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Section A :
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On the Cauchy problem for the Yang-Mills field equations

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RÉSUMÉ. — Nous prouvons que, pour $\overset{\circ}{A}$ une solution donnée des équations du champ de Yang et Mills, appartenant à l'espace fonctionnel $H^{m+1}(V_T^\Omega)$, $m > 2$, où V_T^Ω est un ouvert dans l'espace-temps d'épaisseur temporelle arbitraire finie, il existe une infinité de solutions exactes A dont les données de Cauchy sont proches au sens de l'espace $H^m(\Omega)$ de celles de $\overset{\circ}{A}$, et qui sont elles-mêmes proches de $\overset{\circ}{A}$ au sens de $H^{m+1}(V_T^\Omega)$.

SUMMARY. — We prove that for any given solution $\overset{\circ}{A}$ of the Yang-Mills field equations, which belongs to the functional space $H^{m+1}(V_T^\Omega)$, $m > 2$, where V_T^Ω is an open subset of space-time of arbitrary (but finite) temporal width, there exists an infinite number of exact solutions A such that their Cauchy data are near in the sense of $H^m(\Omega)$ to the Cauchy data of $\overset{\circ}{A}$, and they are near themselves to $\overset{\circ}{A}$ in the sense of $H^{m+1}(V_T^\Omega)$.

1. In this paper we apply a variant of the method developed by Mme Choquet-Bruhat [*1*] for the non-linear partial differential equations to the problem of the global existence of the solutions to the Yang-Mills field equations.

First, let us define the notations here. M_4 will be the Minkowskian space-time with the usual metric tensor $g_{\mu\nu} = \mathrm{diag}\,(+ - - -)$; $\mu, \nu, \ldots = 0, 1, 2, 3$; $x \in M_4$. The Yang-Mills field is given by a potential $A_\mu^a(x)$,

$a, b, \ldots = 1, 2, \ldots, N = \dim G$, where G is a Lie group of the gauge. The field tensor hast the following form:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu \tag{1}$$

where C^a_{bc} are the structure constants of the group G. This tensor satisfies the Yang-Mills field equations:

$$\partial^\mu F^a_{\mu\nu} + C^a_{bc} A^{b\mu} F^c_{\mu\nu} = 0 \tag{2}$$

(the indices μ, ν are raised and lowered by means of the metric tensor $g^{\mu\nu}$ and its inverse $g_{\mu\nu}$).

Hereafter we shall always fix the gauge analogously to the Lorentz gauge for the electromagnetic field, i. e. we shall require that

$$\partial^\mu A^a_\mu = 0 \tag{3}$$

With this condition satisfied, the field equations (2) can be rewritten in terms of A^a_μ only, as follows:

$$\square A^a_\nu + 2C^a_{bc} A^{b\mu} \partial_\mu A^c_\nu - C^a_{bc} A^{b\mu} \partial_\nu A^c_\mu + C^a_{bc} C^c_{de} A^{b\mu} A^d_\mu A^e_\nu = 0\,, \tag{4}$$

$\square$ being the d'Alembertian operator in M_4.

The system (4), which is strictly hyperbolic and semi-linear, is all we shall need for further investigations. In the presence of the external sources $\mathscr{J}^a_\mu$ it is modified to the form

$$\square A^a_\nu + 2C^a_{bc} A^{b\mu} \partial_\mu A^c_\nu - C^a_{bc} A^{b\mu} \partial_\nu A^c_\mu + C^a_{bc} C^c_{de} A^{b\mu} A^d_\mu A^e_\nu = \mathscr{J}^a_\nu \tag{5}$$

2. Let us rewrite the system (5) symbolically as

$$R(A, \partial A) \overset{\text{df}}{=} \square A + U(A, \partial A) = \mathscr{J} \tag{6}$$

Here $U(A, \partial A)$ is the sum of all non-linear terms; it is obviously an analytic function (polynomial of degree 3) of A and its derivatives ∂A. The characteristics of this system are the ordinary light-cones.

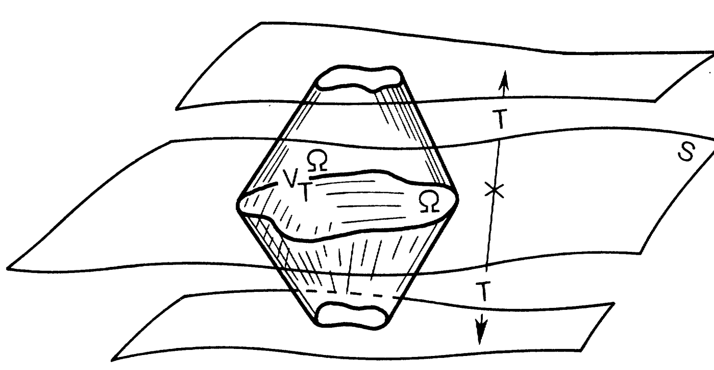

FIG. 1.

Let us now choose some appropriate functional spaces in order to define correctly a continuous mapping R. Let S be a space-like hypersurface in M_4, and Ω an open subset of S, which is supposed to be sufficiently regular. Let V_T^Ω be a subset of M_4 defined as follows: $V_T^\Omega :=$ the union of the domain of dependence and the domain of influence of Ω, intersected with the slice of M_4 cut out by the two hypersurfaces parallel to S at the distance T on both sides of S (see the fig. 1).

Define $H^m(V_T^\Omega)$ as a Hilbert space of the functions $f \in L^2(V_T^\Omega)$ such that

$$\| f \|_{H^m(V_T^\Omega)} = \sum_{|j| \leq m} \| \partial^j f \|_{L^2(V_T^\Omega)} < \infty \tag{7}$$

In the same way we define $H^m(\Omega)$ as the space of such functions $f = f(\vec{x})$, $\vec{x} \in \Omega$, that

$$\| f \|_{H^m(\Omega)} = \sum_{|j| \leq m} \| \partial^j f \|_{L^2(\Omega)} < \infty \tag{8}$$

Now we shall take the Cartesian product of these spaces corresponding to the number of components of the A_μ^a, i. e. 4N times, but we shall maintain the same notation for the sake of brevity.

We make the following hypotheses concerning the functions $A_\mu^a(x)$ and their initial values on the hypersurface S:

HYPOTHESIS 1. — The solutions of (6) we are searching for will be contained in the subspace of $H^{m+1}(V_T^\Omega)$ such that: $f \in H^{m+1}(V_T^\Omega)$, $\Box f \in H^m(V_T^\Omega)$. Let us denote this space by E; the obvious norm in E is

$$\| f \|_E = \| f \|_{H^{m+1}(V_T^\Omega)} + \| \Box f \|_{H^m(V_T^\Omega)}$$

HYPOTHESIS 2. — The initial values $A_\mu^a |_S$ and $\partial_\nu A_\mu^a |_S$ will be contained in the functional space $H^m(\Omega)$.

HYPOTHESIS 3. — $H^m(V_T^\Omega)$ is a Banach algebra. This will be true (cf. Dionne [*5*]) if $m > n/2$, n being the dimension of V_T^Ω. In our case we have the condition $m > 2$. With this condition satisfied $U = U(A, \partial A)$ will be a continuous operator from $H^{m+1}(V_T^\Omega)$ into $H^m(V_T^\Omega)$.

We can always state correctly the Cauchy problem for the d'Alembert equation if the solutions are supposed to be in an appropriate space. As a matter of fact, it has been proved by Leray [*2*] that for the Cauchy data $A|_\Omega = \Phi_1 \in H^m(\Omega)$ and $\partial A|_\Omega = \Phi_2 \in H^{m-1}(\Omega)$ there exists a unique solution of the equation $\Box A = J$ such that

$$A = A(\Phi, J) = G_\Phi(J) \tag{9}$$

where G_Φ is a linear operator containing the Green's function and its derivatives, and satisfying the conditions:

$$[G_\Phi(\mathscr{J})]\,|_\Omega = \Phi_1, \qquad [\partial(G_\Phi(\mathscr{J}))]\,|_\Omega = \Phi_2 \tag{10}$$

Here we denote by Φ the pair of entities Φ_1 and Φ_2.

Because of the regularizing properties of G_Φ, this mapping is continuous from $H^m(V_T^\Omega)$ into $H^{m+1}(V_T^\Omega)$, and it is also obvious that $\square$ is a continuous mapping from the space E, defined in the Hypothesis 1, into $H^m(V_T^\Omega)$.

3. Let us suppose now that $\mathring{A}$ is an exact solution of the system (4), i. e. that

$$R(\mathring{A}) \equiv \square\mathring{A} + U(\mathring{A}, \partial\mathring{A}) = 0 \tag{11}$$

and that $\mathring{A} \in H^{m+1}(V_T^\Omega)$, $\square\mathring{A} \in H^m(V_T^\Omega)$, with $m > 2$.

Let us also denote

$$\mathring{A}\,|_\Omega = \mathring{\Phi}_1, \qquad \partial\mathring{A}\,|_\Omega = \mathring{\Phi}_2 \tag{12}$$

where $\partial\mathring{A}$ means the set of all the first derivatives of $\mathring{A}$.

We can say then that $\mathring{A}$ is a solution of the Cauchy problem for the system (4) with the initial values $\mathring{\Phi}$ on Ω, $\mathring{\Phi}$ meaning the pair $(\mathring{\Phi}_1, \mathring{\Phi}_2)$.

Now we define the mapping P in the following way:

$$A \rightarrow P(A) := \begin{pmatrix} R(A) \\ A\,|_\Omega \\ \partial A\,|_\Omega \end{pmatrix} \tag{13}$$

P is obviously a differentiable mapping from the space E into

$$H^m(V_T^\Omega) \times H^m(\Omega) \times H^{m-1}(\Omega).$$

At the point $A = \mathring{A}$ this mapping takes on the value $(0, \mathring{\Phi}_1, \mathring{\Phi}_2)$. We can apply the inverse mapping theorem in the neighborhood of $A = \mathring{A}$ if we are able to prove that at this point the differential of P is an isomorphism *onto*. The differential is:

$$\delta_A P(X) := \begin{pmatrix} \delta_A R(X) \\ X\,|_\Omega \\ \partial X\,|_\Omega \end{pmatrix} \tag{14}$$

where

$$(\delta_A R)_{A=\mathring{A}}(X) = \square X + \left(\frac{\delta U}{\delta A}\right)_{A=\mathring{A}} X + \left(\frac{\delta U}{\delta(\partial A)}\right)_{A=\mathring{A}} \partial X \tag{15}$$

The mapping $\delta_A P : E \rightarrow H^m(V_T^\Omega) \times H^m(\Omega) \times H^{m-1}(\Omega)$ is a topological isomorphism *onto* by virtue of the existence and uniqueness theorems established for such linear hyperbolic systems by Leray and Dionne. Therefore, we can state the following

THEOREM . — For any exact solution of the Yang-Mills field equations, defined on V_T^Ω and belonging to E, there exists a neighborhood of the Cauchy data $\overset{\circ}{\Phi}$ of this solution, $\| \Phi - \overset{\circ}{\Phi} \|_{H^m(\Omega)} \leqslant \delta$ such that for any Φ contained within this neighborhood there exists one and only one exact solution A of the system (4), contained in a neighborhood of $\overset{\circ}{A}$:

$$\| A - \overset{\circ}{A} \|_E \leqslant \varepsilon \tag{15}$$

This local statement can be generalized for the case $\Omega = S$ because of the validity of a corresponding existence and uniqueness theorem by Dionne, which applies to $\Omega = S$ also V_T^Ω will be then a strip $S \times [-T, T]$ of arbitrary (but finite) temporal width.

The theorem, applied to the simplest case of $\overset{\circ}{A} = 0$ means that there exists infinitely many « weak field » solutions for any finite time. The existence of the solutions in the neighbourhood of other exact solutions will of course depend on the properties of the later ones. We will only mention the fact that most of the static trivial solutions of the electromagnetic type (cf. [3] [4]) do have the required properties, whereas the usual wave solutions, e. g. plane or spherical waves, do not verify our hypotheses.

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