + e_\mathcal{E}(t))^{-1}, & \text{case (ii)}, \\
sl_\mathcal{E}(1 + e_\mathcal{E}(t))^{-1} - z_{\sigma(G_0)}, & \text{case (iii)};
\end{cases} \tag{3.7}
\]

\( g_\mathcal{E} \) is continuous and non-negative for \( t_K \geq 0 \) and \((s, z)\) in the Symanzik region, and is independent of \( z_{\sigma(G_0)} \) in cases (i) and (iii). Thus

\[
F_\mathcal{E}(\mathcal{E}, z; t, v) = \Gamma(-\pi_{G_0}) \int_0^1 \cdots \int_0^1 \prod_{\mathcal{E}_n(G_0)} t_H^{-n_H-1} dt_H \\
\prod_{\mathcal{E}_q} t_Q^{-n_Q-1} dt_Q(1 + e_\mathcal{E})^{-v/2}(\zeta_\mathcal{E} + g_\mathcal{E})^{\pi_{G_0}}. \tag{3.8}
\]

This is our desingularized form of the integral defining \( \mathcal{F}_\mathcal{E} \).

**IV. PROOF OF MAIN RESULT**

We now investigate the behavior of \( F \) for \( s(\mathcal{E}) \) and \( z_h, l \neq w \), restricted to a compact subset of the Symanzik region \( R_{G_0} \), as \( z_w \) varies in \( \{ \text{Re } z_w < 0 \} \cup \{ |z_w| < \epsilon \} \). Consider a single term \( F_\mathcal{E} \) in the decomposition (3.1). From the representation (3.8) \( F_\mathcal{E} \) is singular in this region only if the term \( (\zeta_\mathcal{E} + g_\mathcal{E}) \) vanishes for some \( t \) in the integration region. According to (3.7) this implies that, for \( \epsilon \) sufficiently small, \( F_\mathcal{E} \) is analytic in the region unless \( \zeta_\mathcal{E} = -z_\omega \). We will now study \( s \)-families which satisfy

this condition. For any s-family \( \mathcal{E} \), we let \( \mathcal{E}_m \subset \mathcal{E}_h \) denote the set of MS graphs belonging to \( \mathcal{E} \).

**Lemma 4.1.** — For any s-family \( \mathcal{E} \), \( \mathcal{E}_m \) is totally ordered by inclusion. If \( \mathcal{E}_m = \{ B_1, \ldots, B_r \} \) with \( B_i \subset B_{i+1} \), then \( B_i \in \mathcal{E}_h^0(B_{i+1}) \) and \( B_r \in \mathcal{E}_h^0(G_0) \).

**Proof.** — Since elements of \( \mathcal{E} \) are non-overlapping, the first statement will follow if we show that no two elements of \( \mathcal{E}_m \) are disjoint. Now note that, if \( B \in \mathcal{E}_m \) and \( H \in \mathcal{E}_h \) with \( H \supseteq B \), then either \( H \in \mathcal{E}_m \) or \( \omega \in H \), in which case \( H = H \cup G_0 \). Moreover, we cannot have \( B \subset Q \) for any \( Q \in \mathcal{E}_q \). If \( |\Omega^M| \geq 2 \) this is true because a line \( l \in \Omega^M - \{ \omega \} \) must lie in \( B \) but not in \( Q \); if \( |\Omega^E| \geq 2 \) because by choosing \( Q \) to be the minimal element of \( \mathcal{E}_q \) containing \( B \) and \( B' \) a maximal element of \( \mathcal{E}_h \) with \( B \subset B' \subset Q \) we will have \( B' \in \mathcal{E}_h^0(Q) \), and setting \( Q = G_0/S \) we see that any two external vertices are connected by distinct paths in \( S \) and \( B \), contradicting Definition 3.1 (4).

Suppose then that \( B_1, B_2 \in \mathcal{E}_m \) satisfy \( B_1 \cap B_2 = \emptyset \). Let \( H \) be the minimal element of \( \mathcal{E}_h \) containing \( B_1 \) and \( B_2 \); then there exist \( B'_1, B'_2 \in \mathcal{E}_m \) with \( B'_1, B'_2 \in \mathcal{E}_h^0 \). \( B_1 \cup B_2 \) will then be a link in \( G_0^\omega \), again contradicting Definition 3.1 (4). This proves the first part of the Lemma; the second follows from the observation above that if \( B \in \mathcal{E}_m \) and \( H \supseteq B \) with \( H \in \mathcal{E}_h \), then \( H \in \mathcal{E}_m \) or \( H = G_0 \).

**Lemma 4.2.** — If \( \mathcal{E} \) is an s-family, then \( \zeta_\mathcal{E} = -z_\omega \) if and only if \( \mathcal{E}_m \neq \emptyset \).

**Proof.** — Suppose that \( \zeta_\mathcal{E} = -z_\omega \); we will show that \( \mathcal{E}_h^0(G_0) \) contains an MS graph. For certainly, if we define

\[
G = \bigcup_{\mathcal{E}_h^0(G_0)} H,
\]

then \( \Omega_G \supseteq \Omega^M - \{ \omega \} \), since \( \Omega_{G_0} = \Omega_G \cup \bigcup_{\mathcal{E}_h^0(G_0)} \Omega_Q \cup \{ \omega \} \) and \( \Omega_Q \cap \Omega^M = \emptyset \) for any \( Q \in \mathcal{E}_m \). Moreover, \( G \) must connect all external vertices, since otherwise \( \zeta_\mathcal{E} \) would have the form of (3.5), case (iii). Thus the saturation of \( G \) in \( G_0^\omega \) is \( G_0^\omega \). By discarding those elements from the union (4.1) which are not necessary to make this last statement true, we may find an \( \mathcal{F} \subset \mathcal{E}_h^0(G_0) \) with \( \bigcup H \) a link in \( G_0^\omega \). Then Definition 3.1 (4) implies that \( |\mathcal{F}| = 1 \), i.e., \( \mathcal{E}_h^0 \) contains a graph which is a link in \( G_0^\omega \) and hence an MS graph.

Conversely, suppose that \( \mathcal{E}_m \neq \emptyset \). By Lemma 4.1 we may find a \( B \in \mathcal{E}_m \) with \( B \in \mathcal{E}_h^0(G_0) \). Let \( T \) be the tree of Lemma 3.2 (c). \( B \) connects all external vertices and, since \( T \cap B \) is a tree in \( B \), \( T \cap B \) does also. Then even if \( \sigma(G_0) \in T \), \( T - \{ \sigma(G_0) \} \supseteq T \cap B \) cannot separate the external vertices, so that \( \mathcal{E} \) must belong to case (i) of Lemma 3.2 (c). Then \( \sigma(G_0) \in \Omega^M \), \( \sigma(G_0) \notin \Omega_B \), and \( \Omega_B \supseteq \Omega^M - \{ \omega \} \) imply \( \sigma(G_0) = \omega \), completing the proof.
LEMMA 4.3. — If $\mathcal{E}$ is an $s$-family with $\mathcal{E}_m \neq \emptyset$, then
\[
g_\mathcal{E}(t, s, z) = \prod_{B \in \mathcal{E}_m} t_B h_\mathcal{E}(t, s, z),
\]
with $h_\mathcal{E}$ analytic and nonvanishing for $0 \leq t_K \leq 1$, $s(\chi) > 0$, and $z_l < 0$ ($l \neq \omega$).

Proof. — Since $\zeta_\mathcal{E} = -z_\omega$ (3.6) and (2.3)-(2.5) imply
\[
g_\mathcal{E} = \left[ \frac{1}{2} d(\chi)^{-1} \sum_{\chi} s(\chi) \left( \prod_{L \neq \Omega^m} \prod_{l \neq \omega} t_l \right) - \sum_{l \neq \omega} \alpha_l z_l \right]_{\chi_l = \Omega_0 \Pi \Pi_0} \tag{4.3}
\]
We make the indicated substitutions of the $t$ variables in (4.3), and use the known factorization (3.5) of $d(\chi)$. Observe that since $B \in \mathcal{E}_m$ connects the external vertices, a 2-tree $T_2$ which separates them can intersect $B$ in at most $n(B) - c(B) - 1$ lines, and hence the first term in (4.3) contains a factor $t_B$. Similarly, if $l \in \Omega^M \setminus \{ \omega \}$, then $l \in \Omega_l$, and hence the second term also contains a factor $t_B$; this proves (4.2).

Now let $B_1$ be the minimal element of $\mathcal{E}_m$, which exists by Lemma 4.1, and let $T$ be the tree of Lemma 3.2 (c). Then (as in that Lemma) either (i) $\sigma(B_1) \in \Omega^M$, (ii) $\sigma(B_1) \in T$ and $T - \sigma(B_1)$ separates $\Theta^E$ into non-empty subsets $\psi$ and $\Theta^E - \psi$, or (iii) both; the essential idea of the proof is that otherwise $\mathcal{E}_m^0(B_1)$ would contain an MS graph, contradicting the minimality of $B_1$. Consider case (i): then $x_{\sigma(B_1)} = \prod_{H \in \sigma(B_1)} t_H = \prod_{B \in \mathcal{E}_m} t_B$ by Lemma 4.1, so that
\[
h_\mathcal{E} = -z_{\sigma(B_1)} + \text{non-negative terms}
\]
and hence is nonvanishing. Cases (ii) and (iii) are similar.

Proof of Theorem 2.3. — According to the discussion at the beginning of this section, and Lemma 4.2, the only terms in $F = \Sigma F_\mathcal{E}$ which are singular at $z_\omega = 0$ are those for which $\mathcal{E}_m \neq \emptyset$. For such an $\mathcal{E}$, we have by Lemma 4.3 that
\[
F_\mathcal{E} = \Gamma(-\pi G_0) \int_0^1 \ldots \int_0^1 \int_{\mathcal{E}_m(G_0)} t_H^{-\pi H - 1} \prod_{B \in \mathcal{E}_m} t_B^{-z_B} (1 + e_\mathcal{E})^{-1/2}
\]
\[
\left( \prod_{B \in \mathcal{E}_m} t_B h_\mathcal{E} - z_\omega \right)^{\pi G_0} \tag{4.4}
\]
Now the pinch for $z_\omega \to 0$ in (4.4) is precisely that analyzed in Theorem A.1; specifically, the variables $t_B, B \in \mathcal{E}_m$, correspond to $u_1, \ldots u_n$ of that theo-
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rem, and $s$, $z(l \neq \omega)$, and $t_K$, $K \notin \xi_{m, r}$ to the variables $w$. Thus for $z_\omega$ near 0, (A.2) implies that

$$F_\delta = H_\delta + \sum_{B \in \xi_B} (-z_\omega)^{\nu_B - \nu_0} K_\delta B,$$

and since $\pi_G - \pi_B = \pi_{G/B}$, (2.8) is proved, with

$$H = \sum_{\xi_m = \emptyset} F_\delta + \sum_{\xi_m \neq \emptyset} H_\delta,$$

(4.5)

$$K_B = \sum_{\xi_m \geq B} K_\xi B.$$

From (A.3), $H|_{z_\omega = 0} = F_\delta|_{z_\omega = 0}$, so

$$H|_{z_\omega = 0} = \sum_{\xi} F_\delta|_{z_\omega = 0} = F_{G_0}|_{z_\omega = 0} = F_{G_0},$$

proving (2.9). There remains to prove (2.10), i.e.,

$$K_B|_{z_\omega = 0} = f_Q \prod_{i=1}^r F_{B_i},$$

(4.6)

for $B$ any MS graph, $B_1$, ..., $B_r$ the connected components of $B$, and $Q = G_0/B$.

We introduce the following notation. For any sub or quotient graph $K$, let $P_K = P^{N(K)-1}$ be projective space with homogeneous coordinates indexed by $\Omega_K$; let $V_K = \{ \alpha \in P_K | \alpha_l \geq 0, \forall l \}$, and let $D_K$ be the interior of $V_K$. There is a natural map $\psi_K : P_{G_0} \to P_K$ with $(\psi_K(z))_l = \alpha_l$, $l \in \Omega_K$.

For a fixed $s$-family $\delta$ and MS graph $B \in \xi_{m, r}$, let

$$J_\delta = \{ t_K, K \in \delta^0(G_0) | 0 < t_K \leq 1 \},$$

and let $J_{\delta B}$ be the (half-open) face of $J_\delta$ on which $t_B = 0$. Now (3.3) defines an invertible map $\phi_\delta : J_\delta \to D_{G_0} \cap D_\delta$; moreover, if $H$ is the minimal element of $\delta$ containing $B$ (see Lemma 4.1) then

$$t_B(z) = [\phi_\delta^{-1}(z)]_B = \alpha_{\sigma(B)/\alpha_{\sigma(H)}}$$

is actually well defined for all $z \in D_{G_0}$. Thus we have the diagram

$$\phi_{\delta}$$

\begin{array}{c}
\begin{array}{c}
\phi_\delta \times \chi \rightarrow \psi_B \times \psi_{Q} \times (\phi_{\delta}^{-1})_B = \chi
\end{array}
\end{array}$$

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Here $Q = G_0/B$, and $i : (0,1] \to \mathbb{R}$ is the natural inclusion. The indicated factorization of the composition map, with $\phi_{eB} : J_{eB} \to P_B \times P_Q$, follows from (3.3) since the ratio $\phi_e(l)/\phi_e(l')$ is independent of $t_B$ if $l, l' \in \Omega_B$ or $l, l' \in \Omega_Q$.

We will need certain properties of the map $\phi_{eB}$, described in

**Lemma 4.4.** — a) $\bigcup_{\{e\mid B \in \mathcal{E}_m\}} \phi_{eB}(J_{eB}) = \mathcal{D}_B^0 \times \mathcal{D}_Q^0$. b) If $\mathcal{E} \neq \mathcal{E}'$, then

$$\phi_{eB}(J_{eB}) \cap \phi_{e'B}(J_{e'B})$$

has measure 0.

**Proof.** — If $(\beta, \gamma) \in \mathcal{D}_B^0 \times \mathcal{D}_Q^0$, we normalize by setting $\gamma_l = \beta_{l'} = 1$ for some $l \in \Omega_Q, l' \in \Omega_B$. For $x > 0$ define $\varphi(x) \in P_{G_0}$ by $x_l = x\beta_l, l \in \Omega_B, x_l = \gamma_l, l \in \Omega_Q$, and suppose $x$ is small enough so that $x_l < x_{l'}$ for all $l \in \Omega_B, l' \in \Omega_Q$. By Lemma 3.2 (b) there is an $s$-family $\mathcal{E}$ with $\varphi(x) \in \mathcal{D}_e$; and $\mathcal{E}$ is unique if $\beta_l \neq \beta_{l'}, l, l' \in \Omega_B$, and $\gamma_l \neq \gamma_{l'}, l, l' \in \Omega_Q$. Moreover, it follows from the construction method for $s$-families described in [2] that $B \in \mathcal{E}$. Now if $t(x) = \phi_{e^{-1}}(\varphi(x)) \in J_{e}, (t(x))_K$ is independent of $x$ for $K \neq B$; then the point $\tilde{t} \in J_{eB}$ for which $\tilde{t}_K = (t(x))_K, K \neq B$, satisfies $\phi_{eB}(\tilde{t}) = (\beta, \gamma)$. This proves (a); (b) follows from the uniqueness of $\mathcal{E}$ noted above.

We now continue with the verification of (4.6). Let us write

$$f_Q(\varphi, \psi) = \Gamma(-\pi_Q) \int_{\mathcal{D}_Q} \theta_Q,$$

with $\theta_Q$ the differential form given in (2.7), and

$$\prod_{i=1}^{r} \mathbf{F}_{Bi} = \prod_{i=1}^{r} \biggl\{ \Gamma(-\pi_{Bi}) \int_{\mathcal{D}_{Bi}} \prod_{l \in \Omega_{Bi}} \beta_l^{-d_{Bi}^{v/2}}(\mathcal{D}_{Bi}^*)^{-\pi_{Bi}} \eta \biggr\} = \Gamma(-\pi_{Bi}) \int_{\mathcal{D}_{Bi}} \theta_{Bi},$$

with

$$\theta_B = \prod_{l \in \Omega_B} \beta_l^{-d_{Bi}^{v/2}} \sum_{i=1}^{r} \mathcal{D}_{Bi}^* \eta;$$

the last equality is a projection-space variant of Feynman's formula for the combination of denominators. Thus by Lemma 4.4,

$$f_Q \Pi_{B_i} = \Gamma(-\pi_Q) \Gamma(-\pi_B) \int_{\mathcal{D}_B \times \mathcal{D}_Q} \theta_B \wedge \theta_Q$$

$$= \Gamma(-\pi_Q) \Gamma(-\pi_B) \sum_{\{e\mid B \in \mathcal{E}_m\}} \int_{J_{eB}} \phi_{eB}^*(\theta_B \wedge \theta_Q), \quad (4.8)$$

with $\phi^*_{B}$ the standard pullback map on differential forms. However, from (4.4) and (A.4),

$$K_B |_{z_\omega} = \Gamma(-\pi_Q)\Gamma(-\pi_B) \int_{\theta_B}$$

(4.9)

with

$$\theta_B = \prod_{K \in \mathcal{S}(G_0)} t_K^{\pm \pi_K - 1} dt_K \left\{ [1 + e_\varepsilon(1)]^{-\gamma/2} \left[ h_\varepsilon \prod_{B^* \not= B} t_{B^*} \right]^{-\pi_B} \right\} \bigg|_{t_B = 0}$$

(4.10)

where the exponent is $+\pi_K(-\pi_K)$ if $K \in \mathcal{S}_q(K \in \mathcal{S}_h)]$. Comparing (4.5) and (4.9) with (4.8), we see that (4.6) will follow if we can show that

$$\theta_B = \phi^*_{B}(\theta_B \wedge \theta_Q).$$

(4.11)

Let

$$\theta = \prod_{\alpha \in \Omega} \chi_i^*d(\alpha)^{-\gamma/2}D^*(\xi, \zeta)^{g_{\varepsilon}}$$

be the form whose integral defines $F_{G_0} (2.1)$. Then the calculation leading to (3.4) says that

$$\phi^*_{\varepsilon}(\theta) = \rho \wedge t_B^{-\pi_Q + 1} dt_B,$$

(4.12)

where

$$\rho = \prod_{K \not= B} t_K^{\pm \pi_K - 1} dt_K (1 + e_\varepsilon)^{-\gamma/2} (g_\varepsilon - z_\omega)^{g_{\varepsilon}}.$$  

From (4.2) and (4.10),

$$\theta_B = \left( - z_\omega \right)^{-g_{\varepsilon}} \left\{ \left[ \frac{\partial}{\partial t_B} \phi^*_{\varepsilon}(D^*) \right] \rho \right\} \bigg|_{t_B = 0}$$

(4.13)

(note $\phi^*_{\varepsilon}(D^*) = g_\varepsilon - z_\omega$).

On the other hand, the map $\chi$ of (4.7) is a diffeomorphism onto its range, so we may calculate $(\chi^{-1})^*(\theta)$. In order to make the factors $d$, $D^*$, etc. well defined functions, we normalize coordinates in $\mathbb{P}_{G_0}$ by $\alpha_\omega = 1$, and in $\mathbb{P}_B \times P_Q$ by $\beta_{\sigma(B)} = 1$. (This normalization was adopted in calculating $\phi^*_{\varepsilon}(D^*)$ above.) Then

$$(\chi^{-1})(\theta) = \left[ (\chi^{-1})^*d \right]^{-\gamma/2} \left[ (\chi^{-1})^*D^* \right]^{\sigma_0} \left[ (\chi^{-1})^*(\prod \delta^i) \right],$$

and since $\eta = \prod_{\alpha \not= \omega} d\alpha_i$ with our normalization,

$$\chi^{-1} \left( \prod \chi_i^* \eta \right) = \prod \beta_i^{\delta_i} \prod \gamma_i^* \eta \left( \prod_{j=0}^{\sum (\delta_j + 1) - 1} dt_B. $$

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Now it is shown in [1], Lemma 4.3.4, that
\[ [t_B^{-hB}(\chi^{-1})^*(d)]_{t_B=0} = \rho \sum_{i} (c_{ni}^B) \]

and hence
\[ (\chi^{-1})^*(\theta) = \hat{\rho} \land t_B^{-nB+1}dt_B \]  
(4.14)

with
\[ \theta_B \land \theta_Q = (-z_{\omega})^{-\pi_{\theta}} \left\{ \left[ \frac{\partial}{\partial t_B} ((\chi^{-1})^*(D^*)) \right]^{nB}_{t_B=0} \right\} \]
(4.15)

Since \( \phi_{\epsilon}^*(\theta) = (\phi_{\epsilon B} \times i^*)[(\chi^{-1})^*(\theta)] \) from (4.7), (4.12) and (4.14) imply that
\[ \rho \land t_B^{-nB+1}dt_B = (\phi_{\epsilon B} \times i^*)[\hat{\rho} \land t_B^{-nB+1}dt_B] \]

\[ = [(\phi_{\epsilon B} \times i^*)(\hat{\rho})] \land t_B^{-nB+1}dt_B \]

(i is essentially the identity map.) But then
\[ \rho |_{t_B=0} = [(\phi_{\epsilon B} \times i^*)(\hat{\rho})] |_{t_B=0} = \phi_{\epsilon B}^*(\hat{\rho} |_{t_B=0}), \]

and (4.13) and (4.15) imply (4.11), since
\[ (\phi_{\epsilon B} \times i^*) \left[ \frac{\partial}{\partial t_B} ((\chi^{-1})^*(D^*)) \right] = \frac{\partial}{\partial t_B} [(\phi_{\epsilon B} \times i^*)(\chi^{-1})^*(D^*)] = \frac{\partial}{\partial t_B} \phi_{\epsilon}^*(D^*) \]

and hence
\[ \phi_{\epsilon B}^* \left\{ \left[ \frac{\partial}{\partial t_B} ((\chi^{-1})^*(D^*)) \right]_{t_B=0} \right\} = \left[ \frac{\partial}{\partial t_B} \phi_{\epsilon}^*(D^*) \right]_{t_B=0}. \]

This completes the proof of Theorem 2.3.
APPENDIX

In this appendix we discuss the behavior of a certain analytic function, defined by a multiple integral, near one of its singular points. The singularity is due to a (non-simple) pinch in the integration space which may be described as follows: there are $n$ singular surfaces for the integrand which are in general position; in the pinch configuration, an additional singular surface degenerates into the union of these. The integrand is multiple-valued and infinitely ramified around each singular variety.

Suppose then that $W$ is a compact subset of $\mathbb{R}^n$, that $J \subset \mathbb{R}^n$ is the unit cube $\{ y | 0 \leq u_i \leq 1 \}$, and that $h(y, w)$ and $g(y, w)$ are real analytic on an open neighborhood of $J \times W$, with $g > 0$. For $z < 0$, $w \in W$, and $\underline{x} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{C}^{n+1}$ with $\text{Re} \alpha_i > -1$ for $i = 1, 2, \ldots, n$, define

$$G(z, w, z) = \int_J \Pi_{i=1}^n u_i^{\alpha_i} (h(y, w) \prod_{i=1}^n u_i - z)_{\alpha_0}^n.$$  \hspace{1cm} (A.1)

[In (A.1), and throughout this appendix, it is understood that a complex power of a positive quantity is defined using the principal branch of the logarithm.] We note that $G$ may be analytically continued to a meromorphic function of $\underline{a} \in \mathbb{C}^{n+1}$ by an integration by parts. Here we study the behavior of $G$ under analytic continuation in the variable $z$ throughout a punctured disc $D_\varepsilon = \{ z | 0 < |z| < \varepsilon \}$. We will prove

THEOREM A.1. — For sufficiently small $\varepsilon$, $G$ may be analytically continued along any path in $D_\varepsilon$. For generic $\underline{a}$ there is a decomposition

$$G(z, w, z) = G_0(z, w, z) + \sum_{i=1}^n (-z)^{\alpha_0 + \alpha_i + 1} G_i(z, w, z),$$  \hspace{1cm} (A.2)

where $G_0, G_i$ are analytic at $z = 0$. Moreover, we have the formulae

$$G_0|_{z=0} = G|_{z=0} = \int_J \Pi_{i=1}^n u_i^{\alpha_i + \alpha_0} du h u_0^{\alpha_0},$$  \hspace{1cm} (A.3)

valid for $\text{Re} \alpha_i + \alpha_0 > -1$, and

$$G_i|_{z=0} = \frac{\Gamma(- (\alpha_0 + \alpha_i + 1)) \Gamma(\alpha_i + 1)}{\Gamma(- \alpha_0)} \int_0^1 \cdots \int_0^1 \prod_{j \neq i} u_j^{\alpha_j - \alpha_i - 1} du_j \times [h(y, w) g(y, w)^{-\alpha_i - 1}]_{u_j=0},$$  \hspace{1cm} (A.4)

valid for $\text{Re} \alpha_j > \text{Re} \alpha_0, j \neq i$.

We will prove Theorem A.1 by a sequence of lemmas; our approach will necessitate an explicit construction of the analytic continuation of $G$ throughout $D_\varepsilon$. Let $S_\theta$ denote the operation of analytic continuation clockwise around $z = 0$ by angle $\theta$, so that if $F(z)$ is the germ of some analytic function defined for $\arg z = \phi$, $S_\theta F$ is a germ defined for $\arg z = \phi + \theta$. We write $T \equiv S_{2\pi}$.

Formula (A.1) defines $G$ for $\text{Re} z < 0$. Then clearly

$$(S_\phi - S_{-\phi}) G = (e^{i\pi \alpha_0} - e^{-i\pi \alpha_0}) H,$$  \hspace{1cm} (A.5)

with $H$ defined for $z > 0$ and $\alpha_i > 0, i = 0, 1, \ldots, n$, by

$$H(z, w, z) = \int_J \Pi_{i=1}^n u_i^{\alpha_i} du h(y, w) [z - g(y, w) \Pi u_i]_{\alpha_0}^n \quad J_0 = J \cap \left\{ y | g \left[ \frac{1}{\Pi_{i=1}^n u_i} \leq z \right] \right\}.$$  \hspace{1cm} (A.6)
LEMMA A.2. — Let \( X_i = \exp 2\pi i(a_0 + a_i), \) \( i = 1, \ldots, n. \) Then
\[
(T - X_1)(T - X_2) \ldots (T - X_n)H = 0.
\]

Proof. — Case 1. We first consider the special case \( g(u, w) = 1, \) and construct explicitly the contour deformation necessary for the analytic continuation of \( H. \) For \( R > 0, \theta > 0, \) and \( A \subseteq \{1, \ldots, n\}, A \neq \emptyset, \) the contour \( C(A, \theta, R) \subset \mathbb{C}^n \) is defined in terms of parameters \( r, v_i(i \notin A), \) and \( \phi_i(i \in A) \) as follows:
\[
u_i = \begin{cases} v_i(1 - r) + r & i \notin A, \\ r e^{i\phi_i} & i \in A, \end{cases}
\]
with \( 0 \leq v_i \leq 1, \phi_i \geq 0 \) and \( \sum \phi_i = \theta, \) and \( 0 \leq r \leq \rho(\theta, R), \) where \( \rho(\theta, R) \) is the smallest positive root of
\[
| \Pi u_i | = r^{A} \prod_{i \in A} [v_i(1 - r) + r] = R.
\]

Note that, for fixed \( \theta \) and \( R, \) the contours \( C(A, \theta, R) \) form a polyhedral complex on whose boundary either \( u_i = 0 \) for some \( i, \) \( u_i = 1 \) for some \( i, \) or \( \Pi u_i = \Pi e^{i\theta}. \) Moreover, for \( \theta = 0 \) this complex reduces to \( J_0. \) Thus
\[
S_{\theta}H = \sum_{A} \int_{C(A, \theta, |z|)} \Pi u_i^0 du_i h(u, w)(z - \Pi u_i)^{\rho_0} \tag{A.7}
\]
where on \( C(A, \theta, |z|), u_i^0 \) is defined using \( \arg u_i = \phi_i, i \in A, \) and \( (z - \Pi u_i)^{\rho_0} \) is defined using \( \arg (z - \Pi u_i) = \theta. \)

We now partially evaluate the integral in a typical term of (A.7). Since \( \rho(\theta, R) \to 0 \) as \( R \to 0 \) we may, by choosing \( \varepsilon \) and hence \( |z| \) sufficiently small, expand \( h(u, w) \) in a power series converging uniformly on \( C(A, \theta, |z|): \)
\[
h(u, w) = \sum_{i} h_i(u_0, w) \prod_{j \in A} u_j^i,
\]
the sum running over multi-indices \( i = (i_j), j \in A, \) and \( u_A \) denoting the variables \( (u_j), j \notin A. \) The Jacobian of the variable change on \( C(A, \theta, |z|) \) from \( u \) to \( (u_0, \phi, r) \) is of the form \( F(r)e^{i\theta}, \) so that (A.7) becomes
\[
S_{\theta}H = \sum_{A} \sum_{i} H_{A, i|z|, i} \left\{ \int_{\ldots} e^{i\phi_j + i^0} \prod_{j \in A} u_j^i \right\} \tag{A.8}
\]
with
\[
H_{A, i|z|, i} = \int dr \prod_{j} dv_j F(r) h_j(u_0, w) \prod_{j \notin A} r^j \prod_{j \in A} u_j^i.
\]
The only \( \theta \) dependence in (A.8) is in the bracketed term, which may be evaluated to give
\[
(-i)^{|A|-1} \sum_{j \in A} \frac{e^{i\phi_j + i^0}}{\prod_{k \neq j}(\alpha_j + i_j - \alpha_k - i_k)} \tag{A.9}
\]
(We assume that the \( \alpha \) variables are chosen generically so that the denominators do not vanish). If \( \theta = 2\pi i, \) (A.9) reduces to the form
\[
\sum_{j \in A} f_j(z)X_j.
\]
Thus
\[
\prod_{k=1}^{n} (T - X_k)H = \sum_{i=0}^{n} \sum_{B = \{1, \ldots, n\}} \left( \prod_{k \in B} X_k \right) (T^H)
\]

\[
= \sum_{A,L} H_{A,L} \sum_{j \in A} f_j(x) \sum_{i=0}^{n} X_i \sum_{B \subseteq \{1, \ldots, n\}} \prod_{k \in B} X_k.
\]

But
\[
\sum_{i=0}^{n} X_i \sum_{B \subseteq \{1, \ldots, n\}} \prod_{k \in B} X_k = \prod_{i=1}^{n} (Y - X_i)_{|Y = X_i} = 0,
\]

so that the lemma is proved in the case \( g = 1 \).

**Case 2.** We now assume that
\begin{equation}
\left| \frac{\partial g}{\partial u_i} \right| < \frac{g}{n}
\end{equation}
on \( J \times W \). Let \( U \subset \mathbb{R}^n \) be a convex open neighborhood of \( J - \{(1, 1, \ldots, 1)\} \) on which \( g \) is analytic, (A.10) holds, and
\[
d_i(u) = \frac{1}{n} \left( \sum_{j=1}^{n} (1 - u_j) \right)
\]
satisfies \( |u_i d_i(u)| \leq 1 \) for all \( i \). For \( u \in U \), define
\begin{equation}
x_i(u, w) = u_i g(u, w)^{y/w}(u)
\end{equation}
Then for fixed \( w \), (A.11) is 1 on \( U \), since if \( u_1 \) and \( u_2 = u_1 + 1 \) are in \( U \), and \( i \) is chosen so that \( |s_i| \geq \frac{1}{\sqrt{n}} |s_i| \), an easy calculation using (A.10) shows that \( \sum_{j} \frac{\partial x_i}{\partial u_j} s_j \) has the same sign as \( s_i \), and hence
\[
x_i(u_2, w) - x_i(u_1, w) = \int_{0}^{1} \sum_{j} s_j \frac{\partial x_i}{\partial u_j} (u_1 + t 1) dt \neq 0.
\]

Choosing \( \varepsilon \) small enough so that \( J_0 \subset U \), we may rewrite (A.6) as
\begin{equation}
H = \int_{J_0} \prod_{i=1}^{n} x_i p d x_i h_i(x, w) y w^{y} g_j(x, w)(z - \prod_{i=1}^{n} x_i)_{x_i}^{< 0},
\end{equation}
where
\[
J_0 = \left\{ x \mid 0 \leq x_i \leq 1, \prod_{i=1}^{n} x_i \leq z \right\}.
\]
and \( j(x, w) \) is the Jacobian of (A.11). Applying Case 1 to (A.12) completes the proof of Case 2.

We now discuss the general case. Since \( g \) is positive on \( J \times W \), there is an \( M > 0 \) such that
\begin{equation}
\left| \frac{\partial g}{\partial u_i} \right| \leq Mg
\end{equation}
on \( J \times W \), for all \( i \). Taking \( N \) a positive integer with \( N > nM \), we subdivide the cube \( J \).
into subcubes of side $1/N$; thus $H = \sum K$, where $K$ denotes a subcube and $H_K$ is the integral (A.6) taken over $J_0 \cap K$. By choosing $\varepsilon$ (and hence $|z|$) sufficiently small we may guarantee that $J_0$ intersects only those cubes $K = \{ u \in J_0 / N \leq u_i \leq (j_i + 1)/N \}$ for which at least one $j_i$ is zero. Suppose that $K$ is such a cube, with $j_i = 0$ for $i \in A \subset \{ 1, \ldots, n \}$. In the integral for $H_K$ we introduce new variables by $u_i' = N u_i$, $i \in A$; $u'_J = (w, y_A)$, where $y_A = (u_i), i \notin A$. Then $H_K$ is itself of the form (A.6) [with an additional integration over some $w$ variables, which does not affect the argument], but

$$\left| \frac{\partial \delta}{\partial u'_i} \right| = N^{-1} \left| \frac{\partial \delta}{\partial u_i} \right| < g$$

for $i \in A$, by (A.13). Case 2 then implies that $\prod_{i=1}^n (T - X_i) H_K = 0$, from which the lemma follows.

**Lemma A. 3.** For $\varepsilon$ sufficiently small, and $z \in D_\varepsilon = \{ z \mid 0 < |z| < \varepsilon \}$,

$$H(z, w, z) = \sum_{i=1}^n z^{a_i + a_i + 1} H_i(z, w, z)$$

(A.14)

$$G(z, w, z) = G_0(z, w, z) + \sum_{i=1}^n (-z)^{a_i + a_i + 1} G_i(z, w, z)$$

(A.15)

with $G_0, G_i$ and $H_i$ single valued in $D_\varepsilon$.

**Proof.** We take $x$ a generic point for which $X_i \neq X_j \neq 1$, for any $i, j$. Then

$$f(y) = \prod_{i=1}^n \prod_{j \neq i} \frac{Y - X_i}{X_j - X_i} = 1$$

for all $Y$, since the left hand side is a polynomial in $Y$ of degree $n - 1$, and equality holds at the $n$ points $Y = X_i$. Hence (A.14) will be satisfied with

$$H_i = z_i^{-(a_i + a_i + 1)} \prod_{j \neq i} \frac{(T - X_i)}{(X_i - X_j)} H_i.$$ 

(A.16)

Lemma (A.2) then implies that $T H_i = H_i$, i.e., $H_i$ is single valued.

Now from (A.15),

$$(T - 1) G = (e^{\pi i a_0} - e^{-\pi i a_0}) S_a H.$$ 

Applying $\Pi(T - X_i)$ to this equation gives $(T - 1) \prod_{i=1}^n (T - X_i) G = 0$, and an argument as above yields (A.15).

**Remark A. 4.** — If we insert (A.15) into (A.5), we find

$$(e^{\pi i a_0} - e^{-\pi i a_0}) H = \sum_{i=1}^n [e^{\pi i (a_0 + a_i + 1)} - e^{-\pi i (a_0 + a_i + 1)}] z_i^{a_0 + a_i + 1} S_a G_i.$$ 

Comparison with (A.14) shows that \( \sin (\alpha_0 \pi) H_i = \sin [(\alpha_0 + \alpha_i + 1)\pi] S \alpha_i G_i \). Since these functions are single valued the operator \( S \) is redundant, and

\[
G_i = \frac{\sin \alpha_0 \pi}{\sin (\alpha_0 + \alpha_i + 1)\pi} H_i \tag{A.17}
\]

**Proof of Theorem A.1.** — We first show that the functions \( G_0, G_i, H_i \) of Lemma A.3 have removable singularities at \( z = 0 \). By a straightforward calculation it may be shown that the area of the contour \( C(\alpha, \theta, |z|) \) need to define analytic continuations of \( H \) (Lemma A.2), Case 1) is bounded by a multiple of \( |z|^{1/n} \), if \( \theta \) is bounded. Moreover, if \( \Re \alpha_i \geq 0 \) for \( i = 0, 1, \ldots, n \), the integrand in (A.7) is uniformly bounded on \( C(\alpha, \theta, |z|) \); hence, for bounded \( \theta \),

\[
|S \alpha H| \leq K |z|^{1/n}.
\]

If \( \Re (\alpha_0 + \alpha_i) < 1/n \), (A.16) implies that \( \lim_{z \to 0} z H_i = 0 \), i.e., \( H_i \) has a removable discontinuity for \( z \) in the above range. Since \( H \) is meromorphic in \( z \), the discontinuity is removable for all \( z \). Then (A.17) shows that \( G_i \) is also analytic at \( z = 0 \); an argument similar to the above implies the same conclusion for \( G_0 \).

To verify (A.3), we note from (A.2) that, if \( \Re (\alpha_i + \alpha_0) > -1 \) for all \( i \),

\[
G_0 |_{z=0} = \lim_{z \to 0} G_i.
\]

(A.3) follows from the Lebesque dominated convergence theorem.

It remains to verify (A.4) which, by (A.17), is equivalent to

\[
H_i(z, w, 0) = \frac{\Gamma(\alpha_i + 1)\Gamma(\alpha_0 + 1)}{\Gamma(\alpha_i + \alpha_0 + 2)} \int_0^1 \cdots \int_0^1 \prod_{j \neq i} u_j^{\alpha_j - \alpha_i - 1} du_j \times [h(u, w)g(u, w)^{n+1}]_{u=0}. \tag{A.18}
\]

On the other hand, (A.14) implies that if \( \Re \alpha_i > \Re \alpha_j \) for all \( j \neq i \),

\[
H_i(z, w, 0) = \lim_{z \to 0} z^{-(\alpha_i + \alpha_n + 1)} H_i(z, w, z). \tag{A.19}
\]

We will show that (A.18) and (A.19) agree for

\[
\Re (\alpha_j + 1) > n \Re (\alpha_i + 1) > 0, \quad (j = 1, \ldots, \hat{i}, \ldots, n),
\]

\[
\Re \alpha_k > 0, \quad (k = 0, 1, \ldots, n) \tag{A.20}
\]

the result then follows whenever \( \Re \alpha_j > \Re \alpha_i \) by analytic continuation. We assume \( z \geq 0 \).

Let us write \( H = \int_{J_0} \theta \), with \( \theta \) the \( n \)-form given in (A.6). For fixed \( w \), decompose \( J_0 \) as \( J_0 = J_a \cup J_b \) where \( J_a = J_0 \cap \{ u \mid g(u, w) \int_{J_j} u_j \geq z \} \), and \( J_b = J_0 - J_a \subseteq \bigcup_{j \neq i} J_j \) where \( J_j = \{ u \mid 0 \leq u_k \leq 1, u_j \leq z^{1/n} \} \). Now

\[
\left| z^{-(\alpha_0 + \alpha_i + 1)} \int_{J_j} \theta \right| \leq \left| z^{-(\alpha_i + 1)} \int_{J_j} u_j^{\alpha_j - \alpha_i - 1} du_j \right| \leq \frac{1}{(\Re \alpha_j + 1)} \left| \int_{J_j} u_j^{\alpha_j - \alpha_i - 1} du_j \right| \leq \frac{1}{(\Re \alpha_j + 1)} \left| \int_{J_j} u_j \right| \leq \frac{1}{(\Re \alpha_j + 1)} \left| \int_{J_j} u_j \right| = \frac{1}{(\Re \alpha_j + 1)} \left| \int_{J_j} u_j \right| = 0.
\]

On the other hand,

\[
\int_{J_i} \theta = \int_{J_i} \prod_{j \neq i} u_j^{\alpha_j - \alpha_i - 1} du_j \int_{0}^{1} U^{\alpha_i}(1 - U)^{\alpha_i} dU h^{-\alpha_i + 1}.
\]

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where we have made the substitution
\[ U = u_i g(u, w) \prod_{j \neq i} u_j / z, \]
and
\[ I = \left\{ (u_1, \ldots, \tilde{u}_i, \ldots, u_n) \mid g \prod_{j \neq i} u_j \geq z \right\}. \]

\( \tilde{h}(U, u, w) = h(u, w) \bigg|_{u_i = u \mid \tilde{u}_i = 1} \]

etc. For \( \alpha \) satisfying (A.20) we may apply the Lebesgue dominated convergence theorem to find
\[
\lim_{z \to 0} z^{-(a_0 + a_i + 1)} \int_0^1 \ldots \int_0^1 \prod_{j \neq i} u_j^{a_j - a_i - 1} du_j [h(u, w)g(u, w)^{-i(a_i + 1)}] \bigg|_{u_i = 0} \quad (A.22)
\]

(A.19), (A.21) and (A.22) imply (A.18), completing the proof.

REFERENCES


(Manuscrit reçu le 1er décembre 1975)