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On relativistic multiparticle kinematics in invariant variables

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ABSTRACT. — A complete system of algebraic constraints on the Gram matrices of four-momenta and an analytic reconstruction of any finite set of four-momenta from their Gram matrix are established. The physical regions and phase-space volume elements in general invariant variables of both exclusive and inclusive multiparticle processes are also given.

1. INTRODUCTION

The purpose of this work is to extend some previous analyses [*1*]-[*6*] of relativistic multiparticle kinematics in invariant variables. The present approach is essentially inspired by certain old algebraic results of Jacobi, Hildenfinger, Frobenius, Gauss and Weyl.

The paper is organized as follows. In Section 2, we present two kinematical rules for the Gram matrices of finite sets of four-momenta. The first rule gives a complete system of Gram determinantal equalities and inequalities which guarantees that a real symmetric $n \times n$ matrix is a Gram matrix of n four-momenta. The second rule represents a concrete analytic reconstruction of any finite set of four-momenta from their Gram matrix.

In Section 3, we apply the above kinematical rules to a description of the physical regions of both exclusive and inclusive multiparticle processes in general invariant variables. As applied to this description, some algebraic and geometric properties of the considered physical regions are discussed.

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On the other hand, a convenient unified scheme for finding the ranges of several useful sets of invariant variables is established and the phase-space volume elements are determined with respect to this scheme.

2. KINEMATICAL RULES FOR GRAM MATRICES

We first establish a complete system of algebraic constraints on the Gram matrices of n four-momenta.

It is convenient to start with some notations. The four-momentum q is written as $q = (q^0, \vec{q}) = (q^0, q^1, q^2, q^3)$. The Minkowski scalar product of two four-momenta q and q' is given by $qq' = q^0 q'^0 - \vec{q}\vec{q}'$ (with the notation $(q)^2 = qq$ for the Lorentz square of q).

The Gram matrix of the four-momenta $q_1, \ldots, q_n$ is defined by

$$U = \begin{pmatrix} (q_1)^2 & q_1 q_2 & \cdots & q_1 q_n \\ q_1 q_2 & (q_2)^2 & \cdots & q_2 q_n \\ \vdots & \vdots & & \vdots \\ q_1 q_n & q_2 q_n & \cdots & (q_n)^2 \end{pmatrix} \tag{1}$$

If $U = (u_{ij})$ is a real symmetric $n \times n$ matrix, the minor of U with all rows and columns deleted except for the $i_1, \ldots, i_h$ th lines and $j_1, \ldots, j_h$ th columns is denoted by

$$G^{j_1 \ldots j_h}_{i_1 \ldots i_h}(U) = \det\,(u_{ij})_{i = i_1, \ldots, i_h; j = j_1, \ldots, j_h}. \tag{2}$$

We also use the following notations:

$$G_{i_1 \ldots i_h}(U) = G^{i_1 \ldots i_h}_{i_1 \ldots i_h}(U), \tag{3}$$

$$\sigma_{i_1 \ldots i_h}(U) = \operatorname{sgn}\,(-1)^{h-1} G_{i_1 \ldots i_h}(U), \tag{4}$$

where, for any real number α, $\operatorname{sgn} \alpha = 1$ if $\alpha > 0$, $\operatorname{sgn} \alpha = 0$ if $\alpha = 0$, and $\operatorname{sgn} \alpha = -1$ if $\alpha < 0$.

Let us now state the following rule:

RULE 1. — *Let* U *be a real symmetric* $n \times n$ *matrix. Then the following statements are equivalent:*

a) *there exist n four-momenta such that* U *is their Gram matrix;*

b) U *has at most one positive eigenvalue and at most three negative eigenvalues;*

c) $r = \text{rank } U \leqslant 4$ *and in the case* $r > 0$ *there exist some indices* $i_1, \ldots, i_r \in \{1, \ldots, n\}$ *such that* $\sigma_{i_1 \ldots i_r}(U) \neq 0$ *and*

$$\sigma_{i_1 \ldots i_h}(U) \leqslant \sigma_{i_1 \ldots i_{h+1}}(U), \quad 1 \leqslant h < r, \tag{5}$$

$$\sigma_{i_1 i_2 i_3 i_4}(U) = 1 \quad \textit{if} \quad r = 4, \tag{6}$$

$$\sigma_{i_2}(U) = -1 \quad \textit{if} \quad r = 4, \quad \sigma_{i_1 \ldots i_h}(U) = 0, \quad h = 1, 2, 3. \tag{7}$$

Proof. — Several parts of the above rule are well-known [5]. Suppose that *c*) holds. Let ν denote the number of all indices h such that $\sigma_{i_1 \dots i_h}(\mathrm{U}) = 0$; $1 \leqslant h < r$. According to a remark of Poon [5], the implication *c*) $\Rightarrow$ *b*) represents Jacobi's rule if $\nu = 0$ and Hildenfinger's rule if $\nu = 1$. Moreover, we recall that the implication *c*) $\Rightarrow$ *b*) for $\nu = 2$ represents a rule of Frobenius (for Jacobi's, Hildenfinger's and Frobenius' rules see the full section 3 from [7], Ch. X). The only remaining case is $\nu = 3$. Using (6) and the conditions of (7), we obtain $\sigma_{i_2}(\mathrm{U}) = \sigma_{i_2 i_3}(\mathrm{U}) = -1$ and $\sigma_{i_1 i_2 i_3 i_4}(\mathrm{U}) = 1$. Then (5) holds if the indices i_1, i_2, i_3, i_4 are replaced by i_2, i_3, i_4, i_1, respectively, and the case $\nu = 3$ is reduced to the cases $\nu = 0$ and $\nu = 1$. Hence *c*) implies *b*).

Suppose that *b*) holds. Then it follows from the canonical diagonalization of a real symmetric matrix that there exist a real orthogonal $n \times n$ matrix $\Omega = (\omega_{ij})$ and a real diagonal $n \times n$ matrix $\mathrm{D} = (d_{ij})$ such that $\mathrm{U} = \Omega \mathrm{D} \Omega^t$, $d_{11} \geqslant 0$, $d_{ii} \leqslant 0$, $d_{jj} = 0$, $2 \leqslant i \leqslant 4$, $j > 4$ (see, for example, [7], Ch. X, § 5). The symbol t denotes the matric transpose. Let us denote $q_i^{j-1} = \omega_{ij} \mid d_{jj} \mid^{1/2}$, $i = 1, \dots, n; j = 1, 2, 3, 4$. Then U is the Gram matrix of the four-momenta $q_1, \dots, q_n$. Hence *b*) implies *a*).

In order to prove the implication *a*) $\Rightarrow$ *c*), we introduce certain orthogonal bases for the linear subspaces of the Minkowski space. The basis $\{ e_1, \dots, e_d \}$ is called a *standard basis* if the four-momenta $e_1, \dots, e_d$ satisfy the following relations:

$$e_h e_{h'} = \begin{cases} \varepsilon & \text{if} \quad h = h' = 1 \\ -1 & \text{if} \quad h = h' > 1, \\ 0 & \text{if} \quad h \neq h' \end{cases} \tag{8}$$

where $h, h' = 1, \dots, d$ and $\varepsilon \in \{ -1, 0, 1 \}$. There exists a standard basis for any nonzero linear subspace N of the Minkowski space. Indeed, if N is d-dimensional, then there exists an orthogonal basis $\{f_1, \dots, f_d\}$ for N such that $(f_h)^2 \geqslant (f_{h+1})^2$, $h = 1, \dots, d-1$ [8]. It is easy to see that if two four-momenta are linearly independent and their scalar product is equal to zero, then at least one of these four-momenta has a strictly negative Lorentz square. Then we can define $e_1 = f_1$ if $(f_1)^2 = 0$ and $e_h = \mid (f_h)^2 \mid^{-1/2} f_h$ if either $h > 1$ or $h = 1$ and $(f_1)^2 \neq 0$. Therefore $\{ e_1, \dots, e_d \}$ is a standard basis for N. Notice that if $\varepsilon \leqslant 0$, then $(q)^2 \leqslant 0$ for any $q \in \mathrm{N}$. Hence $\varepsilon = 1$ when $d = 4$ (in this case N is the whole Minkowski space of four-momenta). It is clear that $d \leqslant 4$.

Suppose that U is the Gram matrix of the four-momenta $q_1, \dots, q_n$. Let N be the linear space spanned by the four-momenta $q_{i_1}, \dots, q_{i_h}$, where $i_1, \dots, i_h \in \{ 1, \dots, n \}$. If $q_1 = \dots = q_n = 0$, then $\mathrm{U} = 0$ and it is obvious that the implication *a*) $\Rightarrow$ *c*) is trivial. Therefore we can suppose that $d > 0$. Let $\{ e_1, \dots, e_d \}$ be a standard basis for N. We introduce the $d \times d$ matrix

$V = (e_j e_{j'})$, $1 \leqslant j, j' \leqslant d$, and the $h \times d$ matrix $T = (t_{i_k j})$, $1 \leqslant k \leqslant h$, $1 \leqslant j \leqslant d$, defined by

$$q_i = \sum_{j=1}^{d} t_{ij} e_j, \qquad i = i_1, \ldots, i_h. \tag{9}$$

The Gram matrix of the four-momenta $q_{i_1}, \ldots, q_{i_h}$ is $U' = TVT^t$. If $q_{i_1}, \ldots, q_{i_h}$ are linearly dependent, then it is obvious that $\sigma_{i_1 \ldots i_h}(U) = (-1)^{h-1} \det U' = 0$. Moreover, we have rank $U' \leqslant 4$ and, in particular, $r = \text{rank } U \leqslant d \leqslant 4$. If $q_{i_1}, \ldots, q_{i_h}$ are linearly independent, then $h = d$, $\det T \neq 0$, and (8) implies $\sigma_{i_1 \ldots i_h}(U) = (e_1)^2$. In the case $r = h = 4$ one obtains $\sigma_{i_1 i_2 i_3 i_4}(U) = \det V = 1$ and (6) holds. If $r > 1$, we choose $i_1, \ldots, i_{r'} \in \{1, \ldots, n\}$, $1 < r' \leqslant r$ such that $\sigma_{i_1 \ldots i_{r'}}(U) \neq 0$ and consider that $h \in \{1, \ldots, r' - 1\}$. Then the four-momenta $q_{i_1}, \ldots, q_{i_{r'}}$ are linearly independent and the linear space N' spanned by $q_{i_1}, \ldots, q_{i_{h+1}}$ admits a standard basis $\{e'_1, \ldots, e'_{h+1}\}$ with $\sigma_{i_1 \ldots i_{h+1}}(U) = (e'_1)^2$. Since N is a subspace of N', e_1 is a linear combination of $e'_1, \ldots, e'_{h+1}$ and (8) implies $(e'_1)^2 \geqslant (e_1)^2$. Hence (5) holds for the choice $r = r'$.

Suppose next that $r = r' > 2$ and $\sigma_{i_1}(U) = \sigma_{i_1 i_2}(U) = 0$. It is clear that $(q_{i_1})^2 = q_{i_1} q_{i_2} = 0$. Since q_{i_1} and q_{i_2} are linearly independent, we have $(q_{i_2})^2 < 0$ and (7) also holds. Hence *a*) implies *c*) and Rule 1 is proved.

Let I_n denote the space of all Gram matrices of n four-momenta. Any Gram matrix U satisfies both assertions *b*) and *c*) of Rule 1. Moreover, it follows from the proof of the implication *a*) $\Rightarrow$ *c*) that

$$\sigma_{i_1 \ldots i_h}(U) \leqslant \sigma_{i_1 \ldots i_{h+1}}(U), \qquad h = 1, \ldots, r' - 1, \tag{10}$$

for any $i_1, \ldots, i_{r'} \in \{1, \ldots, n\}$ such that $\sigma_{i_1 \ldots i_{r'}}(U) \neq 0$.

The space I_n is completely defined by the polynomial equalities and inequalities given by (5)-(7) and the condition rank $U \leqslant 4$. If $n > 4$, the last condition is equivalent to the following kinematical constraints of Asribekov [*2*]:

$$G^{j_1 j_2 j_3 j_4 j_5}_{i_1 i_2 i_3 i_4 i_5}(U) = 0, \tag{11}$$

where $1 \leqslant i_1 < \ldots < i_5 \leqslant n$ and $1 \leqslant j_1 < \ldots < j_5 \leqslant n$.

According to a result of Weyl, any identity with respect to the coefficients of U is a consequence of (11) (see, for example, the full section 17 from [*9*], Ch. II). Moreover, according to a result of Kronecker [*10*], the condition rank $U = r$ is equivalent to the following relations:

$$\begin{aligned} &\sum_{1 \leqslant j_1 < \ldots < j_h \leqslant n} G_{j_1 \ldots j_h}(U) = 0, \qquad h = r + 1, \ldots, n, \\ &G_{i_1 \ldots i_r}(U) \neq 0 \end{aligned} \tag{12}$$

for some $i_1, \ldots, i_r \in \{1, \ldots, n\}$.

Hence a real symmetric $n \times n$ matrix U which satisfies (12) for $r \leqslant 4$ and the relation $G_{i_1 \ldots i_r}(U) \neq 0$ for a choice of the indices $i_1, \ldots, i_r \in \{1, \ldots, n\}$ is a Gram matrix of n four-momenta if and only if (5), (6), and (7) hold.

We recall that Rule 1 for any Gram matrix with positive diagonal coefficients has been obtained by Omnes [*1*] and Byers and Yang [*3*].

The reconstruction of a set of four-momenta from their Gram matrix is given by the following rule:

RULE 2. — *Consider a Gram matrix* $U \in I_n$ *with rank* $U = r > 0$. *Then there exists a choice of the indices* $i_1, \ldots, i_r \in \{1, \ldots, n\}$ *such that either* $\sigma_{i_1 \ldots i_h}(U) \neq 0$, $h = 1, \ldots, r$, *or there exists* $k \in \{1, \ldots, r-1\}$ *satisfying the following conditions:*

1) $\sigma_{i_1 \ldots i_h}(U) = -1$ *if* $1 \leqslant h < k$; 2) $\sigma_{i_1 \ldots i_h}(U) = 1$ *if* $k < h \leqslant r$;

3) $\sigma_{i_1 \ldots i_{k-1} i_h}(U) = 0$ *if* $k \leqslant h \leqslant r$; 4) $G^{i_1 \ldots i_{k-1} i_{k+1}}_{i_1 \ldots i_{k-1} i_k}(U) \neq 0$;

5) $i_k < i_{k+1}$ *if and only if* $(-1)^k G^{i_1 \ldots i_{k-1} i_{k+1}}_{i_1 \ldots i_{k-1} i_k}(U) > 0$.

Consider an orthogonal $n \times n$ *matrix* $\widehat{O} = (0_{jj'})$ *such that either* $\widehat{O} = 1$ *for* $\sigma_{i_1 \ldots i_h}(U) \neq 0$, $h = 1, \ldots, r$, *or*

$$o_{jj'} = \begin{cases} 1 & \text{if } j = j' \neq i_k, i_{k+1} \\ 0 & \text{if } j \neq j' \text{ and either } j \neq i_k, i_{k+1} \text{ or } j' \neq i_k, i_{k+1} \\ -1/\sqrt{2} & \text{if } j < j' \text{ and } j, j' \in \{ i_k, i_{k+1} \} \\ 1/\sqrt{2} & \text{if } j \geqslant j' \text{ and } j, j' \in \{ i_k, i_{k+1} \} \end{cases} \tag{13}$$

for $\sigma_{i_1 \ldots i_k}(U) = 0$.

Then the matrix $U' = \widehat{O} U \widehat{O}^t$ *satisfies the relations* $\sigma_{i_1 \ldots i_h}(U') \neq 0$, $1 \leqslant h \leqslant r$. *Moreover,* U *is the Gram matrix of the four-momenta* $q_1, \ldots, q_n$ *defined by*

$$q_i = \sum_{j=1}^{n} o_{ji} q'_j, \quad i = 1, \ldots, n, \tag{14}$$

$$q'^{\mu_{h-1}}_j = \begin{cases} \dfrac{\sigma_{i_1 \ldots i_h}(U') G^{i_1 \ldots i_{h-1} j}_{i_1 \ldots i_{h-1} i_h}(U')}{|G_{i_1 \ldots i_{h-1}}(U') G_{i_1 \ldots i_h}(U')|^{1/2}} & \text{if } h \leqslant r \\ o & \text{if } h > r \end{cases} \tag{15}$$

where $j = 1, \ldots, n$; $h = 1, \ldots, 4$, *and the following convention is used:* $-\sigma_{i_1 \ldots i_{h-1}}(U') = G_{i_1 \ldots i_{h-1}}(U') = 1$ *if* $h = 1$. *Here* $(\mu_0\ \mu_1, \mu_2, \mu_3)$ *is a permutation of* (0, 1, 2, 3) *such that* $\mu_h = 0$ *if* $\sigma_{i_1 \ldots i_{h-1}}(U'). \sigma_{i_1 \ldots i_h}(U') = -1$ *and* $\mu_h < \mu_{h'}$ *if* $h < h'$; $\mu_h, \mu_{h'} \neq 0$.

Proof. — Since $r > 0$, we can choose the indices $j_1, \ldots, j_r \in \{1, \ldots, n\}$ such that $G_{j_1 \ldots j_r}(U) \neq 0$ and $\sigma_{j_1 \ldots j_{h'}}(U) \geqslant \sigma_{j_1 \ldots j_h}(U)$ for $h \geqslant h'$. We now show that there exists a permutation $(i_1, \ldots, i_r)$ of $(j_1, \ldots, j_r)$ which satisfies the conditions of Rule 2. According to Rule 1 *c*), the argument divides into two cases:

Case I. — Suppose that $\sigma_{j_1\dots j_h}(\mathrm{U}) \neq 0$, $h = 1, \dots, r$. Then we choose $i_h = j_h$, $h = 1, \dots, r$.

Case II. — Suppose that there exists $k \in \{1, \dots, r-1\}$ such that $\sigma_{j_1\dots j_h}(\mathrm{U}) = -1$ if $1 \leqslant h < k$ and $\sigma_{j_1\dots j_{k-1}j_h}(\mathrm{U}) = 0$ if $k \leqslant h \leqslant r$. Since $\mathrm{G}_{j_1\dots j_r}(\mathrm{U}) \neq 0$ it follows that there exist two indices $h', h'' \in \{k, \dots, r\}$ such that $\mathrm{G}^{j_1\dots j_{k-1}j_{h''}}_{j_1\dots j_{k-1}j_{h'}}(\mathrm{U}) \neq 0$ and $j_{h'} < j_{h''}$. We choose $i_h = j_h$ if $h \in \{1, \dots, r\}$; $h', h'' \neq h$, and $i_k = j_{h'}$, $i_{k+1} = j_{h''}$ (resp. $i_k = j_{h''}$, $i_{k+1} = j_{h'}$) if $(-1)^k$. $\mathrm{G}^{j_1\dots j_{k-1}j_{h''}}_{j_1\dots j_{k-1}j_{h'}}(\mathrm{U})$ is negative (resp. positive). Note that from the identity

$$\mathrm{G}_{i_1\dots i_{k-1}}(\mathrm{U})\mathrm{G}_{i_1\dots i_{k+1}}(\mathrm{U}) = \mathrm{G}_{i_1\dots i_k}(\mathrm{U})\mathrm{G}_{i_1\dots i_{k-1}i_{k+1}}(\mathrm{U}) - [\mathrm{G}^{i_1\dots i_{k-1}i_{k+1}}_{i_1\dots i_{k-1}i_k}(\mathrm{U})]^2 \tag{16}$$

it follows that $\sigma_{i_1\dots i_{k+1}}(\mathrm{U}) = 1$. Then (5) implies $\sigma_{i_1\dots i_h}(\mathrm{U}) = 1$ if $k < h \leqslant r$.

If k is as in Case II, using (13), we obtain

$$(-1)^{k-1}\mathrm{G}_{i_1\dots j_k}(\mathrm{U}') = \left| \mathrm{G}^{i_1\dots i_{k-1}i_{k+1}}_{i_1\dots i_{k-1}i_k}(\mathrm{U}) \right|, \tag{17}$$

$$\mathrm{G}_{i_1\dots i_h}(\mathrm{U}') = \mathrm{G}_{i_1\dots i_h}(\mathrm{U}), \quad 1 \leqslant h \leqslant r, \quad h \neq k. \tag{18}$$

Therefore $\sigma_{i_1\dots i_h}(\mathrm{U}') \neq 0$, $1 \leqslant h \leqslant r$. Straightforward computations starting from (15) show that $\mathrm{U}' = (q'_j q'_{j'})$; $j, j' = 1, \dots, n$ (see, for example, the Gauss algorithm from [*7*], Ch. II, § 4). Then by (14), we obtain $\mathrm{U} = (q_i q_{i'})$; $i, i' = 1, \dots, n$, and Rule 2 is proved.

We remark that the implication *c*) $\Rightarrow$ *a*) of Rule 1 follows from the proof of Rule 2. Then from the proof of Rule 1 excepting the implication *c*) $\Rightarrow$ *b*), we obtain a direct proof of Jacobi's Hildenfinger's and Frobenius' rules.

Let I_{nr} denote the set of all Gram matrices of rank r belonging to I_n. Consider the set of all matrices $\mathrm{U} \in \mathrm{I}_{nr}$ such that $\mathrm{G}_{i_1\dots i_r}(\mathrm{U}) \neq 0$ for a choice of the indices $i_1, \dots, i_r \in \{1, \dots, n\}$ (in the case $r > 0$). Then Rule 2 shows that $\{u_{i_h j}\}$, $h = 1, \dots, r$; $j \in \{1, \dots, n\}$; $j \neq i_{h'}$, for $h' < h$, is a set of $nr - \frac{1}{2}r(r-1)$ independent kinematical variables with respect to I_{nr}. According to (15) and the identities

$$[\mathrm{G}^{i_1\dots i_{r-1}j}_{i_1\dots i_{r-1}i_r}(\mathrm{U}')]^2 = \mathrm{G}_{i_1\dots i_r}(\mathrm{U}')\mathrm{G}_{i_1\dots i_{r-1}}(\mathrm{U}'), \tag{19}$$

we can choose the independent variables u_{jj}, $u_{i_h j}$ with $h = 1, \dots, r-1$; $j \in \{1, \dots, n\}$; $j \neq i_{h'}$, if $h' \leqslant h$. But the Gram matrix U can be reconstructed from these variables only if the signs of $\mathrm{G}^{i_1\dots i_{r-1}j}_{i_1\dots i_{r-1}i_r}(\mathrm{U}')$, $1 \leqslant j \leqslant n$, $j \neq i_1, \dots, i_r$, are known (see also Röhrlich's dichotomy from [*4*]).

Finally, any representative of a Gram matrix is given by the following completion of Rule 2:

RULE 2'. — *For any matrix* $\mathrm{U} \in \mathrm{I}_n$ *and four-momenta* $q'_1, \dots, q'_n$ *such* *that* U *is their Gram matrix, there exists a standard basis* $\{ e_1, \ldots, e_d \}$ *such that one of the following decompositions holds :*

$$q'_i = \sum_{h=1}^{r} \alpha_{ih} e_h, \qquad i = 1, \ldots, n, \qquad r = d, \tag{20}$$

$$q'_i = \sum_{h=2}^{r+1} \alpha_{ih} e_h + \beta_i e_1, \qquad (e_1)^2 = 0, \qquad \sum_{i=1}^{n} \beta_i^2 = 1, \qquad r = d - 1 \leqslant 2, \tag{21}$$

where $d (0 \leqslant d \leqslant 4)$ *is the dimension of the linear space spanned by* $q'_1, \ldots, q'_n$; r *is the rank of* U, *each* β_i *is a real number,* $\alpha_{ih} = q_i^{\mu_{h-1}}$ *for* $r = d$ *and* $\alpha_{ih} = q_i^{\mu_{h-2}}$ *for* $r = d - 1$ *with* $q_1, \ldots, q_n$ *given by Rule 2. The sums from the r. h. s. of* (19) *and* (20) *are dropped if* $r = 0$.

Proof. — We recall a result of Hall and Wightman [*10*]: if two sets of n four-momenta $\{ q_1, \ldots, q_n \}$ and $\{ q'_1, \ldots, q'_n \}$ have the same Gram matrix U, then there exist a Lorentz transformation Λ, some real numbers $\beta'_1, \ldots, \beta'_n$ and a four-momentum ω such that

$$q'_i = \Lambda q_i + \beta'_i \omega, \qquad (\omega)^2 = \omega(\Lambda q_i) = 0, \qquad i = 1, \ldots, n. \tag{22}$$

Here Λ is identified to the matrix (Λ^μ_ν), $0 \leqslant \mu, \nu \leqslant 3$, such that the four-vectors $f_h = (\Lambda^0_{\mu_{h-1}}, \Lambda^1_{\mu_{h-1}}, \Lambda^2_{\mu_{h-1}}, \Lambda^3_{\mu_{h-1}})$, $h = 1, 2, 3, 4$, satisfy the relations $(f_1)^2 = -(f_2)^2 = -(f_3)^2 = -(f_4)^2 = 1$, $f_h f_{h'} = 0$, $1 \leqslant h' < h \leqslant 4$, with the permutation $(\mu_0, \mu_1, \mu_2, \mu_3)$ of (0, 1, 2, 3) given by Rule 2 with respect to U. Note that the four-vectors f_1, f_2, f_3 and f_4 are linearly independent. Λq is defined by $(\Lambda q)^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu q^\nu$, $0 \leqslant \mu \leqslant 3$, for any four-momentum q.

Suppose that the four-momenta $q_1, \ldots, q_n$ are determined by U as in Rule 2 (in the case $r > 0$). Then by (22), we have

$$q'_i = \sum_{h=1}^{r} q_i^{\mu_{h-1}} f_h + \beta'_i \omega, \qquad i = 1, \ldots, n, \tag{23}$$

where the sum over h is dropped if $r = 0$. If $\omega = 0$ or $\beta'_i = 0$ for any $i = 1, \ldots, n$, we choose $e_h = f_h$, $h = 1, \ldots, r$, and (20) holds. In the opposite case, we set $e_1 = \left(\sum_{i=1}^{n} \beta_i'^2 \right)^{1/2} \omega$ and $e_h = f_{h+1}$, $h = 1, \ldots, r$. According to the proof of Rule 1 a standard basis $\{ e_1, \ldots, e_d \}$ with $(e_1)^2 = 0$ has $d \leqslant 3$. Then (21) holds and Rule 2′ is proved.

Rules 2 and 2′ give an analytical parametrization of any n four-momenta with respect to their scalar products, the parameters of the Lorentz group, and the coordinates of an n-sphere (not all independent).

3. PHYSICAL REGIONS AND PHASE SPACES

Consider a set of n particles $c_1, \ldots, c_n$. By the spectral condition, we attach to each particle c_i a four-momentum p_i such that

$$(p_i)^2 = m_i^2,\ p_i^0 > 0,\ i = 1, \ldots, n, \tag{24}$$

where m_i ($m_i \geqslant 0$) is the mass of c_i.

In order to give a general treatment for the usual choices of invariant kinematical variables (like the multiperipheral momentum transfers squared and multiparticle invariant masses), we introduce the following four-momenta:

$$q_i = \sum_{j=1}^{n} t_{ij} p_j, \qquad i = 1, \ldots, n, \tag{25}$$

where the coefficients t_{ij} are appropriate real numbers or functions of the scalar products $q_i q_j$, $1 \leqslant i, j \leqslant n$.

By (24), the four-momenta q_i satisfy (25) for some t_{ij} only if either all scalar products of q_i vanish or the linear space spanned by $q_1, \ldots, q_n$ has a standard basis $\{ e_1, \ldots, e_r \}$ with $(e_1)^2 = 1$. Then it follows from (20) that r is the rank of the Gram matrix U of $q_1, \ldots, q_n$ and

$$\sigma_{i_1 \ldots i_r}(\mathrm{U}) = 1 \tag{26}$$

if $\mathrm{G}_{i_1 \ldots i_r}(\mathrm{U}) \neq 0$ and $i_1, \ldots, i_r \in \{ 1, \ldots, n \}$. Moreover, according to (20) and to the Sylvester law of inertia, (26) holds if and only if U has one positive eigenvalue (more details can be found in [5]). We denote by I_{n+} the space of all Gram matrices $\mathrm{U} \in \mathrm{I}_n$ such that either $\mathrm{U} = 0$ or U satisfies (26).

Consider now the exclusive reaction

$$c_1 + c_2 \to c_3 + \ldots + c_n, \tag{27}$$

and the inclusive reaction

$$c_1 + c_2 \to c_3 + \ldots + c_{n-1} + \text{anything}. \tag{28}$$

We suppose that the above reactions are not forbidden. The condition of energy-momentum conservation gives $p = p_n$ for (27) and $p^0 \geqslant 0$, $\sqrt{(p)^2} \in \mathrm{S}$ for (28), where $p = p_1 + p_2 - p_3 - \ldots - p_{n-1}$ and S is the spectrum of the invariant mass of the unobserved system from (28).

Suppose next that the $(n-1) \times (n-1)$ matrix $\mathrm{T} = (t_{ij})$, $1 \leqslant i, j \leqslant n-1$, given by (25), is invertible. According to (24) and (25) and using the Gram matrix U of the four-momenta $q_1, \ldots, q_{n-1}$, we obtain the following image of the physical region for reaction (27) (resp. (28)) in the space I_{n-1+} :

$$D_n = \left\{ U \mid U = TU'T^t \in I_{n-1+}, \quad U'_{ii} = m_i^2, \quad \sum_{i,j=1}^{n-1} \varepsilon_i \varepsilon_j U'_{ij} = m_n^2, \quad \sum_{j=1}^{n-1} \varepsilon_j U'_{ij} \leqslant 0, \right.$$

$$\left. U'_{ij} \geqslant 0, \quad \varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \ldots = \varepsilon_{n-1} = 1;\ i, j = 1, \ldots, n-1 \right\}, \tag{29}$$

respectively

$$D'_n = \left\{ U \mid U = TU'T^t \in I_{n-1+}, \quad U'_{ii} = m_i^2, \quad \left[\sum_{i,j=1}^{n-1} \varepsilon_i \varepsilon_j U'_{ij} \right]^{1/2} \in S, \right.$$

$$\left. U'_{ij} \geqslant 0, \quad \sum_{j=1}^{n-1} \varepsilon_j U'_{ij} \leqslant 0, \quad \varepsilon_1 = \varepsilon_2 = -1, \varepsilon_3 = \ldots = \varepsilon_{n-1} = 1; i, j = 1, \ldots, n-1 \right\}. \tag{30}$$

The region D_n can be decomposed into the subregions $D_{nr} = D_n \cap I_{n-1r}$ (each D_{nr} consists of all matrices $U \in D_n$ with rank $U = r$). By Rule 1, we have $r \leqslant r_0 = \min(4, n-1)$. The case $r = 0$ holds only when all masses vanish and all four-momenta are parallel. In the case $T = 1$, the results of Jacobson [6] show that each nonvoid subregion D_{nr} is a connected real analytic manifold of dimension $n(r-1) - r(r+1)/2$, $1 < r \leqslant r_0$. D_{n1} is not empty if and only if $U_{ij} = m_i m_j$, $1 \leqslant i, j \leqslant n$, and $m_1 + m_2 = m_3 + \ldots + m_n \neq 0$. Moreover, the inverse image of D_n in the space of all n-tuples of four-momenta is an analytic manifold if and only if D_{nr} is empty for $r \leqslant 1$. These results persist if the coefficients of T are analytic functions of the coefficients of U. Note that the closure $\overline{D}_{nr}$ of D_{nr}, $1 < r \leqslant r_0$, is not an analytic manifold, but it is a semialgebraic variety if the coefficients of T are polynomials of the coefficients of U (*i. e.* the set $\overline{D}_{nr}$ consisting of all matrices $U \in D_n$ with rank $U \leqslant r$ is completely defined by the polynomial inequalities and equalities given by Rule 1 *c*) and (26)). Notice also that the boundary of the physical region D_n $(n \geqslant 4)$ is $\overline{D}_{nr_0-1}$ and the boundary of D_{nr} is $\overline{D}_{nr-1}$ $(r > 2)$. Finally, we remark that the physical regions of all channels crossed to (27) can be obtained by replacing $\varepsilon_1, \ldots, \varepsilon_{n-1}$ in (29) by $\pm\varepsilon_1, \ldots, \pm\varepsilon_{n-1}$ except for the cases $\varepsilon_1 = \ldots = \varepsilon_{n-1}$ and $\varepsilon_i = -\varepsilon_j$, $j \in \{1, \ldots, n-1\}$, $j \neq i$, for $i \in \{1, \ldots, n-1\}$ fixed with the mass m_i smaller than the sum of the masses m_j, $j \neq i$ (*i. e.* the decay channel of particle c_i).

The region D'_{n-1} can be decomposed into the regions $D'_{n-1}(M)$, $M \in S$, where each $D'_{n-1}(M)$ is defined by the r. h. s. of (29) with m_n replaced by M. Hence any region $D'_{n-1}(M)$ has the same algebraic and geometric properties as D_n. Moreover, a similar behaviour persists for the union D''_{n-1} of the regions $D'_{n-1}(M)$ with $M > M_c$, where M_c is the threshold mass of the continuum.

Thus if $r > 1$, the region $D''_{n-1r} = D''_{n-1} \cap I_{n-1r}$ (resp. its closure $\overline{D}_{n-1r}$) is a connected real analytic manifold of dimension $n(r-1) - r(r+1)/2 + 1$ (resp. a semialgebraic variety) provided the coefficients of T are analytic functions (resp. polynomials) of the coefficients of U. The closure $\overline{D}''_{n-1}$ of D''_{n-1} is the union of $D_{n-1}(M_c)$ and D''_{n-1}. $\overline{D}''_{n-1r}$ consists of all matrices $U \in \overline{D}''_{n-1}$ with rank $U \leqslant r$. Notice that the boundaries of D''_{n-1} and $\overline{D}''_{n-1}$ ($n \geqslant 4$) consist of all matrices $U \in \overline{D}''_{n-1}$ such that either $U \in D'_{n-1}(M_c)$ or rank $U < \min(4, n-1)$.

We now digress a little on phase-space analysis. The phase-space volume element of reaction (27) is defined by

$$dW(P, m_3, \ldots, m_n) = \delta^4\left(P - \sum_{i=3}^{n} p_i\right)\prod_{j=3}^{n}\theta(p_j^0)\delta((p_j)^2 - m_j^2)d^4p_j, \quad (31)$$

where $P = p_1 + p_2$ and the symbols δ and θ denote the usual Dirac and Heaviside distributions.

Let us consider the following transformation of variables:

$$p_i \to \{p_iq_i, p_iq'_i, p_iq''_i, (p_i)^2\}, \; i = 3, \ldots, n-1, \quad (32)$$

where the four-momenta p_i, q_i, q'_i and q''_i are linearly independent and the four-momenta q_i, q'_i and q''_i are differentiable functions of p_j, $1 \leqslant j \leqslant i$. Then the Jacobian of (32) is

$$J = 2^{3-n}\prod_{i=3}^{n-1}[-D(p_i, q_i, q'_i, q''_i)]^{-1/2}. \quad (33)$$

Here and in the remainder of this paper we shall use the notation $D(Q_1, \ldots, Q_h)$ for the determinant of the Gram matrix of $Q_1, \ldots, Q_h$.

Notice that if

$$\gamma_i = \operatorname{sgn}\det\begin{vmatrix} p_i^0 & p_i^1 & p_i^2 & p_i^3 \\ q_i^0 & q_i^1 & q_i^2 & q_i^3 \\ q_i'^0 & q_i'^1 & q_i'^2 & q_i'^3 \\ q_i''^0 & q_i''^1 & q_i''^2 & q_i''^3 \end{vmatrix} \quad (34)$$

is fixed at a nonzero value ($\gamma_i = \pm 1$), then (32) is an one-to-one transformation.

Suppose that

$$D(-q_3, q'_3) < 0, \; q_{n-1} = P - \sum_{i=3}^{n-2} p_i, \; n > 4. \quad (35)$$

Using (32)-(35), the relation

$$\int d(p_3q_3)[-D(p_3, q_3, q'_3, q''_3)]^{-1/2} = \pi[-D(q_3, q'_3)]^{-1/2}, \quad (36)$$

and integrating (31) over p_n, $(p_i)^2$, p_3q_3, and $p_{n-1}q_{n-1}$ $(i = 3, \ldots, n)$, the phase-space volume element can be written as

$$d\tilde{W}(P, m_3, \ldots, m_n) = \chi\pi[-D(q_3, q_3')]^{-1/2}\prod_{i=4}^{n-1}[-D(p_i, q_i, q_i', q_i'')]^{-1/2}$$
$$\cdot \prod_{j=3}^{n-2} d(p_jq_j)\prod_{j'=3}^{n-1} d(p_{j'}q_{j'})\prod_{j''=4}^{n-1} d(p_{j''}q_{j''}), \quad (37)$$

where

$$(p_i)^2 = m_i^2, 3 \leqslant i \leqslant n-1, \quad \text{and} \quad p_{n-1}q_{n-1} = \frac{1}{2}[(q_{n-1})^2 + m_{n-1}^2 - m_n^2]. \quad (38)$$

The function χ gives the ranges of the invariant variables. Thus, if $q_i^0 > 0$, $(q_i)^2 \geqslant 0$, Rule 1 *c*) implies

$$\chi = \theta((q_{n-1})^2 - (m_{n-1} + m_n)^2)\prod_{i=3}^{n-2}\theta(p_iq_i - m_i^2\sqrt{(q_i)^2}).$$
$$\cdot \prod_{j=3}^{n-1}\theta(D(p_j, q_j, q_j'))\prod_{k=4}^{n-1}\theta(-D(p_k, q_k, q_k', q_k'')). \quad (39)$$

Note that if a function invariant to the connected Lorentz group is integrated with respect to the phase-space volume element (31), then the factor $2^{3-n}[\theta(\gamma_3) + \theta(-\gamma_3)] \ldots [\theta(\gamma_{n-1}) + \theta(-\gamma_{n-1})]$ must be included in χ. However, this factor is not necessary for the estimation of cross-sections with parity conservation.

If $(P)^2$ is fixed, we have $3n$-11 essential invariant variables p_jq_j, $p_{j'}q_{j'}$, $p_{j''}q_{j''}$, $3 \leqslant j \leqslant n-2$, $3 \leqslant j' \leqslant n-1$, $4 \leqslant j'' \leqslant n-1$. By Rule 2, the scalar products of $q_3, q_3', q_i, q_i', q_i''$ $(4 \leqslant i \leqslant n-1)$ are functions of the above variables. It is easy to estimate these functions in the scheme of Byers and Yang [3] ($q_3 = q_i' = P$; $q_j = p_3 + \ldots + p_{j-1}$; $q_3' = p_1$; $q_k'' = p_{k-1}$; $4 \leqslant i, k \leqslant n-1$; $4 \leqslant j \leqslant n-2$) or in the scheme of Poon [5] (*i. e.* the following multiperipheral scheme: $q_j' = p_{j-1}$; $q_3 = P$; $q_i = P - p_3 - \ldots - p_{i-1}$; $q_k'' = p_2 - p_3 - \ldots - p_{k-1}$; $4 \leqslant i \leqslant n-2$; $3 \leqslant j \leqslant n-1$; $4 \leqslant k \leqslant n-1$). Notice that (37) is a simple generalization of the phase-space volume element of Byers and Yang [3] (see [11]-[13] for some applications to the results of Byers and Yang). Note also that (37) unifies many usual schemes [14].

Finally, we remark that the phase-space volume element of reaction (28) excepting the discrete part of the spectrum is given by $d\tilde{W}(P, m_3, \ldots, m_{n-1}, M)dM^2$, $M \geqslant M_c$.

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