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On the genericity of nonvanishing instability intervals in Hills equation

by

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ABSTRACT. — We prove that for (Baire) almost every C^∞ periodic function V on $\mathbb{R}$, $-d^2/dx^2 + V$ has *all* its instability intervals non-empty.

In the spectral theory of one dimensional Schrödinger operators [3] [10] with periodic potentials, a natural question occurs involving the presence of gaps in the spectrum. Let $H = -\frac{d^2}{dx^2} + V$ on $L^2(\mathbb{R}, dx)$ where $V(x+1) = V(x)$ for all x . Let A^P (resp. A^Λ) be the operator $-\frac{d^2}{dx^2} + V$ on $L^2([0, 1], dx)$ with the boundary condition $f'(1) = f'(0)$; $f(1) = f(0)$ (resp. $f'(1) = -f'(0)$; $f(1) = -f(0)$). Let E_n^P (resp. E_n^Λ) be the n^{th} eigenvalue, counting multiplicity, of A^P (resp. A^Λ). Finally define

$$\alpha_n = \begin{cases} E_n^P & n = 1, 3, \dots \\ E_n^\Lambda & n = 2, 4, \dots \end{cases}$$

$$\beta_n = \begin{cases} E_n^\Lambda & n = 1, 3, \dots \\ E_n^P & n = 2, 4, \dots \end{cases}$$

$$\mu_n = \alpha_{n+1} - \beta_n$$

It is a fundamental result of Lyapunov that

$$\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \dots \leq \alpha_n < \beta_n \leq \alpha_{n+1} \dots$$

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and one can show [3] [10] that $\sigma(H) = \bigcup_{n=1}^{\infty} [\alpha_n, \beta_n]$. The numbers $\mu_n \geq 0$ enter naturally as the size of gaps in $\sigma(H)$. In the older literature [9], the equation $-f'' + Vf = Ef$ is called Hill's equation and the intervals (β_n, α_{n+1}) (of length μ_n) are called instability intervals.

One has the feeling that for most V 's the gap sizes $\mu_n(V)$ are non-zero. This is suggested in part by a variety of deep theorems that show the vanishing of many μ_n 's places strong restrictions on V : for example, $\mu_n(V) = 0$ all n implies that V is constant [1] [5]; $\mu_n(V) = 0$ all odd n implies that $V\left(x + \frac{1}{2}\right) = V(x)$ [1] [6]; and $\mu_n(V) = 0$ for all but N values of n forces V to lie on a $2N$ -dimensional manifold [5] [4]. On the other hand, some argument is necessary to construct an explicit example of a V with each $\mu_n(V) \neq 0$ [7].

The situation is somewhat reminiscent of that concerning nowhere differentiable functions in $C[0, 1]$. One's intuition is that somehow most functions in $C[0, 1]$ are nowhere differentiable but some argument is needed to construct an explicit nowhere differentiable function. One's intuition in this case is established by a result that also settles the existence question: a dense G_δ (« Baire almost every ») in $C[0, 1]$ consists of nowhere differentiable functions [2].

In this note we wish to prove a similar result that asserts that, for most V , $\mu_n(V) \neq 0$ for all n . We do not claim that that result is of the depth of the above quoted results but we feel it is of some interest especially since it will be a simple exercise in the perturbation theory of eigenvalues [8] [10] [11].

THEOREM. — Let X denote the vector space of real valued C^∞ functions on $\mathbb{R}$ obeying $V(x+1) = V(x)$. Place the Frechet topology on X given by the seminorms

$$\|f\|_n = \sup_x |D^n f(x)|.$$

Then the set of V in X with $\mu_n(V) \neq 0$ for all n is a dense G_δ in X .

Proof. — Fix n . We will show that $\{V \mid \mu_n(V) \neq 0\}$ is a dense open set of X . Thus $\bigcap_n \{V \mid \mu_n(V) \neq 0\}$ is a G_δ which is dense by the Baire category theorem.

Suppose that $\mu_n(V) \neq 0$. Suppose n is even (a similar argument works if n is odd). Thus $E_{n+1}^p(V) \neq E_n^p(V)$. Now, the change of $E_{n+1}^p(V + \lambda W)$ as λ changes can be bounded [8] by $\|W\|_{\text{operator}}$ and the W -independent data of the distance of $E_{n+1}^p(V)$ from $E_n(V)$ and $E_{n+2}(V)$. As a result, there is a constant $\varepsilon(V)$ so that $\mu_n(V + W) \neq 0$ if $\|W\|_\infty \leq \varepsilon(V)$. Since $\|\cdot\|_\infty$ is a continuous seminorm, $\{V \mid \mu_n(V) \neq 0\}$ is open.

Next suppose $\mu_n(V) = 0$ and again suppose that n is even. Since $E_n = E_{n+1}$,

all solutions of $-u'' + Vu = E_n u$ are periodic. Let u_1 be the solution with $u(0) = 0$, $u'(0) = 1$ and u_2 the solution with $u(0) = 1$, $u'(0) = 0$. Since $(u_1(x))^2 \neq (u_2(x))^2$ for x near 0, we can find $W \in X$ with

$$\int W(x) |u_1(x)|^2 dx \neq \int W(x) (u_2(x))^2 dx.$$

It follows [8] that for λ small $E_n(V + \lambda W) \neq E_{n+1}(V + \lambda W)$ and thus that $\mu_n(V + \lambda W) \neq 0$. We conclude that $\{V \mid \mu_n(V) \neq 0\}$ is dense. ■

We conclude by noting that the space $X = C^\infty$ can be replaced by any topological vector space of continuous periodic functions which is a Baire space and which obeys:

a) $\| \cdot \|_\infty$ is a continuous seminorm.

(b) If $\rho_1 \neq \rho_2$ as functions in $L^1([0, 1])$, there is W in the space with

$$\int \rho_1(x) W(x) dx \neq \int \rho_2(x) W(x) dx.$$

In particular, we can take the $C^p([0, 1])$ periodic functions with the C^p topology or the periodic entire analytic functions with the compact open topology.

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