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Section A :
Physique théorique.

On the general relativistic magnetohydrodynamics with a generalized thermodynamical differential equation (*)

by

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SUMMARY. — The system of perfect general relativistic magnetohydrodynamics with a generalized thermodynamical differential equation is examined; the discontinuities, characteristic equations, associated rays, velocities of propagation and suitable hypotesis of compressibility are determined. The exceptional waves are pointed out.

RÉSUMÉ. — On examine le système de la magnétohydrodynamique relativiste avec une équation thermodynamique généralisée dans le cas d'un fluide idéal de conductivité infinie. On étudie les discontinuités en déterminant les équations des caractéristiques, les rayons associés, les vitesses de propagation et on donne des hypothèses de compressibilité. En outre on remarque les ondes exceptionnelles.

1. INTRODUCTION

As is well known, the general relativistic magnetohydrodynamics, in the case of a perfect fluid with an infinite conductivity, is constructed on the basis of the following equations:

(I) the equations of conservation for the energy tensor;

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(II) the Maxwell equations;
(III) the state equation;
(IV) a thermodynamical differential equation.

Usually, in (I) the energy tensor is assumed as the sum of the dynamic energy tensor of the fluid and of the energy tensor of the electromagnetic field; in (III) the fluid is assumed to be adiabatic, or, equivalently, that the proper material density (number of particles) is conserved; in (IV) the thermodynamical differential equation used in classical hydrodynamics is assumed to be valid in a proper frame.

The consequences of this scheme, which later-on will be named « usual scheme », have been exstensively discussed as far as the structure of fundamental differential system, the existence and unicity of solutions, the discontinuities, the characteristic equations, the hypotesis of compressibility and other espects are concerned. The bibliography is very large and here we recall only a fundamental work of Y. Choquet-Bruhat [*1*] and a recent exhaustive Lichnerowicz's monography [*2*].

Now, as observed by Lichnerowicz [*3*], the above scheme is only a first approximation. As a matter of fact, the energy tensor and the thermodynamical differential equation adopted do not take account of a possible influence of the electromagnetic field on the internal structure of the fluid. Recently, a more comprehensive scheme, which accounts for an interaction between the fluid and the electromagnetic field, have been proposed by Maugin [*4*], who has deduced its system from an action principle, making use of a thermodynamical differential equation proposed by Fokker [*5*] in 1939.

In this paper, after stating the notations at the end of this section, in section 2 the fundamental system obtained in [*4*] will be recalled together with some useful consequences. In sections 3, 4, 5, and 6, following the outline given by Lichnerowicz [*6*], infinitesimal discontinuities, characteristic equations, possible exceptionality of corresponding waves and associated rays will be discussed. In section 7, finally, velocities of propagation and reasonable hypotesis of compressibility will be determined. The research carried out leads to the following facts:

a) the well known first condition of compressibility $\gamma > 1$ ($\gamma \equiv c^2 f r'_p$) is again sufficient to ensure the spatial orientation of the waves, but in the expression of γ the index of the fluid is replaced by the socalled modified index, which accounts of the considered interaction between the fluid and the electromagnetic field;

b) the derivative of the proper material density with respect a new thermodynamical variable, necessary to describe the interaction, cannot take arbitrary values: there is a condition of integrability which bounds its variation;

c) of course all velocities of propagation, except for the entropy waves, are changed with respect to the usual scheme;

d) one has, only formally, the same expression for the velocities, that are yelded by the usual scheme, with the substitution of the index of the fluid with the modified index. The interaction is again effective and reflects also on the form of the energy tensor;

e) the entropy waves and the Alfven waves remain exceptional without any restrictive hypotesis, as in both the non relativistic and the usual relativistic cases;

f) the usual scheme is obtained in the particular case of vanishing magnetization, e. g. $\mu = 1$.

NOTATIONS. — The space-time is a four dimensional manifold V^4, whose normal hyperbolic metric ds^2, with signature $+ - - -$, is expressible in local coordinates in the usual form $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$; the metric tensor is assumed to be given of class C^1, piecewise C^2; the four-velocity is defined as $u^\alpha = dx^\alpha/ds$, which implies its unitary character ($u^\alpha u_\alpha = 1$); ∇_α is the operator of covariant differentiation with respect to the given metric. These notations are different from those used by Maugin, but our formulas are converted in its notations by simply making the substitutions

$$u^\alpha \to -\frac{i}{c}u^\alpha, \quad h^\alpha \to -ih^\alpha, \quad \pi_{\alpha\beta} \to -i\pi_{\alpha\beta}, \quad F^{\alpha\beta} \to -iF^{\alpha\beta}, \quad i = \sqrt{-1},$$

and changing all the signs.

2. FIELD EQUATIONS

As said in introduction, Maugin has deduced the system of general relativistic magnetohydrodynamics from an action principle; in its deduction he used the following thermodynamical differential equation

$$(1) \qquad d\varepsilon = TdS - pd\frac{1}{r} + \frac{1}{2}\frac{\pi_{\alpha\beta}}{r}dF^{\alpha\beta}$$

proposed by Fokker in 1939 for a polarized-magnetized fluid.

In eq. (1) ε is the relativistic internal energy, which partly accounts also for interaction between the fluid and the electromagnetic field, T and S are the proper temperature of the fluid and its specific entropy respectively, p is the thermodynamical pressure, r the proper material density (number of particles), $\pi_{\alpha\beta}$ is the polarization-magnetization two-form and $F^{\alpha\beta}$ is electric field-magnetic induction two-form.

In the case of infinite conductivity, $\pi_{\alpha\beta}$ and $F^{\alpha\beta}$ are expressible as

$$(a) \qquad \pi_{\alpha\beta} = E_{\alpha\beta\gamma\delta}m^\gamma u^\delta \qquad F^{\alpha\beta} = E^{\alpha\beta\gamma\delta}b_\gamma u_\delta$$

in which $E_{\alpha\beta\gamma\delta} = \sqrt{-g}\, e_{\alpha\beta\gamma\delta}$, $E^{\alpha\beta\gamma\delta} = -\dfrac{1}{\sqrt{-g}} e_{\alpha\beta\gamma\delta}$, $e_{\alpha\beta\gamma\delta}$ being the Levi-Civita's alternation symbol; b_α is the magnetic induction and $m_\alpha = b_\alpha - h_\alpha$ is the magnetization, h_α being the magnetic field. As is well known, b_α and h_α are given by [2]:

$$b_\alpha = \frac{1}{2} E_{\beta\alpha\gamma\delta} F^{\gamma\delta} u^\beta \qquad h_\alpha = \frac{1}{2} E_{\beta\alpha\gamma\delta} G^{\gamma\delta} u^\beta ,$$

$G^{\gamma\delta}$ being the electric induction-magnetic field two-form.

Of course one has

$$(b) \qquad h_\alpha u^\alpha = b_\alpha u^\alpha = m_\alpha u^\alpha = 0 .$$

The relations (a) are a direct consequence of the general decomposition of $\pi_{\alpha\beta}$ and $F^{\alpha\beta}$ given by Grot-Eringen [7], and of the hypotesis of infinite conductivity.

Supposing that the magnetic induction depend linearly on the field and that the fluid is homogeneous and magnetically isotropic, we have

$$(c) \qquad b_\alpha = \mu h_\alpha \qquad \mu = \text{const.} \qquad m_\alpha = (\mu - 1) h_\alpha ,$$

and eq.(1), taking into account the relations (a), (b) and (c), becomes

$$(2) \qquad d\varepsilon = \mathrm{T}d\mathrm{S} - pd\frac{1}{r} + \frac{1}{r}\frac{\mu(1-\mu)}{2} dh^2 ,$$

with $h^2 \equiv - h^\alpha h_\alpha$. Introducing now the specific magneto-entalpy and the modified index of the fluid, which, in the case considerate, are respectively given by (*)

$$i = \varepsilon + \frac{p}{r} - \frac{\mu(1-\mu)}{r} h^2$$

$$\tilde{f} = 1 + \frac{i}{c^2} + \frac{\mu(1-\mu)}{c^2 r} h^2 ,$$

and taking into account eq. (2), without making, as Maugin has done, a restrictive hypotesis on the functional dependence of ε by S, r and h^2, we find:

$$(3) \qquad c^2 d\tilde{f} = \mathrm{T}d\mathrm{S} + \frac{1}{r} dp + \frac{1}{r}\frac{\mu(1-\mu)}{2} dh^2 .$$

In these conditions the energy tensor takes the form:

$$(4) \qquad \mathrm{T}^{\alpha\beta} = (c^2 r \tilde{f} + \mu h^2) u^\alpha u^\beta - \left[p + \mu\left(1 - \frac{\mu}{2}\right) h^2 \right] g^{\alpha\beta} - \mu h^\alpha h^\beta ,$$

(*) The expressions of i and $\tilde{f}$ given here follow from the formulas (59) and (111) of [4] taking into account the relations (a), (b) and (c).

and the field equations are: the conservation equations for the energy tensor

$$\nabla_\alpha T^{\alpha\beta} = 0, \tag{5}$$

the conservation equation of the proper material density

$$\nabla_\alpha (r u^\alpha) = 0 \tag{6}$$

and the Maxwell equations, which, in this case, are

$$\nabla_\alpha (u^\alpha h^\beta - u^\beta h^\alpha) = 0. \tag{7}$$

At this stage, contracting eq. (7) first with u_β and then with h_β, by virtue of relations (*b*), (*c*) and unitary character of u^β, we have:

$$u^\alpha u^\beta \nabla_\alpha h_\beta - \nabla_\alpha h^\alpha = 0 \tag{8}$$

$$\frac{1}{2} u^\alpha \partial_\alpha h^2 + h^2 \nabla_\alpha u^\alpha - h^\alpha u_\beta \nabla_\alpha h^\beta = 0 \qquad (\partial_\alpha \equiv \partial/\partial x^\alpha). \tag{9}$$

Eq. (5), with $T^{\alpha\beta}$ given by (4), becomes

$$(c^2 r\tilde{f} + \mu h^2) u^\alpha \nabla_\alpha u^\beta + u^\beta \nabla_\alpha [(c^2 r\tilde{f} + \mu h^2) u^\alpha] - g^{\alpha\beta} \partial_\alpha \left[p + \mu \left(1 - \frac{\mu}{2} \right) h^2 \right] - \mu h^\alpha \nabla_\alpha h^\beta - \mu h^\beta \nabla_\alpha h^\alpha = 0.$$

Contracting this with u_β and taking into account eq. (9), we obtain the so-called continuity equation:

$$u^\alpha \partial_\alpha p - \nabla_\alpha (c^2 r\tilde{f} u^\alpha) + \frac{\mu(1-\mu)}{2} u^\alpha \partial_\alpha h^2 = 0. \tag{10}$$

Utilizing now eq. (3) we find:

$$r T u^\alpha \partial_\alpha S + c^2 \tilde{f} \nabla_\alpha (r u^\alpha) = 0.$$

So, eq. (6) implies that the flow is locally adiabatic:

$$u^\alpha \partial_\alpha S = 0. \tag{11}$$

This same result is obtained by Lichnerowicz [*6*] in a different scheme. We observe that Maugin assumes eqs. (6) and (11) as independent constraints on the motion of the fluid. Here the one is a consequence of the other and of the remaining field equations.

After this, taking into account eqs. (9) and (10), we write eq. (5) in the form

$$(c^2 r\tilde{f} + \mu h^2) u^\alpha \nabla_\alpha u^\beta - (g^{\alpha\beta} - u^\alpha u^\beta) \partial_\alpha \left[p + \mu \left(1 - \frac{\mu}{2} \right) h^2 \right] + \mu u^\alpha u^\beta h^\sigma \nabla_\sigma h_\alpha - \mu h^\alpha \nabla_\alpha h^\beta - \mu h^\beta \nabla_\alpha h^\alpha = 0, \tag{12}$$

and, contracting this with h_β, by virtue of eqs. (*b*) and (8), we obtain

$$c^2 r\tilde{f} \nabla_\alpha h^\alpha + h^\alpha \partial_\alpha \left[p + \frac{\mu(1-\mu)}{2} h^2 \right] = 0. \tag{13}$$

3. DISCONTINUITIES

We now suppose that p, S, u^α and h^α are of class C^0, piecewise C^1; that the discontinuities of their first derivatives can take place across an hypersurface Σ of local equation $\varphi(x^\alpha) = 0$, φ of class C^2; that these discontinuities are well determined as differences of the limiting values, of the derivatives of p, S, u^α and h^α, obtained approaching the same point of Σ by the two sides in which Σ divide V^4; that these limiting values are tensor-functions defined on Σ and that the above derivatives are uniformly convergent to these functions, when one tends to the points of Σ on either side. In these hypotesis [6], introducing the operator of infinitesimal discontinuity δ, we study in which conditions the tensor-distributions δp, δS, δu^α and δh^α, supported with regularity by Σ, are not simultaneusly zero. At the same time we obtain the differential equations (i. e. the characteristic equations) which may be satisfied by the functions $\varphi(x^\alpha)$. For this it is sufficient to make the replacement $\nabla_\alpha, \partial_\alpha \to \varphi_\alpha\delta$ ($\varphi_\alpha \equiv \partial\varphi/\partial x^\alpha$) in the differential equations obtained in section 2. So, from eq. (11) we have:

$$\mathrm{U}\delta\mathrm{S} = 0 \qquad (\mathrm{U} \equiv \varphi_\alpha u^\alpha) \tag{14}$$

and from eqs. (8), (9):

$$\mathrm{U}u^\alpha\delta h_\alpha - \varphi_\alpha\delta h^\alpha = 0 \tag{15}$$

$$\frac{1}{2}\mathrm{U}\delta h^2 + h^2\varphi_\alpha\delta u^\alpha - \mathrm{H}u^\alpha\delta h_\alpha = 0 \qquad (\mathrm{H} \equiv \varphi_\alpha h^\alpha) \tag{16}$$

respectively. Eqs. (12) and (13) give

$$\begin{aligned}(c^2r\tilde{f} + \mu h^2)\mathrm{U}\delta u^\alpha - (g^{\alpha\beta} - u^\alpha u^\beta)\varphi_\beta\delta\left[p + \mu\left(1 - \frac{\mu}{2}\right)h^2\right] \\ + \mu u^\alpha\mathrm{H}u^\beta\delta h_\beta - \mu\mathrm{H}\delta h^\alpha - \mu h^\alpha\varphi_\beta\delta h^\beta = 0\end{aligned} \tag{17}$$

and

$$c^2r\tilde{f}\varphi_\alpha\delta h^\alpha + \mathrm{H}\delta\left[p + \frac{\mu(1-\mu)}{2}h^2\right] = 0, \tag{18}$$

while, from eqs. (6) and (7) we deduce:

$$\mathrm{U}\delta r + r\varphi_\alpha\delta u^\alpha = 0 \tag{19}$$

and

$$\mathrm{U}\delta h^\alpha + h^\alpha\varphi_\beta\delta u^\beta - u^\alpha\varphi_\beta\delta h^\beta - \mathrm{H}\delta u^\alpha = 0. \tag{20}$$

4. ENTROPY WAVES

From eq. (14) we see that δS may be different from zero if $U = 0$. In this case we deduce from eqs. (15)-(20) $\delta p = \delta h^2 = \delta u^\alpha = \delta h^\alpha = 0$, from eq. (3) $c^2\delta\tilde{f} = T\delta S$, and, assuming that S, p and h^2 are the independent thermodynamical variables, we have $\delta r = r'_S \delta S$, where the prime denotes partial differentiation with respect to the subscripted variable.

We have so the so-called entropy waves or material waves, and, since $\varphi_\alpha \delta u^\alpha = 0$ is a consequence of $U = 0$, as we can see e. g. from eq. (19), are exceptional waves [8], [9]. The associated rays are the trajectories on Σ of u^α. There are not differences between these results and those obtained in the usual scheme; in order words the considered interaction between the fluid and the electromagnetic field do not affect the behaviour of these waves.

5. HYDRODYNAMICAL WAVES

Excluding now the above case, i. e. supposing $U \neq 0$, we have from eq. (14) $\delta S = 0$. So, from eqs. (15) and (16) we obtain

$$U^2 \delta h^2 = 2(H\varphi_\alpha \delta h^\alpha - h^2 U \varphi_\alpha \delta u^\alpha). \tag{21}$$

Contracting eq. (17) with φ_α, and taking into account eq. (15), we find

$$(c^2 r\tilde{f} + \mu h^2) U \varphi_\alpha \delta u^\alpha - (G - U^2)\delta p - \mu H \varphi_\alpha \delta h^\alpha - (G - U^2)\mu\left(1 - \frac{\mu}{2}\right)\delta h^2 = 0,$$

in which $G = g^{\alpha\beta}\varphi_\alpha \varphi_\beta$, whereas from eq. (19), being $\delta S = 0$, we have

$$r\varphi_\alpha \delta u^\alpha + U r'_p \delta p + U r'_{h^2} \delta h^2 = 0.$$

Substituting δh^2 from eq. (21) in the last two equations and in eq. (18), we obtain the following linear homogeneous system in the distributions $\varphi_\alpha \delta u^\alpha$, δp and $\varphi_\alpha \delta h^\alpha$:

$$\begin{cases} \left\{ \frac{1}{2}[c^2 r\tilde{f} - \mu(1-\mu)h^2]U^2 + \mu\left(1 - \frac{\mu}{2}\right)h^2 G \right\} U\varphi_\alpha \delta u^\alpha - \frac{1}{2}(G - U^2)U^2 \delta p \\ \qquad + \left[\frac{\mu(1-\mu)}{2}U^2 - \mu\left(1 - \frac{\mu}{2}\right)G\right] H\varphi_\alpha \delta h^\alpha = 0 \\ (r - 2h^2 r'_{h^2}) U\varphi_\alpha \delta u^\alpha + U r'_p \delta p + 2r'_{h^2} H \varphi_\alpha \delta h^\alpha = 0 \\ \mu(1-\mu)h^2 U H \varphi_\alpha \delta u^\alpha - U^2 H \delta p - [c^2 r\tilde{f} U^2 + \mu(1-\mu)H^2]\varphi_\alpha \delta h^\alpha = 0. \end{cases}$$

It follows that the distributions $\varphi_\alpha \delta u^\alpha, \delta p$ and $\varphi_\alpha \delta h^\alpha$ can be different from zero only if the determinant

$$\begin{vmatrix} \mathcal{A} & -\frac{1}{2}(G - U^2) & \mathcal{B} \\ r - 2h^2 r'_{h^2} & r'_p & 2r'_{h^2} H \\ \mu(1-\mu)h^2 H & -H & -[c^2 r\tilde{f} U^2 + \mu(1-\mu)H^2] \end{vmatrix},$$

$$\mathcal{A} = \frac{1}{2}[c^2 r\tilde{f} - \mu(1-\mu)h^2]U^2 + \mu\left(1 - \frac{\mu}{2}\right)h^2 G, \quad \mathcal{B} = \left[\frac{\mu(1-\mu)}{2}U^2 - \mu\left(1 - \frac{\mu}{2}\right)G\right]H$$

is zero. In this case δr, $\delta \tilde{f}$, δp, δh^2, $\varphi_\alpha \delta u^\alpha$ and $\varphi_\alpha \delta h^\alpha$ are all known in terms of one of them, e. g., δp.

Expanding the determinant we find

$$\mathrm{N}_4 \equiv c^2 r\tilde{f}(\gamma - 1)\mathrm{U}^4 + (c^2 r\tilde{f} + \mu\gamma h^2)\mathrm{U}^2\mathrm{G} - \mu\mathrm{H}^2\mathrm{G} + \eta[h^2(\mathrm{U}^4 - \mathrm{U}^2\mathrm{G}) - \mathrm{U}^2\mathrm{H}^2]$$

in which

$$(22) \qquad \gamma = c^2 \tilde{f} r'_p, \qquad \eta = \Gamma - \mu(1-\mu)\gamma, \qquad \Gamma = 2c^2 \tilde{f} r'_{h^2}.$$

Now, from eq. (3), we have

$$c^2 \tilde{f}'_p = \frac{1}{r}, \qquad 2c^2 \tilde{f}'_{h^2} = \mu(1-\mu)\frac{1}{r}$$

and the consequent condition of integrability of eq. (3) gives

$$2r'_{h^2} = \mu(1-\mu)r'_p \rightarrow \eta = 0 .$$

Therefore, vanishing the last term of N_4, the differential equation of the hydrodynamical waves is:

$$\mathrm{N}_4 = [c^2 r\tilde{f}(\gamma - 1)]\mathrm{U}^4 + (c^2 r\tilde{f} + \mu\gamma h^2)\mathrm{U}^2\mathrm{G} - \mu\mathrm{H}^2\mathrm{G} = 0 .$$

The associated rays are the trajectories of the vector field

$$\mathcal{N}^\alpha = \frac{1}{2}\frac{\partial \mathrm{N}_4}{\partial \varphi_\alpha} = \rho u^\alpha + \sigma\varphi^\alpha + \tau h^\alpha$$

with

$$\rho = 2c^2 r\tilde{f}(\gamma - 1)\mathrm{U}^3 + (c^2 r\tilde{f} + \mu\gamma h^2)\mathrm{U}\mathrm{G}, \qquad \sigma = (c^2 r\tilde{f} + \mu\gamma h^2)\mathrm{U}^2 - \mu\mathrm{H}^2 ,$$
$$\tau = -\mu\mathrm{G}\mathrm{H} .$$

As $\mathcal{N}^\alpha$ is tangent to the hypersurface Σ of local equation $\varphi(x^\alpha) = 0$ if $\varphi(x^\alpha)$ satisfies $\mathrm{N}_4 = 0$, introducing the components of u^α and h^α tangential to Σ, defined respectively by

$$(23) \qquad v^\alpha = u^\alpha - \frac{\mathrm{U}}{\mathrm{G}}\varphi^\alpha, \qquad t^\alpha = h^\alpha - \frac{\mathrm{H}}{\mathrm{G}}\varphi^\alpha ,$$

we can express $\mathcal{N}^\alpha$ as a combination of v^α and t^α. In fact, replacing in $\mathcal{N}^\alpha u^\alpha$ and h^α given by (23), taking account of $\mathrm{N}_4 = 0$, we find

$$\mathcal{N}^\alpha = \rho v^\alpha + \tau t^\alpha$$

Concluding this section we remark that $\mathrm{N}_4 = 0$ is only formally reduced to the corrisponding equation obtained in the usual scheme. The interaction is again effective and is evidenced by the presence of the modified index. This fact is not at all evident *a priori*, because the energy tensor is given by (4) also being $\eta = 0$, and therefore remains different from the energy tensor usually adopted. This latter is obtained only in the case $\mu = 1$, which does not account for interaction; in fact in this case the magnetization m_α is zero.

6. ALFVEN WAVES

We now suppose $U \neq 0$, $N_4 \neq 0$, and we look if there are hypersurfaces Σ on which the distributions δv^α an δt^α, given by the formulas (23):

$$\delta v^\alpha = \delta u^\alpha - \frac{\varphi_\beta \delta u^\beta}{G} \varphi^\alpha \qquad \delta t^\alpha = \delta h^\alpha - \frac{\varphi_\beta \delta h^\beta}{G} \varphi^\alpha$$

can be different from zero. As $N_4 \neq 0$, $\varphi_\alpha \delta u^\alpha = \varphi_\alpha \delta h^\alpha = \delta p = \delta h^2 = 0$, and from eqs. (17) and (20) we deduce:

$$\begin{cases} (c^2 r\tilde{f} + \mu h^2) U \delta v^\alpha - \mu H \delta t^\alpha = 0 \\ H \delta v^\alpha - U \delta t^\alpha = 0 \end{cases}$$

Therefore δv^α and δt^α can be different from zero only if supported by the hypersurfaces Σ of local equation $\varphi(x^\alpha) = 0$, with $\varphi(x^\alpha)$ solution of

$$N_2 \equiv (c^2 r\tilde{f} + \mu h^2) U^2 - \mu H^2 = 0.$$

We have the so-called Alfven waves and it is possible to prouve, with the same arguments used in [9], that are exceptional waves. Moreover, as N_2 split in two factors: $N_2 \equiv (A^\alpha \varphi_\alpha)(B^\beta \varphi_\beta)$, $A^\alpha = \omega u^\alpha + h^\alpha$, $B^\beta = \omega u^\beta - h^\beta$ with $\omega = [(c^2 r\tilde{f} + \mu h^2)\mu^{-1}]^{\frac{1}{2}}$, we have two types of Alfven waves as in the usual scheme. The respective associated rays are the trajectories of the vector field A^α and B^β, which, being $A^\alpha A_\alpha = B^\beta B_\beta = (c^2 r\tilde{f})\mu^{-1} > 0$, are time-like vectors.

7. VELOCITIES OF PROPAGATION

We recall that, given a regular hypersurface Σ, of local equation $\varphi(x^\alpha) = 0$, its velocity of propagation V, with respect the four-velocity field, is defined by

$$X = \frac{V^2}{c^2} = \frac{U^2}{U^2 - G},$$

and, if Σ is spatial-like, i. e. $G < 0$, it is $X < 1$ (i. e. $V^2 < c^2$).

After this, we first consider the Alfven waves, $N_2 = 0$, and we introduce the useful parameter

$$h_n^2 = \frac{H^2}{U^2 - G} \qquad h_n^2 \leqslant h^2.$$

We have:

$$N_2 = (c^2 r\tilde{f} + \mu h^2) U^2 - h_n^2 (U^2 - G) = 0.$$

So, the velocity of propagation of Alfven waves is given by

$$\frac{V_A^2}{c^2} = \frac{\mu h_n^2}{c^2 r\tilde{f} + \mu h^2}, \qquad V_A^2 < c^2.$$

We consider now the hydrodynamical waves. Introducing h_n^2 in $N_4 = 0$, we obtain

$$c^2 r\tilde{f}(\gamma - 1)U^4 + (c^2 r\tilde{f} + \mu\gamma h^2 - \mu h_n^2)U^2 G + \mu h_n^2 G^2 = 0,$$

which, after moultiplication by $(1 - X)^2/G^2$, can be expressed in terms of X as

$$N_4(X) = (c^2 r\tilde{f} + \mu h^2)\gamma X^2 - (c^2 r\tilde{f} + \mu\gamma h^2 + \mu h_n^2)X + \mu h_n^2 = 0.$$

The values of $N_4(X)$ for $X = 0$, $X = \dfrac{V_A^2}{c^2}$, $X = 1$, are

$$\begin{aligned} N_4(0) &= \mu h_n^2 \\ N_4\left(\frac{V_A^2}{c^2}\right) &= \frac{V_A^2}{c^2}\mu(h_n^2 - h^2)(\gamma - 1) \\ N_4(1) &= c^2 r\tilde{f}(\gamma - 1). \end{aligned}$$

It follows that the condition

$$(24) \qquad \gamma > 1$$

is sufficient for

$$N_4(0) \geqslant 0, \qquad N_4\left(\frac{V_A^2}{c^2}\right) \leqslant 0, \qquad N_4(1) > 0.$$

Therefore, if condition (24) is supposed to be verified, $N_4(X)$ has generally two zeros, $X = V_S^2/c^2$ and $X = V_F^2/c^2$, between 0 and 1, separated by V_A^2/c^2. In this manner we retrouve the slow and the fast hydrodynamical waves, and, to ensure their spatial orientation, as said in introduction, the well known first compressibility condition $\gamma > 1$ is again sufficient with the only replacement, in the expression of γ, of the index of the fluid with the modified index.

Finally we consider the hydrodynamical waves in the following two limiting cases.

First: h^α is orthogonal to the spatial direction of propagation of the waves, i. e. $H = 0$, $h_n^2 = 0$. $N_4(X) = 0$ is then reduced to

$$(c^2 r\tilde{f} + \mu h^2)\gamma X^2 - (c^2 r\tilde{f} + \mu\gamma h^2)X = 0,$$

so that $V_S = 0$ and

$$\frac{V_F^2}{c^2} = \frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\frac{\mu h^2}{c^2 r\tilde{f} + \mu h^2}.$$

Second: h^α is along the spatial direction of propagation of the waves, i. e. $h_n^2 = h^2$. In this case $N_4(X) = 0$ becomes:

$$(c^2 r\tilde{f} + \mu h^2)\gamma X^2 - [c^2 r\tilde{f} + \mu h^2(\gamma + 1)]X + \mu h^2 = 0.$$

We have in this way the two solutions $X = V_A^2/c^2$ and $X = 1/\gamma$, the smaller of which gives V_S, the greater V_F, and the hydrodynamical waves whose velocity of propagation is V_A are also Alfven waves.

Concluding we observe that, as said in introduction, except for the entropy waves, all velocities of propagation are changed with respect to the usual scheme, but the essential features of spatial orientation of the waves and the progressive disposition of the velocities, $O \leqslant V_S^2 \leqslant V_A^2 \leqslant V_F^2 \leqslant c^2$, are respected if condition (24) is imposed. Moreover the entropy waves and the Alfven are exceptional without any restrictive hypotesis. We recall that this fact is also true both in the non relativistic case and in the relativistic usual scheme.

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