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On the dimensional symmetry rearrangement in renormalized quantum field theory

by

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ABSTRACT. — Relation between canonical dilatations of asymptotic fields and dimensional transformations of renormalized fields is discussed for a formally scale invariant version of the Zachariasen-Thirring model. It is shown that a dilatation transformation of asymptotic free fields induces the scale change of dimensional parameters of renormalized theory and thus is rearranged into the dimensional transformation on the level of renormalized Heisenberg fields. In conclusion we point out that within this approach a change of renormalization point cannot be replaced by anomalous transformation of field operators.

I. INTRODUCTION

It is well known that in a formally dilatation invariant renormalizable Lagrangian field theory the canonical scale transformations of field operators cannot be reconciled with scaling equations for renormalized Green's functions [*1*]. This breakdown of canonical scaling is due to the appearance of a new dimensional parameter in renormalized solutions, i. e. the off-mass-shell renormalization point κ. Introduction of this parameter seems to be indispensable in interacting theories without an intrinsic mass unit

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because of infrared divergences, and, in addition, if the on-mass-shell renormalization conditions could be imposed the theory would not contain any dimensional parameter and should be invariant under canonical scaling, which would imply that the theory were a free one [2].

Therefore, if we wish to extend the concept of scale invariance to renormalized theory, we have to construct a transformation that induces a scale change of κ besides the ordinary canonical change of field operators. In this note this idea is illustrated in an example of a « naively » scale invariant version (¹) of the Zachariasen-Thirring model [3]. The model is exactly soluble and requires infinite mass and field renormalizations.

In order to construct the generator of dimensional transformation for renormalized fields which could produce the scale change or renormalization point we have adapted the technique known as « symmetry rearrangement » to this case. This approach, developped by Umezawa and Coworkers, has been applied to spontaneously broken dilatation symmetry [4]. However, the philosophy of Ref. [4] differs considerably from that assumed in this paper. We feel that dynamical rearrangement of dimensional transformations is questionable when applied to unrenormalized operators, and on the other hand the transformation law gets modified when renormalization effects are taken into account.

In order to find the mechanism of rearrangement we solve the model and express the renormalized Heisenberg fields in terms of asymptotic free fields. Then it is found that a scale transformation of asymptotic fields induces also the change of κ in Heisenberg fields and therefore is rearranged into a new symmetry transformation of the interacting theory.

II. THE MODEL

The model is formally defined by the following Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial^{\mu}B(x)\partial_{\mu}B(x) + \frac{1}{2}\int_0^{\infty} d\sigma \{ \partial^{\mu}\varphi(x,\sigma)\partial_{\mu}\varphi(x,\sigma) - \sigma\varphi^2(x,\sigma) \} - g_0 B(x)\int_0^{\infty} d\sigma f(\sigma)\varphi(x,\sigma) \quad (2.1)$$

and the (unrenormalized) ETCCR

$$\begin{aligned} [B(x), \dot{B}(y)]\delta(x^0 - y^0) &= i\delta(x - y) \\ [\varphi(x,\sigma), \dot{\varphi}(y,\sigma')]\delta(x^0 - y^0) &= i\delta(\sigma - \sigma')\delta(x - y) \end{aligned} \quad (2.2)$$

(¹) This model has been extensively discussed by J. Lukierski and the present author in another approach to scale transformation for renormalized field operators [5]. There we have used the Schwinger's action principle (Peierls' formula) to construct generators in terms of Heisenberg operators.

The field $\varphi(x, \sigma)$, $\sigma \in (0, \infty)$, is five-dimensional. The variable σ has the dimension of mass squared.

Mass spectrum being continuous, the model is formally invariant under scale transformations

$$x \rightarrow x' = \lambda x, \qquad \sigma \rightarrow \sigma' = \lambda^{-2}\sigma$$

provided that fields transform as

$$\begin{aligned} B'(x') &= \lambda^{-1}B(x) \\ \varphi'(x', \sigma') &= \varphi(x, \sigma) \end{aligned} \tag{2.3}$$

and $f^2(\sigma) \propto \sigma$. Further we put $f^2(\sigma) = \sigma$.

The model requires infinite renormalization of mass and field operator for the field B (details can be found in [5]). We impose the condition that the renormalized mass of the B particle is zero. At the intermediate step of calculations it is necessary to regularize the Lagrangian (2.1), and this can be done by introduction of a cutoff Λ, viz, by a replacement

$$f(\sigma) \rightarrow \theta(\Lambda^2 - \sigma)f(\sigma) \tag{2.4}$$

Therefore, we get the renormalized field equations

$$\begin{aligned} Z_3\left(\frac{\Lambda}{\kappa}, g\right)\Box B^R(x) &= g\int_0^{\Lambda^2} d\sigma f(\sigma)\varphi^R(x, \sigma) + \delta m^2 B^R(x) \\ (\Box - \sigma)\varphi^R(x, \sigma) &= g\theta(\Lambda^2 - \sigma)f(\sigma)B^R(x) \end{aligned} \tag{2.5}$$

where g is the renormalized coupling constant, and κ is a normalization point

$$p^2\tau(p) = 1 \qquad \text{at} \qquad p^2 = -\kappa^2$$

$\tau(p)$ is a Fourier transform of the renormalized B-propagator.

Équations (2.5) can be easily solved (see e. g. [6]), and we find (²), in the cutoff-free limit

$$\begin{aligned} B^R(x\,;g, \kappa) &= B_0(x) + g\int dx'\tau_{\substack{\text{Ret}\\ \text{Adv}}}(x - x'\,;g, \kappa)\int_0^{\infty} d\sigma f(\sigma)\varphi_{\substack{\text{in}\\ \text{out}}}(x', \sigma) \\ \varphi^R(x, \sigma\,;g, \kappa) &= \varphi_{\substack{\text{in}\\ \text{out}}}(x, \sigma) - gf(\sigma)\int dx'\Delta_{\substack{\text{Ret}\\ \text{Adv}}}(x - x', \sigma)B^R(x'\,;g, \kappa) \end{aligned} \tag{2.6}$$

It will be soon clear why the dependence of renormalized fields on parameters has been displayed in (2.6).

$\tau_{Ret}(\tau_{Adv})$ is the retarded (advanced) Green's function of the field B^R, and its Fourier transform is

$$\tau_{\substack{\text{Ret}\\ \text{Adv}}}(p\,;g, \kappa) = \frac{1}{p^2}\left[1 - g^2 \ln\left(\frac{-p^2}{\kappa^2}\right)\right]^{-1} \tag{2.7}$$

(²) Eq. (2.5) admit also a ghost solution which we discard. It seems that the ghost problem is irrelevant to the topic discussed in this paper.

where the retarded (advanced) pole prescription is understood. The field $B_0(x)$ is massless

$$\Box B_0(x) = 0$$

and in this model we have

$$B_0 = B_{in} = B_{out}$$

while free Licht fields [6] φ_{in} and φ_{out} satisfy

$$(\Box - \sigma)\varphi_{\substack{in\\out}}(x, \sigma) = 0$$

$$\varphi_{in}(x, \sigma), \varphi_{in}(x', \sigma') = i\delta(\sigma - \sigma')\Delta(x - x' : \sigma)$$

and $\varphi_{in} \neq \varphi_{out}$.

III. SYMMETRY REARRANGEMENT

Now we shall demonstrate how scale transformations of asymptotic fields (reviewed in the Appendix) are rearranged into dimensional transformations of renormalized fields. To this end we need only the relation expressing the scaling properties of Green's functions (scaling equation), i. e.

$$\left(x^\mu\partial_\mu + 2 - \kappa\frac{\partial}{\partial\kappa}\right)\tau_{\substack{Ret\\Adv}}(x ; g, \kappa) = 0 \tag{3.1}$$

The action of the « asymptotic » dilatation generator D on renormalized fields can be obtained in a straightforward manner. We have

$$\begin{aligned} i[D, B^R(x ; g, \kappa)] &= i[D, B_0(x)] \\ &\quad + g\int dx'\tau_{Ret}(x - x' ; g, \kappa)\int_0^x d\sigma f(\sigma) i[D, \varphi_{in}(x', \sigma)] = (x^\mu\partial_\mu + 1)B_0(x) \\ &\quad + \left(x^\mu\partial_\mu + 1 - \kappa\frac{\partial}{\partial\kappa}\right)g\int dx'\tau_{Ret}(x - x', g, \kappa)\int_0^x d\sigma f(\sigma)\varphi_{in}(x', \sigma) \\ &= \left(x^\mu\partial_\mu + 1 - \kappa\frac{\partial}{\partial\kappa}\right)B^R(x ; g, \kappa) \end{aligned} \tag{3.2a}$$

and similarly

$$i[D, \varphi^R(x, \sigma ; g, \kappa)] = \left(x^\mu\partial_\mu - 2\sigma\frac{\partial}{\partial\sigma} - \kappa\frac{\partial}{\partial\kappa}\right)\varphi^R(x ; \sigma ; g, \kappa) \tag{3.2b}$$

We see, therefore, that these commutators have an additional term, $-\kappa\frac{\partial}{\partial\kappa}$, compared to the commutators of D with asymptotic fields, and thus we have a dynamical modification of scale symmetry. In the global form relations (3.2) read

$$U(\lambda)B^R(x ; g, \kappa)U^{-1}(\lambda) = \lambda B^R\left(\lambda x ; g, \frac{\kappa}{\lambda}\right) \tag{3.3a}$$

$$U(\lambda)\varphi^R(x, \sigma ; g, \kappa)U^{-1}(\lambda) = \varphi^R\left(\lambda x, \frac{\sigma}{\lambda^2} ; g, \frac{\kappa}{\lambda}\right) \tag{3.3b}$$

The covariance of renormalized field equations can be explicitely verified if we rewrite Eq. (2.5) in a way that would allow to perform the cutoff-free limit. This can be achieved e. g. if we incorporate subtractions by raising the order of field equations [5]. In this way we get

$$\begin{aligned} \square B^{R}(x) &= g\square(\square + \kappa^2)\int_0^{\infty}\frac{d\sigma f(\sigma)}{\sigma(\sigma + \kappa^2)}\varphi^{R}(x, \sigma) \\ (\square - \sigma)\varphi^{R}(x, \sigma) &= g f(\sigma) B^{R}(x) \end{aligned} \tag{3.4}$$

By direct inspection we find that $B^{R'}(x' ; g, \kappa')$ and $\varphi^{R'}(x', \sigma' ; g, \kappa')$ satisfy (3.4) with primed variables (here $\kappa' = \lambda^{-1}\kappa$) (3).

This result, as well as the scaling equation (3.1), show that transformations (3.3) are a symmetry of the renormalized theory.

The family of transformations (3.3) sends σ to σ/λ^2. Such an operation is trivial since we consider φ to be a five-dimensional field. However, unless we adopt a purely passive interpretation, the transformation (3.3) in the (κ, g) plane connects fields belonging apparently to a one-parameter family of physically distinct theories (due to the non-obvious measurability of the parameter κ it is operationally difficult to establish a one to one correspondence between couples (κ, g) and field theories). Let us look at the Gell-Mann-Low renormalization group arguments [7]: the transformation of parameters $(\kappa, g) \mapsto \left(\frac{\kappa}{\lambda}, g\right)$ is equivalent to the change $(\kappa, g) \mapsto (\kappa, \bar{g}(\lambda, g))$, with $\bar{g}$ a *new* coupling constant (4), plus a finite renormalization of fields.

In this particular model that can be demonstrated on the example of the scattering matrix. We have

$$S(g, \kappa) = \exp\left\{\frac{2}{(2\pi)^3}\int_0^{\infty} d\sigma H(\sigma ; g, \kappa)\int dx \varphi_{\text{in}}^{(-)}(x, \sigma)\partial_0\varphi_{\text{in}}^{(+)}(x, \sigma)\right\} \tag{3.5}$$

where [6].

$$H(\sigma ; g, \kappa) = \text{arc tan}\,\frac{\pi g^2}{1 - g^2 \ln\frac{\sigma}{\kappa^2}} \tag{3.6}$$

Explicit calculations give

$$U(\lambda)S(g, \kappa)U^{-1}(\lambda) = S\left(g, \frac{\kappa}{\lambda}\right) = S(\bar{g}(\lambda, g), \kappa) \tag{3.7}$$

Concluding, we would like to comment on the relation between dimensional transformations of renormalized fields and the anomalous scale

(3) In the general case of nonlinear theories which contain nontrivial proper vertex renormalization one must consider the scaling properties of *renormalized products* of field operators.

(4) In this model we have $\bar{g}^2(\lambda, g) = g^2/(1 - g^2 \ln \lambda^2)$.

transformations of renormalized fields. It turns out that the hypothesis adopted in this paper (i. e. that scale transformations of renormalized operators can be described by a generator formed of asymptotic *free* fields) cannot be reconciled with anomalous dimensions, and more generally with the transformation law of the form (5) [5]:

$$U(\lambda)\Phi(x\,;\,g,\,\kappa)U^{-1}(\lambda) = \lambda^{d_{can}}\, Z^{1/2}(\lambda,\,g)\Phi(\lambda x\,;\,\bar{g}(\lambda,\,g),\,\kappa) \qquad (3.8)$$

where Z is the effective field renormalization (it is equal to $\lambda^{2\gamma(g)}$ if g is a fixed point).

To see this discrepancy we point out that if the interacting field is expanded in terms of free fields (which is a hypothesis in a symmetry rearrangement scheme) then under scale transformations the factor multiplying free fields must coincide with that multiplying the whole expression for interacting field, for otherwise the transformation law is not homogeneous.

It is probable that this problem is related to the irrelevance of perturbative approach to scaling of renormalized fields. Anyway, we think that a change of normalization of free fields, which would be necessary for (3.8) to hold, is not a unitarily implementable statement and thus cannot be generated by a conserved local charge.

In the general case of the off-mass-shell normalization the asymptotic fields appearing in solutions should be multiplied by appropriate (finite) effective renormalization constants relating off- and on-mass-shell theories. Then, the composition law for effective field renormalization constants (renormalization group) could allow to maintain the transformation law (3.8) also within the scheme used in the present paper. Unfortunately, infrared divergences (and they do occur in our model) prohibit such an approach, and this is why normalization of our B_0 field may be fixed arbitrarily (6).

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(5) In order to see that it might be so, compare to the scaling equation for renormalized Green's functions (renormalization group equation).

(6) I am grateful to B. Jouvet and E. Tirapegui for a discussion on this subject.

APPENDIX

Scale transformations for Licht fields have been investigated in [8] within a five-dimensional formalism. Below, we write the formulas for the « asymptotic » generator D used in this paper.

The « in » Lagrangian looks as follows

$$\mathscr{L}_{\text{in}} = \frac{1}{2}\partial^\mu B_0(x)\partial_\mu B_0(x) + \frac{1}{2}\int_0^\infty d\sigma \left\{ \partial^\mu \varphi_{\text{in}}(x, \sigma)\partial_\mu \varphi_{\text{in}}(x, \sigma) - \sigma\varphi_{\text{in}}^2(x, \sigma) \right\} \quad (A.1)$$

We choose the following form of the scale current

$$D_\mu(x) = D_\mu^{\text{geom}}(x) + S_\mu(x) = x^\nu\theta_{\mu\nu}^{\text{in}}(x) - 2\int_0^\infty d\sigma\sigma\partial_\mu\varphi_{\text{in}}(x, \sigma)\frac{\partial}{\partial\sigma}\varphi_{\text{in}}(x, \sigma) \quad (A.2)$$

where

$$\theta_{\mu\nu}^{\text{in}}(x) = T_{\mu\nu}^{\text{CAN}}(x) - \frac{1}{6}(g_{\mu\nu}\Box + \partial_\mu\partial_\nu)B_0^2(x)$$

and the second term in (A.2) generates changes of the variable σ.

With the choice (A.3) of the energy-momentum tensor the divergence of the « geometrical » part of the scale current cancels against the divergence of S_μ, and is proportional to these terms in $\mathscr{L}_{\text{in}}$ which break the « geometrical » space-time dilatations. The generator $D = D^{\text{geom}}(t) + S(t)$ obtained from (A.2) is time independent and gives rise to the following transformations

$$\begin{aligned} i[D, B_0(x)] &= (x^\mu\partial_\mu + 1)B_0(x) \\ i[D, \varphi_{\text{in}}(x, \sigma)] &= \left(x^\mu\partial_\mu - 2\sigma\frac{\partial}{\partial\sigma}\right)\varphi_{\text{in}}(x, \sigma) \end{aligned} \quad (A.4)$$

Commutators of D with P_μ^{in} and $M_{\mu\nu}^{\text{in}}$ close up to the Weyl Lie algebra (this is not the case of $D^{\text{geom}}(t)$).

We remark also that it is the combined action of $D^{\text{geom}}(t)$ and $S(t)$ that gives commutators (3.2*a*-*b*).

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