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Spectral properties of one-body relativistic spin-zero hamiltonians


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Spectral properties of one-body relativistic spin-zero hamiltonians (1)

by

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ABSTRACT. — We study the spectral properties of the relativistic spin-zero Hamiltonian \( H = \sqrt{p^2 + \mu^2} + V \) of a spinless particle, by an extension of the method of Aguilar-Combes, for a class of interactions including \( V = -gr^{-\beta}, 0 < \beta < 1 \).

Absence of singular-continuous spectrum is proved, together with the existence of an absolutely-continuous spectrum \( [\mu, \infty) \). In \( \mathbb{R} \setminus \{ \mu \} \) the point spectrum consists of finite-dimensional eigenvalues which are bounded. Properties of resonances are investigated.

RÉSUMÉ. — Nous étudions les propriétés de l’Hamiltonien relativiste de spin zero : \( H = \sqrt{p^2 + \mu^2} + V \) d’une particule sans spin, grâce à une extension de la méthode d’Aguilar-Combes, pour la classe d’interactions comprenant : \( V = -gr^{-\beta}, 0 < \beta < 1 \).

L’absence de spectre singulièrement continu est prouvée, en même temps que l’existence d’un spectre absolument continu \( [\mu, \infty) \). Dans \( \mathbb{R} \setminus \{ \mu \} \) le spectre ponctuel est formé de valeurs propres de dimension finie qui sont bornées.

Les propriétés des résonances sont analysées.

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INTRODUCTION

Recently [1] [2], spectral properties of Schrödinger operators with interactions satisfying analyticity conditions with respect to the dilatation group were studied.

In this work we investigate the spectral properties of the relativistic spin-zero Hamiltonian \( H = \sqrt{p^2 + \mu^2} + V \) of a spinless particle by an extension of the method of [1], for a class of interactions including \( V = - gr^{-\beta} \), \( 0 < \beta < 1 \).

In Section I we study the analyticity properties, with respect to the dilatation group, of the free Hamiltonian \( H_0 = \sqrt{p^2 + \mu^2} \).

In Section II we define the class of interactions, we are considering, and prove that the interactions \( V = - gr^{-\beta} \), \( 0 < \beta < 1 \) are allowed.

In Section III we show that the singular-continuous spectrum is empty together with the existence of an absolutely-continuous spectrum \([\mu, \infty)\). We also show that the point spectrum consists of a bounded set of finite-dimensional eigenvalues different from \( \mu \) (accumulating, at most, at \( \mu \)), and possibly an eigenvalue at \( \mu \). Properties of resonances are also investigated.

We stress the difference with the non-relativistic case, namely the essential-spectrum of the analytic extension of the Hamiltonian is not a straight-line (see fig. 1) but a part of an hyperbola starting at \( \mu \). The basic new technical results for the relativistic case are contained in lemmas 1, 2 and 3. The method of the proof of lemma 4 and theorem 1 were taken from [1] [2] and [6]. For the definitions of vector-valued, and operator-valued analytic functions, and for the classification of the spectrum we refer to T. Kato [3].

In this paper we use the same notation as in [2].

I. THE FREE HAMILTONIAN

Let, the space of wavefunctions of a spinless particle, be the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^3) \) of square-integrable functions in \( \mathbb{R}^3 \). Let \( \mu \), the particle mass, be a strictly positive constant, and let \( \omega(p) = \sqrt{p^2 + \mu^2} \). We define the free Hamiltonian in momentum space by

\[
(H_0 \Psi)(\vec{p}) = \omega(p)\Psi(\vec{p}) ,
\]
on the domain, \( \mathcal{D}(H_0) \), of all \( \Psi \) in \( \mathcal{H} \) such that \( (\omega(p)\Psi(\vec{p})) \) is again in \( \mathcal{H} \). We take, of course, the square root with positive sign. \( H_0 \) is a positive, and selfadjoint operator [3].
Let $U(\Theta)$, $\Theta \in \mathbb{R}$, be the strongly-continuous unitary representation on $\mathcal{H}$ of the dilatation group defined by

$$(U(\Theta)\Psi)(\vec{p}) = e^{-\frac{3\Theta}{2}}\Psi(e^{-\theta}\vec{p}), \quad \Psi \in \mathcal{H}, \quad \Theta \in \mathbb{R}.$$  

Thus, we have

$$(H_0(\Theta)\Psi)(\vec{p}) = (U(\Theta)H_0U(-\Theta)\Psi)(\vec{p}) = \omega(\Theta, p)\Psi(\vec{p}),$$

where

$$\omega(\Theta, p) = \sqrt{e^{-2\Theta}p^2 + \mu^2}, \quad \Theta \in \mathbb{R}.$$  

**Lemma 1.** The family of operators $H_0(\Theta)$, $\Theta \in \mathbb{R}$, can be extended to an analytic family in the strip of the complex-plane

$$S_\frac{\pi}{2} = \left\{ \Theta \in \mathbb{C} \mid \text{Im} \Theta < \frac{\pi}{2} \right\}.$$  

**Proof.** We can write

$$\omega(\Theta, p) = \rho(\Theta, p)e^{i\phi(\Theta, p)},$$

with $\phi(\Theta, p)$, the argument; and a modulus $\rho(\Theta, p) > 0$ for all $p \in [0, \infty)$ and all $\Theta \in S_\frac{\pi}{2}$. There exist two real, positive, and bounded functions of $\Theta(M_1(\Theta)$, and $M_2(\Theta)$) such that

$$0 < \frac{\rho(0, p)}{\rho(\Theta, p)} < M_1(\Theta) < \infty,$$

$$0 < \frac{\rho(\Theta, p)}{\rho(0, p)} < M_2(\Theta) < \infty, \quad p \in [0, \infty), \quad \Theta \in S_\frac{\pi}{2}.$$  

Thus

$$\|H_0\Psi\| \leq M_1(\Theta)\|H_0(\Theta)\Psi\| \quad \Psi \in \mathcal{D}(H_0(\Theta)),$$

$$\|H_0(\Theta)\Psi\| \leq M_2(\Theta)\|H_0\Psi\| \quad \Psi \in \mathcal{D}(H_0)$$

That is to say

$$\mathcal{D}(H_0) = \mathcal{D}(H_0(\Theta)).$$

By a trivial argument, which we omit, we can show that

$$|\omega(\Theta_1, p) + \omega(\Theta_2, p)| \geq \rho(\Theta_1, p) (\cos \text{Im} \Theta_1) > 0, \quad \Theta_1 \Theta_2 \in S_\frac{\pi}{2}.$$  

Then, we have the following estimation

$$\left| \frac{\omega(\Theta_2, p) - \omega(\Theta_1, p)}{\Theta_2 - \Theta_1} \right| \leq 2\left(\frac{\varepsilon + e^{-2R_0\Theta_1}}{\cos \text{Im} \Theta_1}\right)M_1(\Theta_1)\rho(0, p),$$

for $\Theta_1, \Theta_2 \in S_\frac{\pi}{2}$, $|\Theta_2 - \Theta_1| < \eta(\Theta_1)$; $\varepsilon > 0$ and $\eta(\Theta_1) > 0$.  

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Then, as \( \omega(\Theta, p) \) is an analytic function of \( \Theta \) for all \( \Theta \in S_{\frac{\pi}{2}} \), by the Lebesgue's dominated convergence theorem we have

\[
\left( \frac{H_0(\Theta_2) - H_0(\Theta_1)}{\Theta_2 - \Theta_1} \Psi \right)_{\Theta_2 \to \Theta_1} \left( \frac{d}{d\Theta} \omega(\Theta, p) \right)_{\Theta = \Theta_1} \Psi(p), \quad \Psi \in \mathcal{D}(H_0),
\]

in the strong topology in \( \mathcal{H} \).

\textbf{Q. E. D.}

\textbf{Lemma 2.} — The spectrum of \( H_0(\Theta) \), denoted \( \sigma(H_0(\Theta)) \), is a continuous curve, starting at \( \mu \), and tending asymptotically to the straightline \( e^{-i\text{Im} \Theta \cdot \mathbb{R}^+} \) (see fig. 1).

\begin{figure}[h]
\centering
\includegraphics{Fig1.png}
\caption{Fig. 1.}
\end{figure}

\textit{Proof.} — It is an immediate consequence of the definition of the spectrum that

\[
\sigma(H_0(\Theta)) = \{ Z \in \mathbb{C} \mid Z = \sqrt{e^{-2\Theta}p^2 + \mu^2}, \quad p \in [0, \infty) \}
\]

We can write

\[ e^{-2\Theta}p^2 + \mu^2 = \rho(\Theta, p)e^{i\phi(\Theta, p)}, \]
where $\rho(\Theta, p)$ and $\phi(\Theta, p)$ are respectively the modulus and the argument of $e^{-2\Theta}p^2 + \mu^2$.

For $0 \leq \text{Im} \Theta < \frac{\pi}{2}$, $\phi(\Theta, p)$ is a strictly decreasing function of $p$ and $-2 \text{Im} \Theta < \phi(\Theta, p) \leq \phi(\Theta, 0) = 0$, $\rho(\Theta, p)$ is a strictly increasing function of $p$, bounded below by $\mu$, for $0 \leq \text{Im} \Theta \leq \frac{\pi}{4}$; and is a convex-function, with a minimum at $\pi / 2$.

Thus, the spectrum of $H_0(\Theta)$ is, for $0 \leq \text{Im} \Theta < \frac{\pi}{2}$, a continuous curve starting and tending asymptotically to the straight-line $e^{-i\text{Im} \Theta}R^+$.

The proof for $-\frac{\pi}{2} < \text{Im} \Theta \leq 0$ is similar.

Q. E. D.

Remark. — The spectrum of $H_0(\Theta)$ is independent of $\Re(\Theta)$ because $H_0(\Theta_1)$ and $H_0(\Theta_2)$ are unitary equivalents for $\text{Im} \Theta_1 = \text{Im} \Theta_2$.

II. THE CLASS OF DILATATION ANALYTIC INTERACTIONS

We define a dilatation analytic interaction [1] as a symmetric and $H_0$-compact operator [3] $V$ having the following property: the family of operators

$$V(\Theta) = U(\Theta)VU(-\Theta), \quad \Theta \in \mathbb{R},$$

has an $H_0$-compact analytic continuation in an open connected domain $0$ of the complex-plane ($^4$).

We consider, now, the total Hamiltonian

$$H = H_0 + V,$$

where $V$ is a dilatation analytic interaction in the strip $S_a$, $0 < a < \frac{\pi}{2}$.

($^4$) It is clear that the analyticity domain $0$ of $V(\Theta)$ can always be extended to a complex strip $S_a = \{ Z \in \mathbb{C} | |\text{Im} Z| < a \}, a > 0 [1]$. 

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As V is symmetric and \( H_0 \)-compact, \( H \) is selfadjoint, bounded below, and \( \mathcal{D}(H) = \mathcal{D}(H_0) \) [3].

By lemma 1, and the definition of dilatation analytic interactions, the family of operators \( H(\Theta) \), defined for \( \Theta \in \mathbb{R} \) by
\[
H(\Theta) = U(\Theta) H U(-\Theta), \quad \Theta \in \mathbb{R},
\]
has an extension to an analytic and selfadjoint family [3], with
\[
\mathcal{D}(H(\Theta)) = \mathcal{D}(H_0),
\]
in the strip \( S_b \), where \( b = \min \left( a, \frac{\pi}{2} \right) \).

**Lemma 3.** — The multiplication operator (denoted \( V \)) by the function \( -gr^{-\beta}, 0 < \beta < 1 \), where \( g \) is a constant, is dilatation analytic in the entire complex-plane (5).

**Proof.** — \( V \) is a symmetric operator [3]; and
\[
U(\Theta)(-gr^{-\beta})U(-\Theta) = -g(\Theta)r^{-\beta};
\]
where
\[
g(\Theta) = ge^{-\beta\Theta}, \quad \Theta \in \mathbb{R};
\]
which has an analytic extension to the entire complex-plane. Let us define the following operator
\[
(V_n(\Theta)\Psi)(\vec{r}) = (-g(\Theta)r^{-\beta})_n\Psi(\vec{r}),
\]
where
\[
(-g(\Theta)r^{-\beta})_n = \begin{cases} 
-g(\Theta)r^{-\beta}, & r < n \\
0, & r > n,
\end{cases}
\]
(\( n \) is a entire positive number) in the domain \( \mathcal{D}(V_n(\Theta)) \), of all \( \Psi \in \mathcal{H} \), such that \( [(-g(\Theta)r^{-\beta})_n\Psi(\vec{r})] \) is again in \( \mathcal{H} \).
\[
(\sqrt{p^2 + \mu^2} + i)^{-1} \in \mathcal{L}^{3+\frac{\alpha}{2}}(\mathbb{R}^3).
\]
Then \( V_n(\Theta) \) is \( H_0 \)-compact by a theorem of [4].

(5) Clearly, \( V \) is defined as a multiplication operator in configuration-space that is to say
\[
(V\Psi)(r) = -gr^{-\beta}\Psi(\vec{r}),
\]
on the domain, \( \mathcal{D}(V) \); of all \( \Psi \) in \( \mathcal{H} \) such that \( (-gr^{-\beta}\Psi(\vec{r})) \) is again in \( \mathcal{H} \). \( \Psi(\vec{r}) \) is the Fourier transform of the wave-function in momentum space \( \hat{\Psi}(\vec{p}) \). In momentum space \( V \) is an integral operator.

(6) \( \mathcal{L}^{3+\frac{\alpha}{2}}(\mathbb{R}^3) \) is the Banach space of complex valued functions on \( \mathbb{R}^3 \), such that
\[
\int |\hat{\Psi}(\vec{p})|^{3+\alpha}d^3\vec{p} < \infty.
\]
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But as
\[ ||(V(\Theta) - V_{n}(\Theta))(H_{0} + i)^{-1}\Psi|| \leq \frac{|g(\Theta)|}{\eta^{\theta/2}} ||(H_{0} + i)^{-1}|| \ ||\Psi||, \]

\[ V_{n}(\Theta)(H_{0} + i)^{-1} \text{ converges in norm to } V(\Theta)(H_{0} + i)^{-1}. \]

Q. E. D.

We will now study the spectrum of the operators \( H(\Theta) \), \( \Theta \in S_{\theta}/\Re \), and then make the transition to real \( \Theta \). We note that the remark following lemma 2 is also valid for \( H(\Theta) \).

III. SPECTRAL PROPERTIES OF \( H = H_{0} + V \)

Lemma 4. — The spectrum of \( H(\Theta) \) with \( 0 < |\text{Im } \Theta| < b \) consists of:

Essential spectrum: \( \sigma_{e}(H(\Theta)) = \sigma(H_{0}(\Theta)) \).

Real bound state energies (\( \sigma_{b}(H(\Theta)) \)): a bounded set of isolated, finite-dimensional, real-eigenvalues, independent of \( \Theta \), with \( \mu \) as the only possible accumulation point.

Non-real resonance energies (\( \sigma_{r}(H(\Theta)) \)): a bounded set of non-real isolated, finite-dimensional eigenvalues, contained in the sector of the complex-plane bounded by \( \{ \mu, \infty \} \) and \( \sigma_{c}(H(\Theta)) \). The only possible accumulation point is \( \mu \). A given resonance energy is independent of \( \Theta \) as long as it belongs to \( \sigma_{r}(H(\Theta)) \cap \sigma_{c}(H(\Theta)) \).

For \( |\phi| > |\text{Im } \Theta| \) there exist \( C(\phi) > 0 \) such that for \( 0 \leq \rho < \infty \),
\[ ||(H(\Theta) - \lambda_{0} + 1 - \rho e^{i\phi})^{-1}|| \leq C(\phi)\rho^{-1}, \]
where \( \lambda_{0} \) is the minimum of the spectrum (which is independent of \( \Theta \)).

Proof. — By the second resolvent equation [3]
\[ (H(\Theta) - Z)^{-1} = (H_{0}(\Theta) - Z)^{-1}(1 + V(\Theta)(H_{0}(\Theta) - Z)^{-1})^{-1}, \]
for all \( Z \in \mathbb{C}\setminus\sigma(H_{0}(\Theta)) \) such that
\[ (1 + V(\Theta)(H_{0}(\Theta) - Z)^{-1})^{-1} \]
even exists.

But, as \( V(\Theta) \) is \( H_{0}(\Theta) \)-compact and \( ||V(\Theta)(H_{0}(\Theta) + x)^{-1}|| < 1 \) for real \( x \) and \( x > K > 0 \), this holds for all \( Z \in \mathbb{C}\setminus\sigma(H_{0}(\Theta)) \), except for, at most, a set \( S \) of isolated points [3].

Let, for \( \lambda \in S, P_{\lambda} \) be the projection operator defined by [3]
\[ P_{\lambda} = -\frac{1}{2\pi i} \int_{\Gamma} (H_{0}(\Theta) - Z)^{-1}dZ + \frac{1}{2\pi i} \int_{\Gamma} (H(\Theta) - Z)^{-1}V(\Theta)(H_{0}(\Theta) - Z)^{-1}dZ, \]
where \( \Gamma \) is a circle separating \( \lambda \) from \( \sigma(H(\Theta)) \) — \( \{ \lambda \} \).

The first integrand is holomorphic in \( Z = \lambda \), and the second compact; hence \( P_{\lambda} \) is a compact operator.

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Then, $\lambda$ is an isolated finite-dimensional eigenvalue of $H(\Theta)$ [3]. By exchanging the roles of $H_0(\Theta)$ and $H(\Theta)$ we can prove that

$$\sigma_d(H(\Theta)) = \sigma(H_0(\Theta)).$$

Take us $\lambda_0$ in the set $\sigma_d(H(\Theta_0))$ of isolated finite-dimensional eigenvalues of $H(\Theta_0)$. As $H(\Theta)$ is an analytic family with spectrum constant for $\text{Im } \Theta$ constant, $\lambda_0 \in \sigma_d(H(\Theta))$ for $\Theta$ in a neighborhood of $\Theta_0$ [3]. Thus the isolated finite-dimensional eigenvalues of $H(\Theta)$ can accumulate only at $\mu$; and the real bound-state energies are independent of $\Theta$, and the non-real resonance energies are independent of $\Theta$ as long as they belong to $\sigma_d(H(\Theta))) \cap \sigma_d(H(\Theta_0))$, and are contained in the sector of the complex-plane bounded by $[\mu, \infty)$ and $\sigma(\Theta).

The fact that the set $\sigma_d(H(\Theta))$ is bounded (and the validity of the estimation for the resolvent given above) can be proven in the same lines as in [2], then we will omit the proof here.

Q. E. D.

**Theorem 1.** — The point spectrum of $H$ consists of a bounded set of finite-dimensional eigenvalues different from $\mu$ (which are precisely the real eigenvalues of $H(\Theta)$ $\text{Im } \Theta \neq 0$, different from $\mu$) accumulating at most at $\mu$, and possibly an eigenvalue at $\mu$. The projection operators $P(\Theta, \lambda)$, $\Theta \in S_\theta$, on the eigenspace of $H(\Theta)$ corresponding to a fixed-real-eigenvalue $\lambda$ different from $\mu$ form a selfadjoint analytic family in $S_\theta$.

The eigenvectors $\Phi$ of $H$ corresponding to such eigenvalues ($\lambda$) are in the dense set $D_\theta$ of analytic vectors [5] in $S_\theta$, and the analytic extensions $\Phi(\Theta)$ of $\Phi$ are eigenvectors of $H(\Theta)$ corresponding to $\lambda$. The singular-continuous spectrum is empty, i.e.

$$\mathcal{H} = \mathcal{H}_{a.c.} \oplus \mathcal{H}_p,$$

and $\sigma_{a.c.} = [\mu, \infty)$.

**Proof.** — Let $\Phi$ and $\Psi$, be fixed vectors in the dense set $D_\theta$; and $\Phi(\Theta)$ and $\Psi(\Theta)$ their analytic extensions.

By lemma 4 the function

$$F_{\Phi, \Psi}(\Theta, Z) = (\Phi(\Theta), (H(\Theta) - Z)^{-1}\Psi(\Theta)),$$

is analytic in $\Theta$, for fixed $Z$ such that $\text{Im } Z > 0$ and such that

$$- \arg Z \leq \text{Im } \Theta < b.$$

Since

$$F_{\Phi, \Psi}(\Theta, Z) = (\Phi, (H - Z)^{-1}\Psi) \quad \text{for } \Theta \in \mathbb{R},$$

it follows that the equality holds for all $\Theta$ with

$$- \arg Z \leq \text{Im } \Theta < b \quad \text{and } \quad \text{Im } Z > 0.$$
tion from above (Im $Z > 0$) across the line $[\mu, \infty)$ up to the curve $\sigma(H_\lambda(\Theta))$. Let us denote by $E(\lambda)$ the spectral family of $H[3]$, then we have that

$$\langle \Phi, (E_\lambda - E_{\lambda-0})\Psi \rangle = \lim_{Z \to \lambda} \frac{F_{\Phi, \Psi}(\Theta, Z)}{Z - \lambda},$$

where

$$\mathcal{C}_{\lambda, \omega}^+ = \left\{ Z \in \mathbb{C} \mid \text{Im } Z > 0, \omega \leq \arg (Z - \lambda) \leq \pi - \omega, 0 < \omega < \frac{\pi}{2} \right\}.$$

This implies (together with a similar result for Im $\Theta < 0$ and Im $Z < 0$) that the eigenvalues of $H$, different from $\mu$, are precisely the real eigenvalues of $H(\Theta)$, Im $\Theta \neq 0$, different from $\mu$; and that the real poles of $(H(\Theta) - Z)^{-1}$, Im $\Theta \neq 0$, different from $\mu$, are simple. Then, the point spectrum of $H$ is bounded and accumulates, at most, at $\mu$.

Let, $P^z(\Theta, \lambda)$, be the projection operator [3] on the eigenspace of $H(\Theta)$ corresponding to an eigenvalue, $\lambda$, different from $\mu$ ($+$, $-$ corresponds to Im $\Theta > 0$ and Im $\Theta < 0$ respectively).

Setting $P(\lambda) = E_\lambda - E_{\lambda-0}$, we obtain as above:

$$\langle \Phi, P^z(\Theta, \lambda)\Psi \rangle = \langle \Phi(-\Theta), P(\lambda)\Psi(-\Theta) \rangle,$$

where we have used the fact that $\lambda$ is a simple pole.

Then, the function $f_{\Phi, \Psi}(\Theta)$ defined for $\Phi, \Psi \in \mathcal{D}_b$ by

$$f_{\Phi, \Psi}(\Theta) = \begin{cases} 
\langle \Phi, P^z(\Theta, \lambda)\Psi \rangle, & \text{Im } \Theta \neq 0 \\
\langle \Phi, P(\Theta, \lambda)\Psi \rangle, & \text{Im } \Theta = 0 
\end{cases},$$

is analytic in $S_b$, where $P(\Theta, \lambda) = U(\Theta)P(\lambda)U(-\Theta)$ is the projection operator on the eigenspace of $H(\Theta)$ corresponding to the eigenvalue $\lambda$ for $\Theta \in \mathbb{R}$.

Now, it is not difficult to show [1] the fact that the family $P(\Theta, \lambda), \Theta \in \mathbb{R}$, has an analytic extension in $S_b$ which equals $P^+(\Theta, \lambda)$ (resp. $P^-(\Theta, \lambda)$) for Im $\Theta > 0$ (resp. Im $\Theta < 0$).

This implies that the eigenvalues of $H$, different from $\mu$, are finite-dimensional.

By standard arguments [1] it is possible to show that the eigenvectors $\Phi$ of $H$ corresponding to such eigenvalues, $\lambda$, are analytic vectors in $S_b$, and that their analytic extensions $\Phi(\Theta)$ are eigenvectors of $H(\Theta)$ with the same eigenvalue $\lambda$.

Let $\Delta = (a, b)$, be an interval in $(\mu, \infty)$ which contains no-eigenvalue of $H$, then for Im $\Theta > 0$ and $\Phi \in \mathcal{D}_b$ [3]

$$\langle \Phi, E_\lambda \Phi \rangle = \frac{1}{2\pi i} \int_a^b \{ \langle \Phi(\Theta), (H(\Theta) - \lambda)^{-1} \Phi(\Theta) \rangle - \langle \Phi(\Theta), (H(\Theta) - \lambda)^{-1} \Phi(\Theta) \rangle \} \, d\lambda,$$

where $E_\Delta = E_b - E_a$.

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The integrand being analytic, the function \((\Phi, E_x \Phi)\) is absolutely continuous on \(\Delta\); since \(\mathcal{D}_b\) is dense in \(\mathcal{H}\) we obtain
\[
\mathcal{H}_{s.c.} = \emptyset,
\]
that is to say
\[
\mathcal{H} = \mathcal{H}_{a.c.} \oplus \mathcal{H}_p,
\]
and also \(\sigma_{a.c.} = [\mu, \infty)\).

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