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Addendum to « Rigorous absolute bounds for pion-pion scattering. II. Solving modified Szegö-Meiman problems »

by

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In ref. [*1*], use has been made of an asymptotic formula for Legendre polynomials, (see eq. (J. 3), p. 364) which is due to Watson [*2*]. After that article was published, we discovered that Watson's formula indeed is wrong. The object of this note is, firstly, to establish the correct one, secondly, to make the appropriate changes in ref. [*1*].

The starting point of Watson in its original paper [*3*] is the following expression of Legendre polynomials.

$$P_l(\cosh \tau) = \frac{1}{\pi}\int_{-\tau}^{\tau} e^{-lt}\{(1 - e^{t-\tau})(e^{t+\tau} - 1)\}^{-\frac{1}{2}}dt, \tag{1}$$

which can be rewritten as

$$P_l(\cosh \tau) = \frac{1}{\pi}\int_{-\tau}^{\tau} e^{-(l+\frac{1}{2})t}\{2(\cosh \tau - \cosh t)\}^{-\frac{1}{2}}dt. \tag{2}$$

Although this last formula has not been explicitely written in ref. [*3*], it seems to us that Watson has used it with e^{-lt} instead of $e^{-(l+\frac{1}{2})t}$. As a consequence in his asymptotic estimate of P_l (cosh τ), l has to be changed into $l + \frac{1}{2}$, which gives, in place of the wrong formula (5.72.3) of ref. [*2*].

$$P_l(\cosh \tau) = \left(\frac{\tau}{\sinh \tau}\right)^{\frac{1}{2}} I_0\left(\left(l + \frac{1}{2}\right)\tau\right) + c\,|\,\tau\,|^2 \frac{\sinh \operatorname{Re}\left[\left(l + \frac{1}{2}\right)\tau\right]}{\operatorname{Re}\left[\left(l + \frac{1}{2}\right)\tau\right]}, \tag{3}$$

where τ and l may be complex, and*

$$|c| < \frac{\sqrt{10}}{16\,\pi}, \qquad \text{for} \quad |\tau| \leqslant 1\,. \tag{4}$$

From this result, one easily obtains that, when τ complex goes to zero and $|\operatorname{Arg}\tau| \leqslant \frac{\pi}{2} - \varepsilon$, ε being fixed and strictly positive, then

$$\frac{1}{\mathrm{P}_l\,(\cosh\tau)^{\omega}} = \frac{1}{\mathrm{I}_0\left(\left(l+\frac{1}{2}\right)\tau\right)^{\omega}}[1 + \mathcal{O}(\tau^2)]\,, \qquad (\omega > 0) \tag{5}$$

where $\mathcal{O}(\tau^2)$ is uniform in $l(0 \leqslant l \leqslant \infty)$. This modifies Eqs. (J.5) and (J.6) of ref. [1], and leads to the following formula

$$\sum_{l\,\text{even}} \frac{2l+1}{\mathrm{P}_l(z)^{\omega}} = \frac{1}{z^2-1}\int_0^{\infty}\frac{xdx}{\mathrm{I}_0(x)^{\omega}} + \mathrm{R}_{\omega}(z)\,, \tag{6}$$

where $\mathrm{R}_{\omega}(z)$ remains bounded when

$$z \to 1\,,\ |\operatorname{Arg}(z-1)| \leqslant \pi - \varepsilon\,, \tag{7}$$

ε being as above.

Furthermore, by using the Cauchy inequalities, one immediatly obtains that the derivative of $(z-1)\mathrm{R}_{\omega}(z)$ remains bounded under conditions (7), a result much stronger than the Lipschitz continuity of $(z-1)\mathrm{R}_1(z)$ for real $z \geqslant 1$, proved in ref. [1].

Note that the second term in the right hand-side of Eq. (J.6) of ref. [1] has now disappeared in the correct formula (6). As a consequence, the term $\rho_0\sqrt{\cos\frac{\theta}{2}}$ in the decomposition (II, 42) of ref. [1] of the weight function $\bar{\rho}(\theta)$, term which is Lipschitz continuous of order $\frac{1}{2}$, is not there, and $\bar{\rho}(\theta)$ is itself Lipschitz continuous of order 1. This simplifies the proof of Theorem 2 which of course still holds. Nothing is changed concerning the numerical calculation of bounds.

(*) We do not agree with Watson's calculation of the upper bound of $|c|$ in the neighbourhood $|\tau| \leqslant 1$, namely $|c| < 2/(5.6^{3/2})$, which is more than two times smaller than our figure.

REFERENCES

[1] G. AUBERSON, L. EPELE, G. MAHOUX and F. R. A. SIMÃO, *Ann. Inst. Henri Poincaré*, t. **XXII**, 1975, p. 317.

[2] G. N. WATSON, *A treatise on the theory of Bessel functions*. Cambridge University Press, 1958, section 5.72.

[3] G. N. WATSON, *Messenger of Mathematics*, t. **XLVII**, 1918, p. 151.

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