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On the existence of a class of stationary quantum stochastic processes

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ABSTRACT. — Given a classical stationary stochastic process we construct a corresponding quantum stochastic process. As an example we use the Ornstein-Uhlenbeck process to construct the quantum process whose existence was suggested by the work of Ford, Kac and Mazur.

RÉSUMÉ. — Donné un processus aléatoire stationnaire classique, on construit un processus aléatoire quantique correspondant. Par exemple on se sert du processus Ornstein-Uhlenbeck pour construire le processus quantique dont l'existence a été proposée par Ford, Kac et Mazur.

1. INTRODUCTION

The work of Ford, Kac and Mazur [1] suggests the existence of a quantum-mechanical analogue of the Ornstein-Uhlenbeck stochastic process [2]. In this paper we prove a theorem which establishes the existence of a class of such quantum stochastic processes.

Recall that a classical stochastic process $\{X_t : -\infty < t < \infty\}$ is a family of random variables on a probability space Σ with a probability

measure $\mathbb{P}$ given on a field $\mathcal{B}$ of subsets of Σ ; the expectation $\mathbb{E}Y$ of a random variable Y on Σ is given by

$$\mathbb{E}Y = \int_{\Sigma} Y(\sigma) \mathbb{P}(d\sigma). \quad (1)$$

We shall be interested in processes for which for all t

$$\mathbb{E}X_t = 0 \quad , \quad \mathbb{E}X_t^2 < \infty \quad (2)$$

so that $t \rightarrow X_t$ can be regarded as a curve in the real Hilbert space $L^2(\Sigma, \mathbb{P})$ and such a process will be called stationary if the correlation $\mathbb{E}X_t X_{t+\tau}$ depends only on the time-difference τ . The history $\mathfrak{H}^X$ of the process $t \mapsto X_t$ is the closed subspace of $L^2(\Sigma, \mathbb{P})$ spanned by the X_t :

$$\mathfrak{H}^X = V \{ X_t : -\infty < t < \infty \}. \quad (3)$$

It follows that for a stationary process $t \mapsto X_t$ there exists a group $\{ U_t : -\infty < t < \infty \}$ of unitary operators on its history $\mathfrak{H}^X$ such that for all t, τ

$$X_{t+\tau} = U_{\tau} X_t. \quad (4)$$

Thus a classical process involves three objects: an underlying space Σ , a probability measure $\mathbb{P}$ and a family $\{ X_t : -\infty < t < \infty \}$ of random variables. If in addition the process is stationary, its time-development is given by a family of operators which preserve expectations. This suggests the following tentative definition of a quantum stochastic process:

A quantum stochastic process is a family of self-adjoint operators $\{ Q_t : -\infty < t < \infty \}$ on a Hilbert space $\mathfrak{H}$ with a state vector Ω ($\|\Omega\| = 1$). The expectations are given by

$$\mathbb{E}Q_t = \langle \Omega, Q_t \Omega \rangle, \quad (5)$$

$$\mathbb{E}Q_t \circ Q_s = \frac{1}{2} \langle \Omega, (Q_t Q_s + Q_s Q_t) \Omega \rangle. \quad (6)$$

The symmetrized expectation (6) is taken because in general Q_t does not commute with Q_s and so $Q_t Q_s$ is not self-adjoint. For a discussion of quantum correlations see [1] and [3].

A quantum stochastic process $t \mapsto Q_t$ is said to be stationary if there exists a one-parameter group $\{ V_t : -\infty < t < \infty \}$ of unitary operators on $\mathfrak{H}$ such that for all t, τ

$$Q_{t+\tau} = V_{\tau} Q_t V_{\tau}^{-1} \quad (7)$$

and

$$V_{\tau} \Omega = \Omega. \quad (8)$$

It then follows from (5) and (6) that for all t, τ

$$\mathbb{E}Q_t = \mathbb{E}Q_0 \quad (9)$$

and

$$\mathbb{E}Q_t \circ Q_{t+\tau} = \mathbb{E}Q_0 \circ Q_{\tau} \quad (10)$$

so that the correlation depends only on the time-difference τ .

2. DESCRIPTION OF A CLASS OF QUANTUM PROCESSES

It is well-known (for example Doob [4]) that if $\tau \mapsto \gamma(\tau)$ is the correlation of a stationary stochastic process $t \mapsto X_t$ so that

$$\gamma(\tau) = \mathbb{E}X_t X_{t+\tau} \quad (11)$$

then γ is a function of positive type, and that conversely if γ is a function of positive type there exists a stochastic process $(X_t, \Sigma, \mathbb{P})$ such that (11) holds. Given a correlation function γ we define for each $\mu > 0$ a quantum modification γ_μ such that for all τ

$$\lim_{\mu \rightarrow 0+} \gamma_\mu(\tau) = \gamma(\tau), \quad (12)$$

and we prove the existence of a stationary quantum stochastic process $(Q_t, \mathfrak{H}, \Omega)$ such that

$$\gamma_\mu(\tau) = \mathbb{E}Q_t \circ Q_{t+\tau}. \quad (13)$$

One motivation for this is that in the case in which γ is the correlation function of the Ornstein-Uhlenbeck x -process, the quantum modification $\gamma_\hbar$ is the correlation function obtained by Ford, Kac and Mazur. The Ornstein-Uhlenbeck process arises as the stationary solution of a Langevin equation which describes the motion of an oscillator of frequency ω_0 and frictional constant f coupled to a heat-bath at inverse temperature β . Then

$$\gamma(\tau) = \frac{e^{-f|\tau|/2}}{\beta\omega_0^2} \left(\cos \tilde{\omega}\tau + \frac{f}{2\omega_0} \sin \tilde{\omega}|\tau| \right). \quad (14)$$

Since the process X_t is stationary the oscillator is in thermal equilibrium with the heat-bath and so we can regard the inverse temperature β as describing the stationary state of the process. Indeed the joint probability distribution of the position and momentum of the oscillator is just the Maxwell-Boltzmann equilibrium distribution at inverse temperature β . A quantum-analogue of the Maxwell-Boltzmann condition is the Kubo-Martin-Schwinger boundary condition; we use it as formulated by Haag, Hugenholtz and Winnink [5]. The pair (A, B) satisfy the KMS-boundary condition at inverse temperature β if for every pair (A, B) of bounded self-adjoint operators on $\mathfrak{H}$ there exists a function $z \mapsto F_{AB}(z)$ analytic and uniformly bounded on the strip $-\beta\mu \leq \text{Im}z \leq 0$ such that for all t

$$\begin{aligned} F_{AB}(t) &= \langle \Omega, A_t B \Omega \rangle \\ F_{AB}(t - i\beta\mu) &= \langle \Omega, B A_t \Omega \rangle \end{aligned} \quad (15)$$

where

$$A_t = V_t A V_{-t} \quad (16)$$

Suppose $\tau \mapsto \gamma(\tau)$ is a positive definite function given by

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega \tau dG(\omega). \quad (17)$$

Then for $\mu > 0$ the quantum modification at inverse temperature β is defined to be

$$\gamma_\mu(\tau) = \gamma(0) \int_0^\infty b_\mu(\omega) \cos \omega \tau dG(\omega) \quad (18)$$

where

$$b_\mu(\omega) = \frac{\beta \mu \omega}{2} \coth \frac{\beta \mu \omega}{2}. \quad (19)$$

Notice that, because of the inequality $x \coth x < 1 + x$ for $x > 0$, the function γ_μ is bounded provided $\int_0^\infty \lambda dG(\lambda) < \infty$.

THEOREM. — *Let $(X_t, \Sigma, \mathbb{P})$ be a real stationary stochastic process with correlation function*

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega \tau dG(\omega)$$

such that $\Delta G(0) = 0$ and $\int_0^\infty \omega dG(\omega) > \infty$.

Then there exists a stationary quantum stochastic process $(Q_t, \mathfrak{H}, \Omega)$ with $Q_t = V_t Q_0 V_t^{-1}$ such that

$$(i) \quad \langle \Omega, \exp \left\{ i \sum_{j=1}^n k_j Q_{t_j} \right\} \Omega \rangle = \exp \left\{ -\frac{1}{2} \sum_{j,l=1}^n k_j k_l \gamma_\mu(t_j - t_l) \right\} \quad (20)$$

where γ_μ is the quantum modification (18) of γ at inverse temperature β .

$$(ii) \quad \exp(iQ_s) \exp(iQ_t) = \exp\{i\mu\beta\gamma'(t-s)\} \exp(iQ_t) \exp(iQ_s). \quad (21)$$

(iii) The pair (V_t, Ω) satisfies the KMS boundary condition (15) at inverse temperature β .

Remark. — The properties (i) and (ii) are precise formulations of the more familiar looking

$$(i)' \quad \mathbb{E} Q_s \circ Q_t = \gamma_\mu(t-s), \quad (22)$$

$$(ii)' \quad [Q_s, Q_t] = i\mu\beta\gamma'(s-t)1. \quad (23)$$

The theorem is proved by constructing a process with the claimed properties.

3. THE CONSTRUCTION OF THE QUANTUM PROCESS

The construction proceeds in three parts. First we prove a lemma about classical processes which states that a real process can be complexified in

such a way that time-translation acts as a unitary group with positive generator. Next we use the Araki-Woods construction [6], [7] to construct a quantum process and finally we verify the properties claimed for the process.

LEMMA. — *Let $(X_t, \Sigma, \mathbb{P})$ be the real process of the theorem. Then there exists a complex process $(\xi_t, \Sigma, \mathbb{P})$ and a one-parameter unitary group*

$$U_t = \int_{\mathbb{R}} e^{it\lambda} E(d\lambda) \quad (24)$$

on the complex Hilbert space $\mathfrak{H}^\xi$ such that

$$(i) \quad X_t = \sqrt{2} \Re e \xi_t, \quad (25)$$

$$(ii) \quad \xi_t = U_t \xi_0, \quad (26)$$

$$(iii) \quad U_t = \exp(iC^2 t), \quad (27)$$

with $C \geq 0$,

$$(iv) \quad \|E(d\lambda) \xi_0\|^2 = 2\gamma(0) dG(\lambda). \quad (28)$$

Proof. — By Bôchner's theorem there exists a distribution function F such that

$$\gamma(\tau) = \gamma(0) \int_{-\infty}^{\infty} e^{it\omega} dF(\omega). \quad (29)$$

For $\omega \geq 0$ define $G(\omega)$ by

$$G(\omega) = \begin{cases} F(\omega) - F(-\omega - 0) & , \quad \omega > 0, \\ 0 & , \quad \omega = 0 \end{cases} \quad (30)$$

Then

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega \tau dG(\omega) \quad (31)$$

since γ is real. Further, X_t has a representation

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} \zeta(d\lambda) \quad (32)$$

as the Fourier transform of a stochastic process ξ_λ with orthogonal increments such that

$$\mathbb{E} |\zeta(d\lambda)|^2 = \gamma(0) dF(\lambda). \quad (33)$$

Let $\tilde{X}_t$ be the Hilbert transform of X_t :

$$\tilde{X}_t = \int_{-\infty}^{\infty} e^{it\lambda} g(\lambda) \zeta(d\lambda) \quad (34)$$

where

$$g(\lambda) = \begin{cases} i & , \quad \lambda < 0, \\ 0 & , \quad \lambda = 0, \\ -i & , \quad \lambda > 0. \end{cases} \quad (35)$$

Then we have

$$\mathbb{E}\tilde{X}_t\tilde{X}_{t+\tau} = \gamma(\tau), \quad (36)$$

$$\mathbb{E}X_t\tilde{X}_{t+\tau} = \tilde{\gamma}(\tau) = \gamma(0) \int_0^\infty \sin \lambda \tau dG(\lambda). \quad (37)$$

Put

$$\xi_t = \frac{1}{2^{1/2}} (X_t + i\tilde{X}_t) \quad (38)$$

so that

$$\mathbb{E}\bar{\xi}_t\xi_{t+\tau} = \gamma(\tau) + i\bar{\gamma}(\tau). \quad (39)$$

Then there exists a unitary operator

$$U_t = \int_{-\infty}^\infty e^{it\lambda} E(d\lambda) \quad (40)$$

on $\mathfrak{H}^\xi$ such that

$$\xi_t = U_t \xi_0 \quad (41)$$

and so

$$\int_{-\infty}^\infty e^{it\lambda} \|E(d\lambda)\xi_0\|^2 = \gamma(0) \int_0^\infty e^{it\lambda} dG(\lambda). \quad (42)$$

Thus the spectrum of the generator of U_t is positive and

$$\|E(d\lambda)\xi_0\|^2 = \gamma(0)dG(\lambda). \quad (43)$$

Put

$$C = \int_0^\infty \lambda^{1/2} E(d\lambda) \quad (44)$$

then

$$U_t = e^{itC^2}, \quad C \geq 0. \quad (45)$$

For the Araki-Woods construction we follow Chaiken [7]. A Weyl system over a complex vector space $\mathcal{V}$ is a map W from $\mathcal{V}$ to the unitary operators $\mathcal{U}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$ together with a vector Ω in $\mathfrak{H}$ such that

$$\bigvee \{ W(v)\Omega : v \in \mathcal{V} \} = \mathfrak{H},$$

which satisfies

$$(i) \quad W(v_1)W(v_2) = \exp \left\{ -\frac{i}{2} \mathcal{J}m \langle v_1, v_2 \rangle \right\} W(v_1 + v_2), \quad (46)$$

(ii) for all $v \in \mathcal{V}$ the map $\lambda \mapsto W(\lambda v)$ is continuous from $\mathbb{C}$ to $\mathcal{U}(\mathfrak{H})$ in the strong operator topology.

It follows from (ii) by Stone's theorem that for each $v \in \mathcal{V}$ there exists a self-adjoint operator $R(v)$ such that

$$W(v) = \exp \{ iR(v) \}. \quad (47)$$

A Weyl system is determined up to equivalence by its generating functional

$$\mu(v) = \langle \Omega, W(v)\Omega \rangle. \quad (48)$$

THEOREM (Chaiken [7]). — *Let $\mathcal{V}$ be a complex Hilbert space. Let T be a self-adjoint operator on $\mathcal{V}$ with domain $\mathcal{D}(T)$ and such that $Y \geq 1$. Then there exists a Weyl system (W, ξ, Ω) over $\mathcal{D}(T)$ with generating functional*

$$\mu(v) = \exp \left(-\frac{1}{4} \|Tv\|^2 \right). \quad (49)$$

Proof. — Let $A = \frac{1}{2}(T^2 - 1)$ and let J be a conjugation on $\mathcal{V}$ which commutes with A . Let $\mathcal{M}$ be the closure of the range of $A^{1/2}$; let $(W_1, \mathcal{I}(\mathcal{V}), \Omega_1)$ be the Fock-Cook Weyl system over $\mathcal{M}$ and let $(W_2, \mathcal{I}(\mathcal{M}), \Omega_2)$ be the Fock-Cook Weyl system over $\mathcal{M}$. For $v \in \mathcal{D}(T)$ put

$$W(v) = W_1((1 + A)^{1/2}v) \otimes W_2(A^{1/2}Jv) \quad (50)$$

on

$$\xi = \mathcal{I}(\mathcal{V}) \otimes \mathcal{I}(\mathcal{M})$$

with

$$\Omega = \Omega_1 \otimes \Omega_2.$$

Then direct calculation shows that

$$\langle \Omega, W(v)\Omega \rangle = \exp \left(-\frac{1}{4} \|Tv\|^2 \right). \quad (51)$$

The methods of Araki-Woods [6] show that Ω is cyclic for W and that W is a factor representation; it is irreducible if and only if $A = 0$.

With the notation of the lemma take, in the Araki-Woods construction, $\mathcal{V} = \xi^\xi$, $v_t = (\mu\beta)^{1/2}CU_{t\zeta_0}$, and let T be the unique positive operator such that

$$T^2 = \coth \frac{\mu\beta}{2} C^2. \quad (52)$$

Then $T^2 > 1$ since $\coth x < 1$ for $x > 0$, and $C\zeta_0 \in \mathcal{D}(T)$ since

$$\begin{aligned} \|TC\zeta_0\|^2 &= \int_0^\infty \lambda \coth \frac{\mu\beta\lambda}{2} \|E(d\lambda)\zeta_0\|^2 \\ &= \gamma(0) \int_0^\infty \lambda \coth \frac{\mu\beta\lambda}{2} dG(\lambda) = \frac{2}{\mu\beta} \gamma_\mu(0) < \infty. \end{aligned} \quad (53)$$

A similar calculation gives

$$\operatorname{Re} \langle Tv_s, Tv_t \rangle = 2\gamma_\rho(t - s), \quad (54)$$

and

$$\operatorname{Im} \langle v_s, v_t \rangle = -\mu\beta\gamma'(t - s). \quad (55)$$

Putting

$$Q_t = R(v_t) \quad (56)$$

we check that (20) and (21) follow from (46) and (49).

Let $(W, \mathcal{I}(\mathcal{H}), \Omega)$ be the Fock-Cook Weyl system over a Hilbert space $\mathcal{H}$;

then to each self-adjoint operator B on $\mathcal{H}$ there corresponds a self-adjoint operator $\Gamma(B)$ on $\mathcal{I}(\mathcal{H})$ such that

$$W(e^{iBt}v) = e^{i\Gamma(B)t}W(v)e^{-i\Gamma(B)t} \quad (57)$$

for all t and all v in $\mathcal{H}$. It is straightforward to check that

$$V_t \exp(i\lambda Q_s) V_t^{-1} = \exp(i\lambda Q_{s+t}) \quad (58)$$

and

$$V_t \Omega = \Omega \quad (59)$$

where

$$V_t = e^{i\Gamma(C^2)} \otimes e^{-i\Gamma(C^2)} \quad (60)$$

so that the conditions (7) and (8) are satisfied. It remains to check that (V_t, Ω) satisfy the KMS boundary condition at inverse temperature β . It is enough to consider the function defined for $v, w \in \mathcal{D}(T)$ by

$$\begin{aligned} & F_{vw}(t + iy) \\ &= \exp \left[-\frac{1}{4} \left\{ \|T(U_t e^{-yC^2}v + w)\|^2 + \langle U_t e^{-yC^2}v, w \rangle - \langle w, U_t e^{-yC^2}v \rangle \right\} \right] \end{aligned} \quad (61)$$

which is analytic and bounded in the strip $-\beta\mu \leq y \leq 0$.

Using (52), we find that

$$T^2(e^{\mu\beta C^2} - 1) = (e^{\mu\beta C^2} + 1) \quad (62)$$

which together with (46) and (49) gives

$$F_{vw}(t) = \langle \Omega, W(U_t v) W(w) \Omega \rangle, \quad (63)$$

$$F_{vw}(t - i\mu\beta) = \langle \Omega, W(w) W(U_t v) \Omega \rangle, \quad (64)$$

so that the KMS boundary condition is satisfied and the proof of the theorem is complete.

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