ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 25, nº 2 (1976), p. 177-182
<http://www.numdam.org/item?id=AIHPA_1976__25_2_177_0>

Ann. Inst. Henri Poincaré
Vol. XXV, n° 2, 1976, p. 177-182.

Section A :
Physique théorique.

On the principle of stability of invariance of physical systems

by

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ABSTRACT. — We take the point of view :
(*i*) that the invariance principles are fundamental laws of nature;
(*ii*) that the qualitative aspects of a physical system are equally important. With this in mind we study a physical system. The concept of « invariance stability » is introduced analogous to structural stability. As a consequence, we find that only those conservation laws which are due to abelian symmetries can not be violated. For instance energy-momentum and charge conservation. All other conservation laws due to non-abelian (continuous) symmetries can always be violated by the variation of some parameter of the theory concerned.

0. INTRODUCTION

The purpose of this note is to point out the qualitative aspects of the behavior of a physical system using only its invariance properties. To accomplish this, we abondon the equations of motion (or field equations) and structural stability and instead introduce the concept of *invariance stability* with invariance principles as basic laws of physics.

Historically, H. Poincaré [*1*] was one of the first to point out the importance of qualitative analysis in connection with his studies on dynamical systems. He visualized a dynamical system as a field of vectors on phase space in which a solution of the differential equations of motion is a smooth curve, tangent at each of its point to the vector based at that point. The first important aspect of this qualitative theory is the study of geometrical properties (from the global point of view) of the phase portrait. A phase portrait is the family of solution curves which fill up the entire phase space. The second aspect is the replacement of analytical methods by differential topological one and the most important aspect of the qualitative point of view is the study of structural stability [2]. In 1937 this problem was posed by Andronove and Pontraigin and later on Smale, Peixoto, Zeeman, Thom and many others [2] studied it extensively. Especially R. Thom [4] has applied this idea of qualitative analysis to biology, linguistics, and many other branches of science.

The laws of nature can be described in many ways. They can be expressed either in the form of differential equations of motion of the changes of the physical system or in terms of certain conditions on the physical quantities like scattering matrix. There are also symmetry properties of the physical systems which one consider to describe the system. Invariance principles were shown to be of very fundamental nature in the form of theory (special) of relativity. The special theory of relativity is an invariance principle which is applicable to all systems irrespective of their energies. However, at low energies it is not possible to measure the relativistic effects with the present days experiments. It is only at high energies that we know the effects are measurable. More recently, these principles of invariance [3] were exploited to explain the properties of elementary particles. It is here in elementary particles physics that the equations of motion (or exact Lagrangian and field equations) are not well known and one has to rely on other methods like studying the analytic properties of measurable quantities, for instance, dispersion relations. However, most of these models describe more of a quantitative aspect than a qualitative one. We are concern with the qualitative behaviour of the system from the point of of its symmetry group as the parameters of the theory are varied. For a general reference in this direction we refer to R. Thom's book on this subject [4].

As we know that in the theory of dynamical systems, one study the stability of vector fields under small variations of some parameter [11]. The topological properties of vector fields reflects the behaviour of a given dynamical system. In an analogous fashion we consider the topological properties of Lie-algebras of symmetry groups under parametric variations. Thus we replace the equation of motions (mathematically vector fields on manifold) with groups of symmetry and their Lie-algebras, the topological properties of vector fields with those of Lie-algebras, and structural stability with invariance stability. In the light of success of group theoretical methods in high energy physics [3] and failure of field theory, our hypothesis is quite reasonable.

In the next section we formulate mathematical definition of invariance stability and the last section describes few propositions and remarks in this connection.

1. PRINCIPLE OF STABILITY OF INVARIANCE

Consider a physical system with certain symmetry properties in the usual sense. Let G be the group of invariance. That is under the action of group elements the new systems obtained are indistinguishable from the original one. We do not need to know the dynamics of the system for this purpose. It is sufficient that we label our system by the invariants of G. Now consider the effect of variation of some parameter on the symmetry group G. For example, a relativistic system becomes an action-at-a distance system when Poincaré group goes to Galilei group due to variation in the velocity of light [*12*], (in fact this corresponds to $c \to \infty$, but it is a matter of which units one chooses). To say it mathematically, under what condition and how a group changes to an non-isomorphic group? When it is stable with respect to parametric variations? For our purpose two systems are different if their symmetry groups are different and hence the group invariants which label them. The mathematical technique we use here allows us these consideration with fixed dimension for all symmetry group. Whatever non-isomorphic group one arrive at, starting from G, the dimension remains the same. Due to this technical limitation we can consider a symmetry breaking but not symmetry enlargement. The group G is stable if the new group obtained through parametric variation is isomorphic to G. Otherwise it is unstable. Symmetry breaking is therefore due to unstability of group of invariance.

To define invariance stability topologically we need the concept of a bifurcation set. The problem of bifurcation arises whenever topological stability of any set, vector field, dynamical system, differentiable mappings, or algebraic variety is considered. In any of these cases one always start with a given continuous family of geometrical objects O_x which are all represented by points in a parameter space S. The space S can be an Euclidean space or a manifold (differentiable or otherwise) of finite or infinite dimensions. Let $x \in S$ and the corresponding geometrical object O_x; if $y \in S$ in the neighborhood of x in S with respect to some given topology, then O_x and O_y are of the same topological type. One may also say that O_x is topologically stable or generic. All such generic points of S form an open set of S. A bifurcation set is defined as the complement of generic set of points in S. It is a closed subset of S. It is a very general problem of differential topology to determine whether a bifurcation set is dense subset of S for given problem. This means that in the space of parameters, the bifurcation set of points are those set of points where the topological structure changes abruptly. If we consider dynamical systems, it is the set of points, that is bifurcation set, where the structural stability breaks down [*11*]. The concept of invariance stability for physical systems is analogous to the concept of structural stability. From the mathematical point of view, we are interested in the properties of algebraic varieties formed by the set of all n-dimensional Lie-algebras corresponding to group G.

Let $g = (V, \mu)$ be a Lie-algebra over a commutative field K (K = R or C) with the underlying vector space V of dimension n. Let $\mathscr{M}$ be the set of all Lie-algebra multiplications on the vector space V and let $A^2(V, V)$ be the bilinear alternate maps having a vector space structure and the set $\mathscr{M}$ is an algebraic set in $A^2(V, V)$. The general linear group $GL(V) = G$ has a natural representation on the space $A^2(V, V)$ under the action of which the set $\mathscr{M}$ is stable. The orbits of G on $\mathscr{M}$ define the isomorphism classes of Lie-algebras with the same underlying vector space V. A Lie-algebra $g_1 = (V, \mu')$ is a contraction of $g = (V, \mu)$ if $\mu' \in \overline{G(\mu)}$, where $\overline{G(\mu)}$ is the closure of the orbit of μ (for details see ref. 7). The topology on $\mathscr{M}$ is Zariski topology. The Lie-algebras g and g_1 are not isomorphic to each other. If we consider the Grassmanian variety $\Gamma_n(V)$ of all m-dimensional subspaces of V, then the algebraic subset $\gamma \subset \mathscr{M} \times \Gamma_n(V)$ of all pairs (γ, W), such that W is the subspace of V, is the set in which we are essentially interested in. To study the bifurcation set

$$\Sigma = \overline{G(\mu)} - G(\mu)$$

one studies the stability properties of the set γ in the neighborhood of (μ, W). The Richardson [*10*], condition of rigidity with respect to deformation of Lie-algebras, in fact, guarantees the existence of open orbits of G at μ and hence the existence of bifurcation set Σ. If the second cohomology group $H^2(g, g)$ is zero [*9*], then the set Σ is non-empty. So we can see that the problem of determining the set of contracted Lie-algebras is a problem of finding out the bifurcation set in the space of either structure constants or the algebraic set of all Lie-algebra multiplications on a given vector space V. Therefore, if the physical system in consideration is invariance stable, then we may say that the Lie-algebra g is generic in the space $A^2(V, V)$. We can now formulate rigoursly the definition of invariance stability as follows.

DÉFINITION. — Let $g = (V, \mu)$ is the Lie-algebra corresponding to invariance Lie-group G of a physical system, the system is invariance stable if the set Σ is a null set or empty.

Some consequences

Given the definition of invariance stability we may remark that for an unstable system (in our sense) it is always possible to break the symmetry of the system by varying some parameter of the theory [*6*]. This type of symmetry breaking does not corresponds to any of the four types mentioned by Thom [*8*]. The symmetry breaking in our case corresponds roughly to what is called a catastrophy. One may note that our definition of stability is applicable to dynamical or non-invariance groups [*5*] as well, however the basic philiosphy is different. The same is true if we consider a model where the internal and external symmetries are combined togather. To end the discussion, we relate few propositions and remarks whereas the proofs can esaily be obtained using the definition and theorems in references [*7*], [*9*] and [*10*].

REMARK 1. — For an invariance stable system the set of Lie-algebras g_t form a dense set in the space $\mathcal{M}$. Here g_t is a one parameter family of g.

PROPOSITION 1. — All abelian gauge theories are invariance stable. (This is because an abelian g has bifurcation set null.)

PROPOSITION 2. — Let $H^2(g, g)$ be the second cohomology group of g with coefficients in itself. If $H^2(g, g) = 0$, then the system is invariance unstable.

As corollaries of proposition 1, we have :

COROLLARY 1. — Charge conservation can not be violated by any means.

COROLLARY 2. — Energy-momentum conservation can not be violated in a homogeneous Minkowaski universe.

ACKNOWLEDGMENT

I thank very much to Pr. H. D. DOEBNER for hospitality at the Institute for Theoretical Physics, Clausthal, Germany.

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[11] M. PEIXOTO, (*Topology*, t. **1**, 1962, p. 101, *Ann. of Math.*, t. **87**, 1968, p. 422) has shown that if the dimension of the manifold is two, then the set of structurally stable vector field or dynamical system is a dense set on the set of all vector fields on this two dimensional manifold. In the case of differentiable maps i. e. C^{∞}-maps, one can define stability as follows. Let M^n and N^p be the two C^{∞}-manifolds and let C^r (,) be the space of all maps from M^n to N^p provided with the C^r-topology. A map is called stable if all'nearby maps' k are of the same type topologically as f and the diagram

$$\begin{tikzcd} M^n \arrow[r, "f"] \arrow[d, "h"'] & N^p \arrow[d, "h'"] \\ M^n \arrow[r, "k"'] & N^p \end{tikzcd}$$

is commutative, i. e. $fh = h'k$ where h and h' are ε-homeomorphisms of M^n and N^p. For a general reference, see M. Golubitsky and Guillemin, Stable mappings and their Singularities, Springer-Verlag, 1973.

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(Manuscrit reçu le 23 mai 1975)