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Alternative axioms for statistical physical theories

by

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ABSTRACT. — In the usual partially ordered vector space approach to the theory of statistical physical systems the set of states of the system under consideration is represented by a norm closed generating cone V^+ in a Banach space V with base norm. In this sense the Banach space V can be said to represent the system. With any such system is associated a classical system which is represented by the Banach space $Z(V^*)_*$, the predual of the centre $Z(V^*)$ of the dual space V^* of V. The set of questions or propositions associated with the system is represented by the set $E(\mathcal{Q})$ of extreme points of the weak* compact convex set $\mathcal{Q}$, the order unit interval in the order unit space V^*. The set of classical propositions is therefore represented by the set $E(\mathcal{Q}_c)$ of extreme points of the order unit interval $\mathcal{Q}_c$ in $Z(V^*)$. In this case $E(\mathcal{Q}_c)$ forms a complete Boolean algebra. In the standard model for classical probability theory the set $\mathcal{L}$ of propositions forms a Boolean σ-algebra not necessarily complete. A new set of axioms for statistical physical systems is introduced. It is shown that the associated classical systems are precisely those which arise in classical probability theory.

RÉSUMÉ. — Dans l'approche conventionnelle par les espaces vectoriels partiellement ordonnés à la théorie des systèmes physiques statistiques, l'ensemble des états du système est représenté par un cône V^+ qui engendre un espace V de Banach avec une norme de base. Dans ce sens, on peut dire que l'espace V représente le système. Avec tout système de cet ordre on peut rapprocher un système classique qui est représenté par l'espace $Z(V^*)_*$ de Banach, le prédual du centre $Z(V^*)$ de l'espace V^* qui est le dual de V. L'ensemble des questions ou des propositions du système est représenté par l'ensemble $E(\mathcal{Q})$ des points extrêmaux de l'ensemble $\mathcal{Q}$

convexe compact par rapport à la topologie $\sigma(V^*, V)$, l'intervalle d'unité d'ordre dans l'espace V^*. L'ensemble des propositions classiques est donc représenté par l'ensemble $E(\mathcal{Q}_c)$ des points extrêmaux de l'intervalle $\mathcal{Q}_c$ d'unité d'ordre dans $Z(V^*)$. Dans ce cas $E(\mathcal{Q}_c)$ est une algèbre Boolean complet. L'ensemble des propositions suivant la théorie classique des probabilités forme une σ-algèbre Boolean, qui n'est pas nécessairement complet. On introduit un ensemble nouveau d'axiomes qui décrivent des systèmes physiques statistiques. On démontre que les systèmes classiques associés sont précisément ceux qui apparaissent dans la théorie classique des probabilités.

1. INTRODUCTION

In the operational approach to the theory of statistical physical systems as formulated in [3] [4] [6] [7] [10] [12] [14] [19] the set of *states* of a system is represented by a norm closed generating cone V^+ in a certain partially ordered Banach space V, the norm in which is a base norm in the sense that

$$\| x + y \| = \| x \| + \| y \|, \ \forall x, \ y \in V^+$$

and

$$\| x \| = \inf \{ \| x_1 \| + \| x_2 \| : x_1, x_2 \in V^+, \ x = x_1 - x_2 \}, \ \forall x \in V.$$

The dual space V^* of V with dual cone V^{*+} is a GM-*space* [25] (F-*space* [23]) *with unit* e defined by $e(x_1 - x_2) = \| x_1 \| - \| x_2 \|$, $x_1, x_2 \in V^+$. The set of *operations* on the system is represented by the set $\mathcal{P}$ of positive norm non-increasing linear operators on V or alternatively by the set $\mathcal{P}^*$ of weak* continuous positive nor non-increasing linear operators on V^* the mapping $T \mapsto T^*$ being an isometric affine isomorphism from $\mathcal{P}$ onto $\mathcal{P}^*$. The set of *simple observables* (*effects* [19], *tests* [12]) is represented by the set $\mathcal{Q} = V^{*+} \cap (e - V^{*+})$ and the simple observable measured by the operation T is T^*e. The set of *extreme simple observables* (*decision effects* [19], *question*) is represented by the set $E(\mathcal{Q})$ of extreme points of the weak* compact convex set $\mathcal{Q}$. The method of obtaining this description from physical axioms is discussed in detail elsewhere [4] [6] [19].

Recall that the *ideal centre* $\mathfrak{O}(V)$ of V consists of linear operators T on V for which there exists $\lambda \geq 0$ such that $\lambda x \pm Tx \in V^+$, $\forall x \in V^+$. $\mathfrak{O}(V)$ is a uniformly closed commutative subalgebra of the algebra $\mathfrak{L}(V)$ of bounded linear operators on V, relative to the cone $\mathfrak{O}(V)^+$ of positive operators in $\mathfrak{O}(V)$ and the operator norm, $\mathfrak{O}(V)$ is a GM-space with unit 1, the identity operator, and for T, $S \in \mathfrak{O}(V)^+$, $TS \in \mathfrak{O}(V)^+$. A result of Kadison [16] shows that there exists a compact Hausdorff space Ω_V such that $\mathfrak{O}(V)$ is algebraically and order isomorphic to the algebra $C(\Omega_V)$ of real-valued continuous functions on Ω_V. Further $\mathfrak{O}(V)$ is boundedly complete (i. e.

uniformly bounded increasing nets in $\mathfrak{D}(V)$ have least upper bounds in $\mathfrak{D}(V)$) which implies that Ω_V is *stonean* and $\mathfrak{D}(V)$ is separated by its positive normal linear functionals which implies that Ω_V is *hyperstonean* [*1*] [*2*] [*5*] [*13*]. Hence the set $\mathscr{S}(V)$ of idempotents in $\mathfrak{D}(V)$ forms a complete Boolean algebra uniformly generating $\mathfrak{D}(V)$. $\mathscr{S}(V)$, the set of *split projections*, is the set of extreme points of the set $\mathfrak{D}(V)^+ \cap (1 - \mathfrak{D}(V)^+)$. The ideal centre $\mathfrak{D}(V^*)$ of V^* has identical properties to those of $\mathfrak{D}(V)$ and the mapping $T \mapsto T^*$ is an algebraic isomorphism and hence a normal order isomorphism from $\mathfrak{D}(V)$ onto $\mathfrak{D}(V^*)$. In particular its restriction to $\mathscr{S}(V)$ is a complete Boolean algebra isomorphism onto $\mathscr{S}(V^*)$. The *centre* $Z(V^*)$ of V^* is the image of $\mathfrak{D}(V^*)$ under the mapping $T^* \mapsto T^*e$. $Z(V^*)$ is a weak* closed subspace of V^* and the mapping $T^* \mapsto T^*e$ is a normal order isomorphism from $\mathfrak{D}(V^*)$ onto $Z(V^*)$. In particular its restriction to $\mathscr{S}(V^*)$ is a complete Boolean algebra isomorphism onto the complete Boolean algebra $E(\mathscr{Q}_c)$ of extreme points of the weak* compact convex set $\mathscr{Q}_c = Z(V^*)^+ \cap (e - Z(V^*)^+)$. Moreover $\mathfrak{D}(V)$, $\mathfrak{D}(V^*)$ and $Z(V^*)$ are all weak* isomorphic to the dual space of the Banach space V/N where

$$N = \{ x : x \in V,\ Te(x) = 0,\ \forall T \in \mathfrak{D}(V) \} \quad [11].$$

Denote by $\mathscr{P}_c$, $\mathscr{P}_c^*$ the sets $\mathscr{P} \cap \mathfrak{D}(V)$, $\mathscr{P}^* \cap \mathfrak{D}(V^*)$ respectively and notice that $\mathscr{P}_c$, $\mathscr{P}_c^*$ are weak* compact convex sets such that

$$\mathscr{S}(V) = E(\mathscr{P}_c), \quad \mathscr{S}(V^*) = E(\mathscr{P}_c^*).$$

Elements T of $\mathscr{P}_c$ are operations on the system which have the property that 1-T is also an operation. Hence every state x possesses a decomposition into states Tx, $(1 - T)x$. Therefore the effect of such an operation is to divide any state according to a well-defined classical prescription. Therefore $\mathscr{P}_c$ or $\mathscr{P}_c^*$ is referred to as the set of *classical operations* on the system. Each of these sets is affine isomorphic and weak* homeomorphic to the set $\mathscr{Q}_c$ of *classical simple observables (effects, tests)*. It follows that the classical part of the system can be described by replacing V by V/N. Elements of $\mathscr{S}(V)$, $\mathscr{S}(V^*)$ or $E(\mathscr{Q}_c)$ are regarded as *classical questions (propositions)* associated with the system. Alternatively they can be regarded as *superselection rules* for the system. In the case $N = \{ 0 \}$ or equivalently $Z(V^*) = V^*$ the system must be regarded as being itself classical.

In the standard model for classical probability theory the set of propositions forms a Boolean σ-algebra $\mathscr{L}$ and it is usually supposed that there are enough σ-additive positive measures on $\mathscr{L}$ to separate points. This is not consistent with the conclusion arrived at above that the set of classical propositions forms a complete Boolean algebra. Two points of view are now possible. The first would be to suppose that in actual physical situations it is always possible to enlarge the Boolean σ-algebra of classical equations in such a way that it becomes a complete Boolean algebra. Investigations based on assumptions of this kind have been made by Neumann [21]. The second point of wiew, which is adopted here, is that the original axioms used to produce the model described above were not the best possible and that an alternative set of axioms may more accurately reflect the real situation. The main purpose of this paper is to list such a set of axioms much in the spirit of [20] [24]. The principal result is that the classical part of a system obeying the new set of axioms forms a classical system in the sense in which it has been understood hitherto. That is the set of classical propositions forms a Boolean σ-algebra.

The conventional model for quantum mechanics also provides a model satisfying the new axioms. The only conclusion to be arrived at in this case is that the set of positive trace class operators on a Hilbert space H should more properly be regarded as defining σ-normal rather than normal linear functionals on the W*-algebra $\mathfrak{L}(H)$ of bounded linear operators on H.

In § 2 certain preliminary results concerning the centres of a class of GM-spaces are proved. In § 3 the axioms and a discussion of their plausibility are given. Also in § 3 the main theorems are stated and their implications are examined. In § 4 a few remarks are made about classical systems and their interactions with arbitrary systems. Proofs of the main results are given in § 5. In § 6 the situation in which the set of states of the system is supposed to possess a « physical » topology [15] is examined. It is shown that in this case the Boolean σ-algebra of classical propositions is σ-isomorphic to the Borel sets of some Borel space.

2. PRELIMINARIES

A partially ordered Banach space W with norm closed cone W^+ is said to be a GM-*space with unit* $e \in W^+$ provided that the closed unit ball in W is $(-e + W^+) \cap (e - W^+)$. Such a space is said to be *monotone σ-complete* if every uniformly bounded monotone increasing sequence $\{ a_n \} \subset W$ possesses a least upper bound in W. An example of such a space is the self-adjoint part of a monotone σ-complete C*-algebra with identity. If W, W′ are two such spaces a positive linear mapping $T: W \to W'$ is said to be σ-normal if for each uniformly bounded monotone increasing sequence $\{ a_n \} \subset W$, $T(\operatorname{lub} a_n) = \operatorname{lub} Ta_n$.

THEOREM 2.1. — *Let* W *be a monotone σ-complete* GM-*space with unit e the set* K^μ *of σ-normal linear functionals on which satisfies the condition that* $a \in W$, $x(a) \geqslant 0$, $\forall x \in K^\mu$ *implies that* $a \in W^+$. *Then,*

(i) *The ideal centre* $\mathfrak{O}(W)$ *of* W *is monotone σ-complete and the set* C^μ *of σ-normal linear functionals on* $\mathfrak{O}(W)$ *satisfies the condition that* $T \in \mathfrak{O}(W)$, $g(T) \geqslant 0$, $\forall g \in C^\mu$ *implies that* $T \in \mathfrak{O}(W)^+$.

(ii) *The set* $\mathscr{S}(W)$ *of idempotents in* $\mathfrak{D}(W)$ *forms a Boolean* σ*-algebra uniformly generating* $\mathfrak{D}(W)$.

(iii) *If* $T \in \mathfrak{D}(W)^+$ *then* T *is* σ*-normal.*

(iv) *The mapping* $T \mapsto Te$ *from* $\mathfrak{D}(W)$ *into* W *is* σ*-normal.*

Proof. — Let $\{ T_n \} \subset \mathfrak{D}(W)$ be monotone increasing and suppose that $\| T_n \| \leqslant k < \infty$, for all n. Then $\{ T_n + k1 \} \subset \mathfrak{D}(W)^+$ is monotone increasing and $\| T_n + k1 \| \leqslant 2k < \infty$, for all n. In order to prove monotone σ-completeness it is therefore sufficient to assume that $\{ T_n \} \subset \mathfrak{D}(W)^+$. For $a \in W^+$, $\{ T_n a \}$ is a uniformly bounded monotone increasing sequence in W^+. If $Ta = \text{lub}\, T_n a$ since K^μ separates points in W simple limit arguments show that T is a positive linear operator from W^+ to itself which therefore extends to a positive linear operator on W. Moreover, since for $a \in W^+$, $Ta = \text{lub}\, T_n a$ it follows that $0 \leqslant T_n \leqslant T \leqslant k1$, for all n and hence that $T \in \mathfrak{D}(W)^+$. Further if $S \in \mathfrak{D}(W)^+$ and for all n, $T_n \leqslant S$ then for all $a \in W^+$, $T_n a \leqslant Sa$ which implies that $Ta \leqslant Sa$, $T \leqslant S$. Therefore $T = \text{lub}\, T_n$ and hence $\mathfrak{D}(W)$ is monotone σ-complete.

Notice also that by choosing $a = e$ above (*iv*) has been proved. For $x \in K^\mu$ define $g_x(T) = x(Te)$, $\forall T \in \mathfrak{D}(W)$. Then, using (*iv*) it follows that $x \mapsto g_x$ maps K^μ into C^μ. If $T \in \mathfrak{D}(W)$, $g_x(T) \geqslant 0$, $\forall x \in K^\mu$ then $Te \in W^+$ and since the mapping $T \mapsto Te$ is bipositive on $\mathfrak{D}(W)$, $T \in \mathfrak{D}(W)^+$. This completes the proof of (*i*).

By standard properties of ideal centres there exists a compact Hausdorff space Ω_W such that $\mathfrak{D}(W)$ is algebraically and hence isometrically order isomorphic to $C(\Omega_W)$. It follows that $C(\Omega_W)$ is monotone σ-complete and hence that Ω_W is basically disconnected [*13*]. Therefore the set of idempotents in $C(\Omega_W)$ forms a Boolean σ-algebra which uniformly generates $C(\Omega_W)$. The same therefore applies to $\mathfrak{D}(W)$ completing the proof of (*ii*).

Let $T \in \mathfrak{D}(W)^+$ and let $\{ a_n \} \subset W^+$ be a uniformly bounded monotone increasing sequence with least upper bound a. If $T \leqslant k1$, $k \geqslant 0$, then $\{ Ta_n \}, \{ ka_n - Ta_n \}$ are uniformly bounded monotone increasing sequences. If $b = \text{lub}\, Ta_n$, $c = \text{lub}\, (ka_n - Ta_n)$ simple limit arguments show that for $x \in K^\mu$, $x(ka - b) = x(c)$ and hence that $ka - b = c$. But since T, $k1 - T$ are positive, $Ta_n \leqslant Ta$, $ka_n - Ta_n \leqslant ka - Ta$ which imply that $b \leqslant Ta$, $ka - b \leqslant ka - Ta$ and hence that $Ta = b$ as required.

Notice that an example of a space W satisfying the conditions of Theorem 2.1 is the self-adjoint part of a Baire* algebra with identity [*18*] [*22*].

The next result which follows immediately from the properties of the mapping $T \mapsto Te$ summarises the structure of the centre Z(W) of W.

THEOREM 2.2. — *Let* W *be a monotone* σ*-complete* GM*-space with unit* e *the set* K^μ *of* σ*-normal linear functionals on which satisfies the condition that* $a \in W$, $x(a) \geqslant 0$, $\forall x \in K^\mu$ *implies that* $a \in W^+$. *Then,*

(i) *The centre* Z(W) *of* W *is monotone* σ*-complete and the set* K^μ *of* σ*-nor-*

mal linear functionals on $Z(W)$ *satisfies the condition that* $z \in Z(W)$, $g(z) \geqslant 0$, $\forall g \in K_c^\mu$ *implies that* $z \in Z(W)^+$.

(ii) *The set* $E(\mathcal{Q}_c)$ *of extreme points of the convex set*

$$\mathcal{Q}_c = Z(W)^+ \cap (e - Z(W)^+)$$

forms a Boolean σ*-algebra.*

3. THE AXIOMS

The main features of previous axiomatic approaches are maintained here. There are two basic sets, $\mathcal{Q}$ the set of simple observables (effects, tests) and K the set of states. Elements of K are thought of as equivalence classes of ensembles of the system under consideration and elements of $\mathcal{Q}$ are thought of as equivalence classes of operations on the system. For $a \in \mathcal{Q}$, $x \in K$, let $p(a, x)$ be the stength of the state produced by an operation T corresponding to a on the state x. Notice $p(a, x)$ is only defined up to multiplication by a positive constant which is supposed to be fixed once and for all. There follows a statement of the axioms along with some discussion of their physical motivation. It will be supposed that $\mathcal{Q}$, K are abstract sets and that p is a mapping from $\mathcal{Q} \times K$ to the set of non-negative real numbers.

AXIOM 1. — *For* $a_1, a_2 \in \mathcal{Q}$, $p(a_1, x) = p(a_2, x)$, $\forall x \in K$ *implies that* $a_1 = a_2$. *For* $x_1, x_2 \in K$, $p(a, x_1) = p(a, x_2)$, $\forall a \in \mathcal{Q}$ *implies that* $x_1 = x_2$. This axiom merely describes what is meant by « identical » for states and simple observables.

AXIOM 2. — *There exist* $0, e \in \mathcal{Q}$ *such that*

$$0 = p(0, x) \leqslant p(a, x) \leqslant p(e, x), \ \forall x \in K, \ \forall a \in \mathcal{Q}.$$

This axiom asserts the existence of absurd and certain events. The uniqueness of 0, e follows from Axiom 1.

AXIOM 3. — *For* $\{ x_n \} \subset K$, $\{ \alpha_n \} \subset R^+$ *satisfying the condition that*

$$\sum_{n=1}^{\infty} \alpha_n p(a, x_n) < \infty, \qquad \forall a \in \mathcal{Q},$$

there exists $x \in K$ *such that*

$$p(a, x) = \sum_{n=1}^{\infty} \alpha_n p(a, x_n), \qquad \forall a \in \mathcal{Q}.$$

This axiom asserts that countable mixtures of states exist.

Notice that from Axiom 2 the convergence of the sum for every $a \in \mathcal{Q}$

is ensured if the sum converges with $a = e$. Again the element $x \in K$ defined in Axiom 3 is unique by Axiom 1.

AXIOM 4. — *For $a \in \mathcal{Q}$, $\alpha \in [0, 1]$ there exists $a_1 \in \mathcal{Q}$ such that*

$$p(a_1, x) = \alpha p(a, x), \qquad \forall x \in K.$$

If T is a filtering operation designed to measure a the axiom asserts that a filtering operation T_1 can be constructed the effect of which on any state x is to produce a state the strength of which is diminished in the ratio α: 1 compared with that produced by T. The uniqueness of a_1 follows from Axiom 1.

AXIOM 5. — *For $a_1, a_2 \in \mathcal{Q}$ satisfying*

$$p(a_1, x) \leqslant p(a_2, x), \qquad \forall x \in K,$$

there exists $a \in \mathcal{Q}$ such that

$$p(a_1, x) + p(a, x) = p(a_2, x), \qquad \forall x \in K.$$

Suppose that T_1, T_2 are operations corresponding to a_1, a_2 respectively and suppose also that the effect of T_2 on any state x is to produce a stronger state than that produced by the operation T_1 on x. The axiom asserts the existence of an operation T the effect of which on x is to produce a state the strength of which is the difference between the stengths of the states obtained by operating with T_2, T_1 on x. The uniqueness of a follows from Axiom 1.

AXIOM 6. — *For $a_1 \in \mathcal{Q}$, $\alpha \in R^+$ satisfying*

$$p(a_1, x) \leqslant \alpha p(e, x), \qquad \forall x \in K,$$

there exists $a \in \mathcal{Q}$ such that

$$p(a_1, x) = \alpha p(a, x), \qquad \forall x \in K.$$

If $\alpha \geqslant 1$, Axiom 6 follows immediately from Axiom 4. Suppose that $\alpha < 1$ and that T_1 is an operation corresponding to a_1. Suppose also that under T_1 the strength of a state x is decreased by a proportion not exceeding α. The axiom asserts the existence of an operation T the effect of which on any state x is to produce a state the strength of which is a multiple $1/\alpha$ of that of the state produced by T_1. The uniqueness of a follows from Axiom 1.

AXIOM 7. — *For $\{ a_n \} \subset \mathcal{Q}$ satisfying*

$$\sum_{n=1}^{\infty} p(a_n, x) \leqslant p(e, x), \qquad \forall x \in K$$

there exists $a \in \mathcal{Q}$ such that

$$p(a, x) = \sum_{n=1}^{\infty} p(a_n, x), \qquad \forall x \in K.$$

This is the analogue of Axiom 3 for simple observables. Suppose that $\{ T_n \}$ is a sequence of operations such that T_n corresponds to a_n. Suppose also that the strength of the state obtained by mixing the states obtained by applying each operation T_n to any state x is less than the strength of x itself. The axiom then asserts the existence of an operation T the effect of which on any state x is to produce a state the strength of which is equal to the strength of the mixed state obtained by combining the states obtained by applying each T_n to x. Again a is unique.

The first main result is the following.

THEOREM 3.1. — *Let $\mathcal{Q}$, K, p satisfy Axioms 1, 2, 4-7. Then there exists a monotone σ-complete GM-space W with unit e the set K^μ of σ-normal linear functionals on which satisfies the condition that $a \in W$, $x(a) \geqslant 0, \forall x \in K^\mu$ implies that $a \in W^+$ and a bijection ϕ from $\mathcal{Q}$ onto $W^+ \cap (e - W^+)$ satisfying,*

(i) $\phi(0) = 0$, $\phi(e) = e$.

(ii) *For a, a_1, α as in Axiom 4 or Axiom 6,*

$$\phi(a_1) = \alpha\phi(a).$$

(iii) *For a_1, a_2, a as in Axiom 5,*

$$\phi(a_1) + \phi(a) = \phi(a_2).$$

(iv) *For $\{ a_n \}$, a as in Axiom 7, let $\{ b_n \} \subset \mathcal{Q}$ be defined by*

$$p(b_n, x) = \sum_{r=1}^{n} p(a_r, x), \qquad \forall x \in K.$$

Then,

$$\phi(a) = \operatorname{lub} \phi(b_n).$$

From now on it will be supposed that $\mathcal{Q}$ and $W^+ \cap (e - W^+)$ are identical. For $x \in K$, the mapping $a \mapsto p(a, x)$ on $\mathcal{Q}$ clearly extends to a σ-normal linear functional $\psi(x)$ on W. It follows from Axiom 1 that the mapping ψ is an injection from K into K^μ.

THEOREM 3.2. — *Let $\mathcal{Q}$, K, p satisfy Axioms 1-7. Then, under the conditions of Theorem 3.1 there exists a bijection ψ from K onto a weak* dense subcone $\psi(K)$ of K^μ defined for $x \in K$, $a \in \mathcal{Q}$ by*

$$\psi(x)(\phi(a)) = p(a, x)$$

satisfying the condition that if $\{ x_n \}$, $x \subset K$, $\{ \alpha_n \} \subset R^+$ are as in Axiom 3 then

$$\lim_{n\to\infty} || \psi(x) - \sum_{r=1}^{n} \alpha_r \psi(x_r) || = 0$$

where $||\,.\,||$ is the norm in the dual space of W.

In the sequel it will be supposed that K and $\psi(K)$ are identical and for $a \in \mathcal{Q}$, $x \in K$, $p(a, x) = x(a)$.

Notice that the set $B = \{ x : x \in K, x(e) = 1 \}$ is a base for the cone K. Let $V = K - K$ and for $x \in V$ define

$$\| x \|_B = \inf \{ x_1(e) + x_2(e) : x_1, x_2 \in K, x = x_1 - x_2 \}.$$

The first part of the next result follows immediately from [*10*].

THEOREM 3.3. — *Under the conditions of Theorem* 3.2 *with* $\mathcal{Q}$, K *respectively identified with* $\phi(\mathcal{Q})$, $\psi(K)$ *let* $V = K - K$, $B = \{ x : x \in K, x(e) = 1 \}$ *and let* $\| . \|_B$ *be the base semi-norm on* V *defined above. Then* $\| . \|_B$ *is a norm on* V *with respect to which* V *is complete. Further, the mapping* $a \mapsto a'$ *defined for* $x \in V$ *by* $a'(x) = x(a)$ *is a* σ*-normal isometric order isomorphism from* W *onto a monotone* σ*-closed weak* dense subspace* W′ *of* V*.

In the sequel W and W′ will also be identified. Then, K can either be regarded as a subcone of the cone of σ-normal linear functionals on the monotone σ-complete GM-space W with unit e or alternatively as a generating cone for a complete base norm space V. In this case W must be regarded as a monotone σ-closed subspace of the dual space V* of V.

In the preceding discussion the presence of « operations » has been tacitly assumed. The final axiom describes the properties of operations.

AXIOM 8. — *Let* $T : \mathcal{Q} \to \mathcal{Q}$ *be a mapping satisfying*

(i) $T(0) = 0$.

(ii) *For* a, a_1, α *as in Axiom* 4 *or Axiom* 6,

$$p(Ta_1, x) = \alpha p(Ta, x), \qquad \forall x \in K.$$

(iii) *For* a_1, a_2, a *as in Axiom* 5,

$$p(Ta_1, x) + p(Ta, x) = p(Ta_2, x), \qquad \forall x \in K$$

(iv) *For* $\{ a_n \}$, a *as in Axiom* 7,

$$p(Ta, x) = \sum_{n=1}^{\infty} p(Ta_n, x), \qquad \forall x \in K.$$

Then, there exists a mapping $T' : K \to K$ *such that*

$$p(a, T'x) = p(Ta, x), \qquad \forall x \in K, \forall a \in Q.$$

The set $\mathcal{P}$ of mapping T satisfying the conditions of Axiom 8 is said to be the set of *operations* on the system. Clearly each $T \in \mathcal{P}$ has a unique extension to a norm non-increasing σ-normal linear operator on W. The axiom ensures that the adjoint T* of T leaves K invariant. T′ is merely the restriction of T* to K. The next result describes the image of $\mathcal{P}$ under the mapping $T \mapsto T'$.

THEOREM 3.4. — *The mapping* $T \mapsto T'$ *defined for* $x \in V$, $a \in W$ *by*

$$T'x(a) = x(Ta)$$

is an isometric affine isomorphism from $\mathscr{P}$ onto the set $\mathscr{P}'$ of norm non-increasing positive linear operators T' *on the complete base norm space* V *the adjoints* T'^* *of which leave* W *invariant.*

It is now possible to give a complete discussion of classes of operations as in [6]. However the main purpose of this paper is to examine only one class of operation. An element $T \in \mathscr{P}$ is said to be a *classical operation* if and only if

$$p(Ta, x) \leqslant p(a, x), \qquad \forall a \in \mathscr{Q},\ \forall x \in K.$$

Let $\mathscr{P}_c$ denote the set of classical operations and let $\mathscr{Q}_c$ be the image of $\mathscr{P}_c$ under the mapping $T \mapsto Te$. $\mathscr{Q}_c$ is the set of *classical simple observables (effects, tests)*. Notice that alternatively the set of classical operations can be identified with the set $\mathscr{P}'_c$ of elements of $\mathscr{P}'$ satisfying the condition

$$p(a, T'x) \leqslant p(a, x), \qquad \forall a \in \mathscr{Q},\ \forall x \in K.$$

For $x_1, x_2 \in K$ define $x_1 \sim x_2$ if and only if

$$p(a, x_1) = p(a, x_2), \qquad \forall a \in \mathscr{Q}_c.$$

Clearly $\sim$ is an equivalence relation on K. The set K_c of equivalence classes of elements of K under $\sim$ is said to be the set of *classical states* of the system.

The results of § 2 show that $\mathscr{Q}_c$ is the set $Z(W)^+ \cap (e - Z(W)^+)$. Moreover K_c can clearly be identified with a cone of σ-normal linear functional on Z(W). Most of the following result is an immediate consequence of Theorem 2.2.

THEOREM 3.5. — *Under the conditions of Theorem* 3.2 *let* $\mathscr{Q}_c, K_c$ *be defined as above and let the mapping* p_c *be defined by*

$$p_c(a, x^{\sim}) = p(a, x), \qquad \forall a \in \mathscr{Q},\ \forall x \in K$$

where $x^{\sim}$ *is the equivalence class containing* $x \in K$. *Then,* $0, e \in \mathscr{Q}_c$ *and Axioms* 1-7 *are satisfied with* $\mathscr{Q}$, K, p, 0, e *replaced by* $\mathscr{Q}_c$, K_c, p_c, 0, e *respectively.*

Recall that in [6] the set $E(\mathscr{Q})$ of extreme points of $\mathscr{Q}$ is identified with the set of extreme simple observables or questions. The next result follows immediately from Theorem 2.2.

COROLLARY 3.6. — *Under the conditions of Theorem* 3.5, *the set* $E(\mathscr{Q}_c)$ *of classical questions is a Boolean σ-algebra the points of which are separated by the set of σ-additive positive measures on* $E(\mathscr{Q}_c)$.

The system described in Theorem 3.5 is said to be the *centre* of the original system. The question arises of which set of operations on the system described by Z(W) and K_c is relevant. The assumption made here is that every operation on the centre must arise from an operation on the whole system. In this case the set of operations on the centre is defined to be the set of restrictions to Z(W) of those elements of $\mathscr{P}$ which leave Z(W) invariant. It is not clear when this set is the set of *all* norm non-increasing σ-normal linear operators on Z(W). However the set does contain the restrictions of elements of $\mathscr{P}_c$ to Z(W).

In the case Z(W) = W the system is said to be *classical*. It follows that the centre of any system is classical.

4. CLASSICAL SYSTEMS

Recall that a compact Hausdorff space Ω is said to be basically disconnected if and only if the closure of every co-zero set in Ω is open. Alternatively Ω is basically disconnected if and only if the algebra $C(\Omega)$ is monotone σ-complete. Ω is said to be *σ-hyperstonean* if Ω is basically disconnected and the set K_1^μ of σ-normal linear functionals on $C(\Omega)$ satisfies the condition that $f \in C(\Omega)$, $x(f) \geqslant 0$, $\forall x \in K_1^\mu$ implies $f \geqslant 0$. The set $\mathscr{L}$ of closed and open sets of a σ-hyperstonean space forms a Boolean σ-algebra the points of which are separated by the set of positive σ-additive measures on $\mathscr{L}$. Conservely, given any Boolean σ-algebra $\mathscr{L}$ the points of which are separated by the set of positive σ-additive measures on $\mathscr{L}$ then the Stone space Ω of $\mathscr{L}$ is σ-hyperstonean and $\mathscr{L}$ can be identified with the Boolean σ-algebra of closed and open sets in Ω. The results of § 3 imply that there is a bijection from the set of classical systems onto the set of σ-hyperstonean spaces.

Next the interaction between a classical system described by a σ-hyperstonean space Ω and an arbitrary system is considered. Let $\mathscr{Q}_1$ be the set $\{ f : f \in C(\Omega),\ 0 \leqslant f(\omega) \leqslant 1,\ \forall \omega \in \Omega \}$ and let $\mathscr{Q}$, K, p, 0, e satisfy Axioms 1-7. Then the interaction is described by a mapping $\mathscr{E}$ called an *instrument* [3] [4] from $\mathscr{Q}_1 \times \mathscr{Q}$ to $\mathscr{Q}$ satisfying

(*i*) $\mathscr{E}$ is bilinear.

(*ii*) $\mathscr{E}(1_\Omega, e) = e$.

(*iii*) If $\{ f_n \} \subset \mathscr{Q}_1$ satisfies the condition that

$$\sum_{n=1}^{\infty} f_n(\omega) = f(\omega) \leqslant 1, \qquad \forall \omega \in \Omega,$$

then,

$$p(\mathscr{E}(f, a), x) = \sum_{n=1}^{\infty} p(\mathscr{E}(f_n, a), x), \qquad \forall a \in \mathscr{Q},\ \forall x \in K.$$

(*iv*) If $\{ a_n \}$, $a \in \mathscr{Q}$ satisfy the conditions of Axiom 7 then

$$p(\mathscr{E}(f, a), x) = \sum_{n=1}^{\infty} p(\mathscr{E}(f, a_n), x), \qquad \forall f \in \mathscr{Q}_1,\ \forall x \in K.$$

Essentially conditions (*iii*), (*iv*) mean that $\mathscr{E}$ is σ-normal in each variable.

In particular each element $f \in \mathcal{Q}_1$ defines an operation on the system. Notice that the instrument $\mathcal{E}$ gives rise to an observable $\mathcal{A}$. Again following [3] [4] an *observable* $\mathcal{A}$ is a mapping from $\mathcal{Q}_1$ to $\mathcal{Q}$ satisfying,

(*i*) $\mathcal{A}$ is linear,

(*ii*) $\mathcal{A}(1_\Omega) = e$.

(*iii*) If $\{f_n\} \subset \mathcal{Q}_1$ satisfies the condition that

$$\sum_{n=1}^{\infty} f_n(\omega) = f(\omega), \qquad \forall \omega \in \Omega,$$

then,

$$p(\mathcal{A}(f), x) = \sum_{n=1}^{\infty} p(\mathcal{A}(f_n), x), \qquad \forall x \in \mathrm{K}.$$

The instrument $\mathcal{E}$ gives rise to the observable $\mathcal{A}$ defined by

$$\mathcal{A}(f) = \mathcal{E}(f, e), \qquad \forall f \in \mathcal{Q}_1.$$

In particular if $\mathcal{L}$ is the Boolean σ-algebra of closed and open sets in Ω, $\mathcal{E}$ gives rise to the σ-additive measure $\mathcal{A}$ on $\mathcal{L}$ defined by

$$\mathcal{A}(\mathrm{M}) = \mathcal{A}(\chi_\mathrm{M}), \qquad \forall \mathrm{M} \in \mathcal{L}$$

where χ_M is the characteristic function of M.

It is possible to go on to examine when two instruments can be composed along the lines of [4]. Using the results of [26] it can be shown that instruments corresponding to a fairly general class of classical systems can be composed.

5. PROOFS

Proof of Theorem 3.1. — Using Axioms 1, 2, 4 and the finite form of Axiom 7 it follows easily that $\mathcal{Q}$ is an abstract convex set and can therefore be embedded into an abstract convex cone C. The embedding i is then an affine isomorphism from $\mathcal{Q}$ into C. Further, from Axiom 2, $i(0) = 0$, the vertex of the cone C and

$$\mathrm{C} = \bigcup_{\alpha > 0} \alpha(\mathcal{Q}).$$

Again using standard techniques C can be embedded as a generating cone W^+ for a real vector space W. The embedding j is an affine isomorphism from C onto W^+ such that $j(0) = 0$. Let $\phi = j \circ i$ be the embedding of $\mathcal{Q}$ into W. It follows from Axiom 5 that relative to the cone W^+, $\phi(a_1) \leqslant \phi(a_2)$ if and only if $p(a_1, x) \leqslant p(a_2, x)$, $\forall x \in \mathrm{K}$. Let $\phi(e)$ also be denoted by e. Then clearly $\phi(\mathcal{Q}) \subseteq \mathrm{W}^+ \cap (e - \mathrm{W}^+)$ but an application of Axiom 6 is required to show that the reverse inclusion also holds. A further routine verification shows that e is an Archimedean order unit for W. The associated order unit norm $\|\,.\,\|$ is defined for $a \in W$ by

$$\| a \| = \inf \{ \lambda : \lambda \geqslant 0, - \lambda e \leqslant a \leqslant \lambda e \}.$$

To show that W is a GM-space with unit it remains to show that W is complete for then W^+ is automatically closed. However, Axiom 7 ensures that $\phi(\mathscr{Q})$ contains the least upper bounds of monotone increasing sequences in $\phi(\mathscr{Q})$. Since

$$W^+ = \bigcup_{\alpha > 0} \alpha\phi(\mathscr{Q})$$

it follows that W is monotone σ-complete and therefore by a standard argument (see [9]) complete. This completes the proof.

Proof of Theorem 3.2. — Axiom 3 ensures that $\psi(K)$ is a cone in W^* and since ψ is a positive mapping Axiom 1 ensures that $\psi(K)$ is weak* dense in W^{*+} and hence in K^μ. If $\{ x_n \}$, $x \subset K$, $\{ \alpha_n \} \subset R^+$ are as in Axiom 3 then

$$\left\| \psi(x) - \sum_{r=1}^{n} \alpha_r \psi(x_r) \right\| = (\psi(x) - \sum_{r=1}^{n} \alpha_r \psi(x_r))(\phi(e))$$
$$= p(e, x) - \sum_{r=1}^{n} \alpha_r p(e, x_r) \rightarrow 0$$

as $n \rightarrow \infty$. This completes the proof.

Proof. of Theorem 3.3. — The dual space of V can be identified with the space $A^b(B)$ of bounded affine functions on B endowed with the supremum norm and the natural ordering. Clearly the mapping $a \mapsto a'$ defined for $x \in B$ by $a'(x) = x(a)$ is an order isomorphism from W into $A^b(B)$. It follows that the mapping is isometric and σ-normal with monotone σ-closed range weak* dense in V^*.

Proof of Theorem 3.4. — By Axiom 8 the mapping $T \mapsto T'$ is an isometric affine isomorphism from $\mathscr{P}$ into $\mathscr{P}'$. However, for $S \in \mathscr{P}'$, $S^*W \subseteq W$ and since S^* is weak* continuous it follows that the restriction of S^* to W is σ-normal. This completes the proof.

6. THE CASE $W = A(B)^\mu$

In [*15*] it was suggested that the set K of states of the system should possess a physical topology to describe the lack of accuracy in measurements. By requiring that all the information about a particular physical system could be obtained in a finite number of measurements it was shown in [*14*] that the physical topology extended to V could be assumed to be locally convex Hausdorff and that K could be assumed locally compact. Equivalently the base B could be assumed compact.

Let B be a compact convex set regularly embedded in a locally convex Hausdorff topological vector space V, let A(B) be the space of continuous affine functions on B and let $A^b(B)$ as before be the space of bounded affine functions on B. Then V can be identified with $A(B)^*$ and $A^b(B)$ can be identified with $A(B)^{**}$. $A^b(B)$ is monotone σ-complete and hence a smallest monotone σ-closed subset of $A^b(B)$ containing A(B) exists. This set is a subspace $A(B)^\mu$ of $A^b(B)$ called the monotone σ-envelope of A(B). With the supremum norm and the natural ordering $A(B)^\mu$ is a monotone σ-complete GM-space with unit 1_B. Further the set of σ-normal linear functionals on $A(B)^\mu$ can be identified with $A(B)^{*+}$. Hence by choosing $W = A(B)^\mu$ and $K = A(B)^{*+}$ a possible model for a physical system is obtained. The main result of [8] shows that $Z(A(B)^\mu)$ is σ-normally and algebraically isomorphic to the algebra $F_{\mathscr{G}}(\Gamma)$ of bounded $\mathscr{G}$-measurable functions on a space Γ equipped with a σ-algebra $\mathscr{G}$ of subsets. Various choices of Γ are available. Let $\widehat{A(B)}$ denote the set of non-zero minimal elements of the complete Boolean algebra $\mathscr{S}(A(B)^*)$ and let

$$GP(B) = \{ x : x \in B, Px = x, \quad \text{some} \quad P \in \widehat{A(B)} \}.$$

Notice that GP(B) is the union of the family $\{ PA(B)^* \cap B : P \in \widehat{A(B)} \}$ of disjoint subsets of B. Further, for $T \in \mathfrak{O}(A(B)^*)$, $P \in \widehat{A(B)}$, the function T^*1_B is constant on $PA(B)^* \cap B$. The basic result is the following. The proof is given in [8].

THEOREM 6.1. — *There exists a σ-algebra $\mathscr{G}$ of subsets of $\widehat{A(B)}$ and a σ-normal algebraic isomorphism π from $Z(A(B)^\mu)$ onto the algebra $F_{\mathscr{G}}(\widehat{A(B)})$ of bounded $\mathscr{G}$-measurable functions on $\widehat{A(B)}$ defined for $z \in Z(A(B)^\mu)$ by*

$$\pi(z)(P) = z(x), \qquad \forall x \in PA(B)^* \cap B.$$

Hence the Boolean σ-algebra $E(\mathscr{Q}_c)$ of extreme points of the convex set $\mathscr{Q}_c = Z(A(B)^\mu)^+ \cap (1_B - Z(A(B)^\mu)^+)$ is σ-isomorphic to $\mathscr{G}$.

The conclusion is that in this special case the centre of the system is described by means of a Borel space. Notice that in Theorem 6.1 $\widehat{A(B)}$ could be replaced by GP(B). In fact if E(B) denotes the set of extreme points of B, and if $\overset{\wedge}{A(B)}$ denotes the subset of $\widehat{A(B)}$ consisting of elements P such that $PA(B)^* \cap E(B) \neq \varnothing$ then either $\overset{\wedge}{A(B)}$ or E(B) could also replace $\widehat{A(B)}$ in the theorem.

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REFERENCES

[1] E. M. ALFSEN, E. G. EFFROS, Structure in real Banach spaces I. *Ann. Math.*, t. **96**, 1972, p. 98-128.

[2] E. M. ALFSEN, E. G. EFFROS, Structure in real Banach spaces II. *Ann. Math.*, t. **96**, 1972, p. 129-173.

[3] E. B. DAVIES, On the repeated measurement of continuous observables in quantum mechanics. *J. Funct. Anal.*, t. **6**, 1970, p. 318-346.

[4] E. B. DAVIES, J. T. LEWIS, An operational approach to quantum probability. *Commun. math. Phys.*, t. **17**, 1970, p. 277-304.

[5] J. DIXMIER, Sur certains espaces considérés par M. H. Stone. *Summa Brasil. Math.*, t. **2**, 1951, p. 151-182.

[6] C. M. EDWARDS, Classes of operations in quantum theory. *Commun. math. Phys.*, t. **20**, 1971, p. 26-56.

[7] C. M. EDWARDS, The theory of pure operations. *Commun. math. Phys.*, t. **24**, 1972, p. 260-288.

[8] C. M. EDWARDS, Spectral theory for A(X), *Math. Ann.*, t. 207, 1974, p. 67-85.

[9] C. M. EDWARDS, The spectrum of a real Banach space. *Proc. London Math. Soc.*, t. **28**, 1974, p. 654-670.

[10] C. M. EDWARDS, M. A. GERZON, Monotone convergence in partially ordered vector spaces. *Ann. Inst. Henri Poincaré*, t. A **12**, 1970, p. 323-328.

[11] D. A. EDWARDS, On the ideal centres of certain partially ordered Banach spaces. *J. London Math. Soc.*, t. **6**, 1973, p. 656-658.

[12] R. GILES, Foundations of quantum mechanics. *J. Math. Phys.*, t. **11**, 1970, p. 2139-2160.

[13] L. GILMAN, M. JERISON, *Rings of continuous functions*, New Jersey, Van Nostrand, 1960.

[14] J. GUNSON, On the algebraic structure of quantum mechanics. *Commun. math. Phys.*, t. **6**, 1967, p. 262-285.

[15] R. HAAG, D. KASTLER, An algebraic approach to quantum field theory. *J. Math. Phys.*, t. **5**, 1964, p, 846-861.

[16] R. V. KADISON, A representation theory for commutative topological algebras. *Mem. Amer. Math. Soc.*, t. **7**, 1951.

[17] R. V. KADISON, Unitary invariants for representations of topological algebras. *Ann. Math.*, t. **66**, 1957, p. 304-379.

[18] E. T. KEHLET, On the monotone sequential closure of a C*-algebra. *Math. Scand.*, t. **25**, 1969, p. 59-70.

[19] G. LUDWIG, Deutung des Begriffs « physikalische Theorie » und axiomatische Grundlegung der Hilbertraumstruktur der Quantenmechanik durch Hauptsätze des Messens. *Lecture Notes in Physics*, 4. Berlin, Heidelberg, New York, Springer, 1970.

[20] G. W. MACKEY, *Mathematical foundations of quantum mechanics*, New York, Benjamin, 1963.

[21] H. NEUMANN, *Classical systems in quantum mechanics and their representations in topological spaces.* Preprint.

[22] G. K. PEDERSEN, On the weak and monotone σ-closures of C*-algebras. *Commun. math. Phys.*, t. **11**, 1969, p. 221-226.

[23] F. PERDRIZET, Espaces de Banach ordonnés et idéaux. *J. Math. Pures Appl.*, t. **49**, 1970, p. 61-98.

[24] R. J. PLYMEN, C*-algebras and Mackey's axioms. *Commun. math. Phys.*, t. **8**, 1968, p. 132-146,

[25] W. WILS, The ideal center of partially ordered vector spaces. *Acta Math.*, t. **127**, 1971, p. 41-79.

[26] J. D. M. WRIGHT, Products of positive vector measures. *Quart. J. Math.*, t. **24**, 1973, p. 189-206.

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