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Section A :
Physique théorique.

Algebra of observables and quantum logic

by

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SUMMARY. — An algebraic formalism for classical and quantum mechanics is developed and observables, superobservables, states and complete sets of observables are investigated by applying the theory of von Neumann algebras. A characterization of the pure states is given and a representation of the algebra of observables is introduced in correspondence with any state. The conditions for the faithfulness and the irreducibility of such a representation are found. If an assumption for the quantum case is made, which amounts to « the possibility of neglecting the superselection rules », then the present algebraic picture can be easily drawn from the formalism of quantum logic.

INTRODUCTION

The interest in the algebraic approach to quantum mechanics has been increasing since the original proposal of Segal [*1*]. This approach has been mainly developed in both statistical mechanics and quantum field theory. The point of view we adopt here is to assume that the « algebra of observables » is a von Neumann algebra (W*-algebra) [*2*], as we want to develop a picture well supported by the logic approach to the foundations of quantum mechanics. The set of projections of a W*-algebra is in fact a « quantum logic » [*3*] and moreover it has been shown that « a concrete

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C*-algebra with identity satisfies Ludwig's axiom of sensitivity increase of effects [4] if and only if it is a W*-algebra » [5].

We have attempted to make clear what are the assumptions of our scheme, with particular care for the physical interpretation of the terms introduced. In our construction in fact the axioms and the rules of interpretation are introduced explicitly, while the rest of the picture is worked out in the form of theorems. Among the results which we obtain, the following topics are of particular interest. After defining the concept of representation of the algebra of observables induced by a state in a framework which embodies the superselection rules in a natural way, we are able to introduce a physically sensible axiom which amounts to « the possibility of neglecting the superselection rules » which is usually assumed in elementary quantum mechanics. From this axiom it follows that only discrete superselection rules can be present. In connection with this problem we notice that a somewhat general discussion about superselection rules has been performed by Guenin [6] and Piron [7] in the framework of the quantum logic approach. An extensive and thorough discussion of the concept of pure state is carried out, one of the results being that we are able to show that the pure states of our approach are in fact the pure states of the logic approach. A classical pure state results for instance to be the characteristic function of a point of the phase space of the system. Finally, we show that our algebraic description can be thought of to be a representation of a quantum logic description if only a suitable assumption about superobservables is made.

After collecting some mathematical tools in section 1, we introduce in section 2 the algebra of observables, the states, the superobservables and the concept of complete set of commuting observables. In section 3 we discuss pure states and in section 4 we find how a representation of the algebra of observables can be induced by a state. In section 5 we show that the picture of the foregoing sections can be drawn from the quantum logic approach.

1. MATHEMATICAL TOOLS

In this section we collect all the results concerning the theory of W*-algebras we need for the development of the following sections. Most of these results are very well known and we refer to the classical Dixmier's book [8] for the proof (the first edition will be denoted by D. A. and the second one by D. B.). Only for some original results about maximal abelian W*-algebras we will add the proof.

Let Λ be a Hausdorff locally compact space and $\mathbb{B}$ the σ-algebra generated by the topology of Λ. We shall call $\mathbb{B}$ the Borel σ-algebra of Λ and its elements Borel sets. A positive measure μ on $\mathbb{B}$ will be referred to simply as measure on Λ. A subset A of Λ is said to be μ-negligible if it is contained in an element B of $\mathbb{B}$ such that $\mu(B) = 0$. The σ-algebra $\mathbb{B}_\mu$ is defined to be the family of the subsets A of Λ which can be written as $A = A_1 \cup A_2$, with $A_1 \in \mathbb{B}$ and A_2 μ-negligible. It can be always assumed that $A_1 \cap A_2 = \emptyset$. The σ-algebra $\mathbb{B}_\mu$ can be equivalently defined to be the family of the subsets A of Λ for which two subsets A′ and A″ in $\mathbb{B}$ exist such that $A' \subset A \subset A''$ and $\mu(A'' - A') = 0$. The σ-algebra $\mathbb{B}_\mu$ is called μ-completion of $\mathbb{B}$. A function $f : \Lambda \to \mathbb{C}$ is called μ-measurable if $f^{-1}(\Delta) \in \mathbb{B}_\mu$, $\forall \Delta \in \mathbb{B}_\mathbb{C}$, where $\mathbb{B}_\mathbb{C}$ is the Borel σ-algebra of the complex field $\mathbb{C}$. A property is said to hold μ-a. e. on a set $A \in \mathbb{B}$ if a μ-negligible subset B of A exists such that the property holds for every point of $A - B$.

If $\mathscr{A}$ is an abelian C*-algebra of operators on the Hilbert space $\mathscr{H}$, a Hausdorff locally compact space Λ and an isomorphism S from the C*-algebra $L_\infty(\Lambda)$ onto $\mathscr{A}$ exist, where $L_\infty(\Lambda)$ is the involutive algebra of the continuous complex functions on Λ vanishing at infinity, equipped with the norm of the supremum (S is in fact the Gel'fand-Naimark isomorphism). Moreover a projection valued measure P from the Borel σ-algebra $\mathbb{B}$ of Λ into the set $\mathscr{P}(\mathscr{H})$ of the projections of $\mathscr{H}$ exists such that

$$(x, Sfy) = \int_\Lambda f d\nu_{x,y}, \qquad \forall f \in L_\infty(\Lambda), \qquad \forall x, y \in \mathscr{H},$$

where $\nu_{x,y}$ is the complex measure on $\mathbb{B}$ defined as

$$\nu_{x,y}(E) = (x, P(E)y), \qquad \forall E \in \mathbb{B}.$$

A measure ν on Λ is called basic if, for $E \in \mathbb{B}$,

$$\nu(E) = 0 \Leftrightarrow \nu_{x,x}(E) = 0, \qquad \forall x \in \mathscr{H}.$$

We notice that, if ν is a basic measure, then the support of ν (supp ν) is Λ, a measure μ on Λ equivalent to ν ($\mu \simeq \nu$) is basic and for any $x \in \mathscr{H}$ the measure $\nu_{x,x}$ is absolutely continuous with respect to ν ($\nu_{x,x} \ll \nu$).

Remark 1.1. — For an abelian C*-algebra $\mathscr{A}$ of operators on the Hilbert space $\mathscr{H}$ a bounded basic measure ν exists iff a vector $x \in \mathscr{H}$ exists which is cyclic for the commutant $\mathscr{A}'$ of $\mathscr{A}$. If this is the case we have $\nu \simeq \nu_{x,x}$ (D. B., I, § 7, prop. 3 and 2). When $\mathscr{H}$ is separable a cyclic vector for $\mathscr{A}'$ always exists (D. B., I, § 2, cor. at p. 19 and § 1, cor. at p. 6).

DEFINITION 1.1. — Let $\mathscr{H}$ be a direct integral of Hilbert spaces,

$$\mathscr{H} = \int_\Lambda^\oplus \mathscr{H}(\lambda) d\nu(\lambda),$$

where Λ is a Hausdorff locally compact space and ν a measure on Λ, and

$\mathscr{L}(\mathscr{H})$ the algebra of the bounded operators on $\mathscr{H}$ (D. B., II, § 1). With $D_c(\mathscr{H})$, $D(\mathscr{H})$, $R(\mathscr{H})$ we denote the sets of continuously diagonal, diagonal, decomposable operators of $\mathscr{L}(\mathscr{H})$ respectively. Namely (D. B., II, § 2)

$$D_c(\mathscr{H}) = \left\{ \int_\Lambda^\oplus f(\lambda) \mathbb{1}_\lambda d\nu(\lambda) \,;\, f \in L_\infty(\Lambda) \right\},$$

$$D(\mathscr{H}) = \left\{ \int_\Lambda^\oplus f(\lambda) \mathbb{1}_\lambda d\nu(\lambda) \,;\, f \in L^\infty(\Lambda, \nu) \right\},$$

where the C*-algebra $L^\infty(\Lambda, \nu)$ is the involutive algebra of the equivalence classes of the ν-essentially bounded complex functions on Λ equipped with the norm of the essential supremum,

$$R(\mathscr{H}) = \left\{ \int_\Lambda^\oplus A(\lambda) d\nu(\lambda);\ \lambda \to A(\lambda) \ \begin{array}{c}\text{essentially bounded and measurable}\\ \text{field of operators}\end{array} \right\}.$$

PROPOSITION 1.1. — *Let* $D_c(\mathscr{H})$, $D(\mathscr{H})$ *and* $R(\mathscr{H})$ *be as in definition* 1.1. *Then we have:*

(a) $D(\mathscr{H})$ *is an abelian* W*-*algebra and it is the weak closure of* $D_c(\mathscr{H})$, $R(\mathscr{H})$ *is a* W*-*algebra and* $R(\mathscr{H}) = D_c(\mathscr{H})' = D(\mathscr{H})'$;

(b) *if supp* $\nu = \Lambda$ *and* $\mathscr{H}(\lambda) \neq \{0\}$ ν-a. e., *then* $D_c(\mathscr{H})$ *is a* C*-*algebra which is isomorphic to* $L_\infty(\Lambda)$ *and the* W*-*algebra* $D(\mathscr{H})$ *is isomorphic to* $L^\infty(\Lambda, \nu)$. *The measure* ν *is basic for* $D_c(\mathscr{H})$.

Proof:

(*a*) D. B., II, § 2, proposition 7 and corollary at page 164.

(*b*) D. B., II, § 2, remark at page 162 and proposition 6.

Remark 1.2. — From the very definition of $R(\mathscr{H})$ we have (D. B., II, § 2, prop. 5; II, § 3, def. 1 and def. 2)

$$R(\mathscr{H}) = \int_\Lambda^\oplus \mathscr{L}(\mathscr{H}(\lambda)) d\nu(\lambda)$$

and also (D. B., II, § 3, th. 2)

$$R(\mathscr{H}) = W^* \{ D(\mathscr{H}), T_1, T_2, \dots \},$$

namely $R(\mathscr{H})$ is a W*-algebra generated by $D(\mathscr{H})$ and by a sequence $\{ T_n \}$ of decomposable operators.

If in proposition 1.1 we take $\mathscr{H} = L^2(\Lambda, \nu)$, then $R(\mathscr{H}) = D(\mathscr{H})$.

PROPOSITION 1.2. — *Let* $\mathscr{A}$ *be an abelian* W*-*algebra of operators on the Hilbert space* $\mathscr{H}$ *such that* $\mathscr{A}' = W^* \{ \mathscr{A}, T_1, T_2, \dots \}$. *Then there exist:*

(a) *a Hausdorff locally compact space* Λ *and a measure* ν *on* Λ *such that* supp $\nu = \Lambda$;

(b) *a Hilbert space* $\hat{\mathscr{H}} = \int_\Lambda^\oplus \mathscr{H}(\lambda)d\nu(\lambda)$, *with* $\mathscr{H}(\lambda) \neq \{0\}$ ν-a. e.;

(c) *an isomorphism* U *from* $\mathscr{H}$ *onto* $\hat{\mathscr{H}}$ *such that the restriction to* $\mathscr{A}$ *of the isomorphism* Φ_U *of* $\mathscr{L}(\mathscr{H})$ *with* $\mathscr{L}(\hat{\mathscr{H}})$, *defined as*

$$\Phi_U : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\hat{\mathscr{H}}), \qquad \Phi_U(A) = UAU^{-1},$$

is an isomorphism of $\mathscr{A}$ *with* $D(\hat{\mathscr{H}})$.

Proof. — D. A., II, paragraph 6, theorem 2.

The decomposition of $\mathscr{A}$ which is furnished by proposition 1.2 is essentially unique in the sense of D. A., II, paragraph 6, theorem 4.

Remark 1.3. — If in proposition 1.2 the Hilbert space $\mathscr{H}$ is separable, it can be shown that a selfadjoint operator $B \in \mathscr{L}(\mathscr{H})$ exists such that Λ can be assumed to be the spectrum Sp B of B (hence Λ is a topological subspace of the real line $\mathbb{R}$ contained in a finite interval $[m, M]$) and the measure ν to be defined as $\nu(\Delta) = (x, P(\Delta)x)$ for any Borel set Δ of $\mathbb{R}$, where x is a cyclic vector for $\mathscr{A}'$ and P the spectral measure of B.

From proposition 1.2 we have that $\mathscr{A} \sim D(\hat{\mathscr{H}})$, where « $\sim$ » means isomorphism. Hence from proposition 1.1 it follows that $\mathscr{A} \sim L^\infty(\Lambda, \nu)$. Since in this isomorphism the operator B corresponds to the identity function on Sp B, we can say that all the elements of $\mathscr{A}$ are functions of B.

The condition that $\mathscr{A}'$ is generated by $\mathscr{A}$ and by a sequence of operators is not necessary in order to prove proposition 1.2 when $\mathscr{H}$ is separable. A proof of proposition 1.2 in the case of a separable Hilbert space can be found also in [9].

DEFINITION 1.2. — Let $\mathscr{A}$ be a W*-algebra. An abelian sub-W*-algebra of $\mathscr{A}$ is called maximal in $\mathscr{A}$ when it is maximal with respect to the order relation which is defined by inclusion on the set of the abelian sub-W*-algebras of $\mathscr{A}$. When $\mathscr{A} = \mathscr{L}(\mathscr{H})$, then a maximal abelian sub-W*-algebra of $\mathscr{A}$ will be called simply maximal abelian.

Remark 1.4. — A sub-W*-algebra $\mathscr{B}$ of a W*-algebra $\mathscr{A}$ is maximal abelian in $\mathscr{A}$ iff $\mathscr{B}' \cap \mathscr{A} = \mathscr{B}$ (D. B., I, § 1, prop. 13). Hence a W*-algebra $\mathscr{B}$ is maximal abelian iff $\mathscr{B} = \mathscr{B}'$.

If $\mathscr{A}$ is an abelian W*-algebra, then a maximal abelian W*-algebra $\mathscr{B}$ exists such that $\mathscr{A} \subset \mathscr{B}$ (D. B., I, § 1, prop. 12).

PROPOSITION 1.3. — *Let* $\mathscr{A}$ *be an abelian* W*-*algebra of operators on the Hilbert space* $\mathscr{H}$. *The following are equivalent conditions:*

(a) $\mathscr{A}$ *is maximal, namely* $\mathscr{A} = \mathscr{A}'$;

(b) *the Hilbert space* $\hat{\mathscr{H}}$ *which is constructed in proposition* 1.2 *to decompose* $\mathscr{A}$ *is* $L^2(\Lambda, \nu)$, *namely* $\mathscr{H}(\lambda) = \mathbb{C}$, $\forall \lambda \in \Lambda$.

Proof. — Since Φ_U is an isomorphism, the condition $\mathscr{A} = \mathscr{A}'$ is equivalent to $D(\hat{\mathcal{H}}) = D(\hat{\mathcal{H}})'$. The result then follows from D. B., II, paragraph 2, example 1 and from the essential unicity of the decomposition furnished by proposition 1.2.

PROPOSITION 1.4. — *Let $\mathcal{A}$ be a* W*-*algebra of operators on the Hilbert space $\mathcal{H}$ and $\mathcal{C}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$ its center. The following are equivalent conditions:*

(a) *$\mathcal{C}(\mathcal{A})$ is a maximal abelian sub-*W*-*algebra of $\mathcal{A}'$;*

(b) *$\mathcal{C}(\mathcal{A}) = \mathcal{A}'$;*

(c) *$\mathcal{A}' \subset \mathcal{A}$, namely the commutant of $\mathcal{A}$ is abelian;*

(d) *for each selfadjoint element* A *of $\mathcal{A}$ a maximal abelian* W*-*algebra $\mathcal{B}$ exists which contains* A *and is contained in $\mathcal{A}$, namely $\mathcal{B} = \mathcal{B}'$, $A \in \mathcal{B}$, $\mathcal{B} \subset \mathcal{A}$;*

(e) *a maximal abelian* W*-*algebra $\mathcal{B}$ exists which is contained in $\mathcal{A}$, namely $\mathcal{B} = \mathcal{B}'$, $\mathcal{B} \subset \mathcal{A}$.*

Proof:

$(a) \Leftrightarrow (b)$: from remark 1.4 it follows that $\mathcal{C}(\mathcal{A})$ is a maximal abelian sub-W*-algebra of $\mathcal{A}'$ iff $\mathcal{C}(\mathcal{A})' \cap \mathcal{A}' = \mathcal{C}(\mathcal{A})$. Moreover

$$\mathcal{C}(\mathcal{A})' = (\mathcal{A} \cap \mathcal{A}')' = W^* \{ \mathcal{A}, \mathcal{A}' \}$$

(D. B., I, § 1, prop. 1). Hence we get

$$\mathcal{C}(\mathcal{A})' \cap \mathcal{A}' = \mathcal{C}(\mathcal{A}) \quad \text{iff} \quad W^* \{ \mathcal{A}, \mathcal{A}' \} \cap \mathcal{A}' = \mathcal{C}(\mathcal{A}) \quad \text{iff} \quad \mathcal{A}' = \mathcal{C}(\mathcal{A});$$

$(b) \Rightarrow (c)$: trivial;

$(c) \Rightarrow (d)$: setting $\mathcal{A}_A = W^* \{ \mathcal{A}', A \}$, we have that $\mathcal{A}_A$ is an abelian W*-algebra since $\{ \mathcal{A}', A \}$ is a selfadjoint subset of $\mathcal{L}(\mathcal{H})$ consisting of normal operators which mutually commute. From remark 1.4 it follows that a maximal abelian W*-algebra $\mathcal{B}$ exists such that $\mathcal{A}_A \subset \mathcal{B}$. Hence we have $A \in \mathcal{B}$ and $\mathcal{A}' \subset \mathcal{B}$, which implies $\mathcal{B} \subset \mathcal{A}$;

$(d) \Rightarrow (e)$: trivial, since the unit element of $\mathcal{L}(\mathcal{H})$ is an element of $\mathcal{A}$;

$(e) \Rightarrow (b)$: taking the commutants, from $\mathcal{B} \subset \mathcal{A}$ we get $\mathcal{A}' \subset \mathcal{B}' = \mathcal{B} \subset \mathcal{A}$ and this implies that $\mathcal{C}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}' = \mathcal{A}'$.

Remark 1.5. — Let $\mathcal{A}$ be a W*-algebra of operators on the Hilbert space $\mathcal{H}$ for which the conditions of proposition 1.4 hold. If $\{ \mathcal{B}_i \}$ $(i \in \mathcal{I})$ is the family of the maximal abelian W*-algebras which are contained in $\mathcal{A}$, then the equality $\mathcal{C}(\mathcal{A}) = \bigcap_{i \in \mathcal{I}} \mathcal{B}_i$ holds. In fact from $\mathcal{C}(\mathcal{A}) = \mathcal{A}'$ and $\mathcal{B}_i \subset \mathcal{A}$, $\forall i \in \mathcal{I}$, it follows that $\mathcal{C}(\mathcal{A}) \subset \mathcal{B}_i$, $\forall i \in \mathcal{I}$, namely

$$\mathcal{C}(\mathcal{A}) \subset \bigcap_{i \in \mathcal{I}} \mathcal{B}_i .$$

We can show the converse relation simply proving that an element A of $\bigcap_{i \in \mathcal{I}} \mathcal{B}_i$ results to be an element of $\mathcal{A}'$. For this it is sufficient to show that $[A, S] = 0$ for each selfadjoint element S of $\mathscr{A}$ and this easily follows since a $\mathscr{B}_i$ exists which contains S and $\mathscr{B}_i$ is abelian.

Remark 1.6. — Let $\mathscr{A}$ be a W*-algebra of operators on the Hilbert space $\mathscr{H} = \int_\Lambda^\oplus \mathscr{H}(\lambda)d\nu(\lambda)$. The algebra $\mathscr{A}$ is decomposable, namely $\mathscr{A} = \int_\Lambda^\oplus \mathscr{A}(\lambda)d\nu(\lambda)$ (D. B., II, § 3, def. 2), iff $\mathscr{A} = W^*\{ D(\mathscr{H}), T_1, T_2, \dots \}$, where $\{ T_n \}$ is a sequence of decomposable operators (D. B., II, § 3, th. 2). When $\mathscr{A}$ and $\mathscr{A}'$ are decomposable, if $\mathscr{C}(\mathscr{A}) = D(\mathscr{H})$ then $\mathscr{A}(\lambda)$ is factor ν-a. e., namely $\mathscr{C}(\mathscr{A}(\lambda)) = \{ K\mathbb{1}_\lambda ; K \in \mathbb{C} \}$ ν-a. e. (D. B., II, § 3, th. 3).

PROPOSITION 1.5. — *Let $\mathscr{A}$ be a* W*-*algebra of operators on the Hilbert space $\mathscr{H}$ such that the following conditions are satisfied:*

(*a*) $\mathscr{C}(\mathscr{A})' = W^*\{ \mathscr{C}(\mathscr{A}), T_1, T_2, \dots \}$;

(*b*) $\mathscr{A} = W^*\{ \mathscr{C}(\mathscr{A}), A_1, A_2, \dots \}$, $\mathscr{A}' = W^*\{ \mathscr{C}(\mathscr{A}), B_1, B_2, \dots \}$;

where $\{ T_n \}$, $\{ A_n \}$ and $\{ B_n \}$ are sequences of operators of $\mathscr{L}(\mathscr{H})$.

Since $\mathscr{C}(\mathscr{A})$ is an abelian W*-*algebra and condition* (a) *is satisfied, from proposition 1.2 it follows the existence of a Hausdorff locally compact space Λ, of a measure ν on Λ with* supp $\nu = \Lambda$, *of a Hilbert space*

$$\hat{\mathscr{H}} = \int_\Lambda^\oplus \mathscr{H}(\lambda)d\nu(\lambda)$$

with $\mathscr{H}(\lambda) \neq \{ 0 \}$ ν-a. e. *and of an isomorphism Φ_U of $\mathscr{L}(\mathscr{H})$ with $\mathscr{L}(\hat{\mathscr{H}})$ such that $\Phi_U(\mathscr{C}(\mathscr{A})) = D(\hat{\mathscr{H}})$. The following assertions are true:*

(1) *the algebra $\Phi_U(\mathscr{A})$ is decomposable,*

(2) $\Phi_U(\mathscr{A})(\lambda)$ *is a factor* ν-a. e.

Proof:

(1) from $\mathscr{C}(\mathscr{A}) \subset \mathscr{A}'$ we get $\mathscr{A} \subset \mathscr{C}(\mathscr{A})'$. Since Φ_U is an isomorphism, we have

$$\Phi_U(\mathscr{A}) \subset \Phi_U(\mathscr{C}(\mathscr{A})') = \Phi_U(\mathscr{C}(\mathscr{A}))' = D(\hat{\mathscr{H}})' = R(\hat{\mathscr{H}}).$$

Hence $\Phi_U(A_n)$ is a decomposable operator for each operator A_n which appears in condition (*b*). Moreover

$$\begin{aligned}\Phi_U(\mathscr{A}) &= \Phi_U(\{ \mathscr{C}(\mathscr{A}), A_1, A_1^+, A_2, A_2^+, \dots \}'') \\ &= \{ D(\hat{\mathscr{H}}), \Phi_U(A_1), \Phi_U(A_1)^+, \Phi_U(A_2), \Phi_U(A_2)^+, \dots \}'' \\ &= W^*\{ D(\hat{\mathscr{H}}), \Phi_U(A_1), \Phi_U(A_2), \dots \}.\end{aligned}$$

Then from remark 1.6 it follows that $\Phi_U(\mathscr{A})$ is decomposable.

(2) the result follows from remark 1.6 since

$$\mathscr{C}(\Phi_U(\mathscr{A})) = \Phi_U(\mathscr{C}(\mathscr{A})) = D(\hat{\mathscr{H}})$$

and $\Phi_U(\mathscr{A})'$ is decomposable. The decomposability of $\Phi_U(\mathscr{A})'$ can in fact be shown with the same procedure used for $\Phi_U(\mathscr{A})$.

We notice that when the Hilbert space is separable the condition of remark 1.6 for the decomposability of $\mathscr{A}$ is replaced simply by the condition $D(\mathscr{H}) \subset \mathscr{A} \subset R(\mathscr{H})$ (D. B., I, § 7, ex. 3 *b*). Then conditions (*a*) and (*b*) of proposition 1.5 can be omitted and the proof can be performed as in D. B., II, paragraph 6, corollary at page 210.

PROPOSITION 1.6. — *Using the same symbols and under the same assumptions as in proposition* 1.5, *the following are equivalent conditions:*

(*a*) $\Phi_U(\mathscr{A}) = R(\hat{\mathscr{H}})$;

(*b*) $\mathscr{A}' = \mathscr{C}(\mathscr{A})$.

Proof:

(*a*) $\Rightarrow$ (*b*): from $\Phi_U(\mathscr{A}) = R(\hat{\mathscr{H}})$ we get

$$\Phi_U(\mathscr{A}') = \Phi_U(\mathscr{A})' = D(\hat{\mathscr{H}}) = \Phi_U(\mathscr{C}(\mathscr{A})),$$

whence $\mathscr{A}' = \mathscr{C}(\mathscr{A})$.

(*b*) $\Rightarrow$ (*a*): from $\mathscr{A}' = \mathscr{C}(\mathscr{A})$ we get

$$\Phi_U(\mathscr{A}) = \Phi_U(\mathscr{A}')' = D(\hat{\mathscr{H}})' = R(\hat{\mathscr{H}}).$$

From proposition 1.6 it follows that, if the conditions of proposition 1.4 are satisfied for $\mathscr{A}$, then $\mathscr{A}$ is isomorphic to $R(\hat{\mathscr{H}})$.

2. THE ALGEBRA OF OBSERVABLES

A physical system is characterized giving the set of observables and the set of states. To get physical results, obviously these two sets must be equipped with some mathematical structure.

A classical system can be represented in fact by an abelian C*-algebra $\mathscr{A}$ [*10*]. Since any C*-algebra is isomorphic with a C*-algebra of operators on a Hilbert space, from the Gel'fand-Naimark theorem it follows that $\mathscr{A}$ can be identified with $L_\infty(\Lambda)$, where Λ is a Hausdorff locally compact space. Usually Λ is assumed to be the phase space of the system. The states are taken to be the continuous positive linear functionals on $\mathscr{A}$ with unit norm. From the Riesz representation theorem, for any state φ a measure μ on Λ exists such that $\varphi(f) = \int_\Lambda f d\mu$, $\forall f \in \mathscr{A}$. The measure μ results to be a regular Borel measure such that $\mu(\Lambda) = 1$.

If a state φ exists such that supp $\mu = \Lambda$, then we can construct the Hilbert space $L^2(\Lambda, \mu)$ and from proposition 1.1 it follows that $\mathscr{A}$ is isomorphic to the C*-algebra $D_c(L^2(\Lambda, \mu))$. It has been shown that Ludwig's axiom of sensitivity increase [*4*] holds for a C*-algebra of operators on a Hilbert space iff the algebra is in fact a W*-algebra [*5*]. Therefore it is convenient to assume $L^\infty(\Lambda, \mu)$ instead of $L_\infty(\Lambda)$ as the algebra of the observables of the system. From proposition 1.1 we have in fact that $L^\infty(\Lambda, \mu)$

is isomorphic to $D(L^2(\Lambda, \mu))$, which is the W*-algebra generated by $D_c(L^2(\Lambda, \mu))$.

On these lines we shall now introduce a mathematical structure in both the sets of observables and states for a classical as well as quantum mechanical system. We need first some definitions.

DEFINITION 2.1. — Let $\mathscr{A}$ be a W*-algebra in $\mathscr{L}(\mathscr{H})$. A selfadjoint operator A in $\mathscr{H}$ is said to be affiliated to $\mathscr{A}$ and it is then written $A\eta\mathscr{A}$ if its spectral projections belong to $\mathscr{A}$.

We notice that if $A \in \mathscr{L}(\mathscr{H})$ then $A\eta\mathscr{A}$ iff $A \in \mathscr{A}$ (D. B., I, § 1, ex. 10).

DEFINITION 2.2. — Let $\mathscr{A}$ be a W*-algebra. A positive linear functional on $\mathscr{A}$ is said to be normal if it is continuous in the ultraweak (French: ultrafaible) topology of $\mathscr{A}$.

Of course, since the norm topology is finer than the ultraweak one, a normal functional is bounded.

AXIOM 1. — *If a physical system is given, then we have a Hilbert space and a W*-algebra $\mathscr{A}$ in $\mathscr{L}(\mathscr{H})$ such that conditions* (a) *and* (b) *of proposition* 1.5 *are satisfied. The observables of the system are the selfadjoint operators which are affiliated to $\mathscr{A}$. The states are the normal functionals on $\mathscr{A}$ with unit norm.*

In the sequel, $\mathscr{A}$ will be called the algebra of observables. A possible motivation to assume as the algebra of observables a W*-algebra instead of a C*-algebra is furnished by the afore-mentioned argument of Chen [5].

To assume that the states are continuous in the ultraweak topology amounts to consider the « physical topology » to be defined not only by matrix elements but also by convergent series of matrix elements (D. B., I, § 3). It should be remarked that, for a positive linear functional φ, the ultraweak continuity is equivalent to the complete additivity, namely if $\{ P_i \}$ $(i \in \mathscr{I})$ is a family of mutually orthogonal projections of $\mathscr{A}$ then the equality $\varphi\left(\sum_{i\in\mathscr{I}} P_i\right) = \sum_{i\in\mathscr{I}} \varphi(P_i)$ holds (D. B., I, § 4, ex. 9). Since the projections of $\mathscr{A}$ are a logic $\mathscr{L}_{\mathscr{A}}$ [3], the restriction of φ to $\mathscr{L}_{\mathscr{A}}$ is then a state in the sense of the quantum logic approach to quantum mechanics. A state of this approach will be hereafter referred to as Q. L. state. Its definition can be found in section 5. Moreover, a positive linear functional φ is normal iff $\varphi = \sum_{n=1}^{\infty} \omega_{x_n}$, where $\omega_{x_n} : \mathscr{A} \to \mathbb{C}$, $\omega_{x_n}(A) = (x_n, Ax_n)$ and $\sum_{n=1}^{\infty} \| x_n \|^2 = 1$ (D. B., I, § 4, th. 1). Hence the states introduced by axiom 1 have exactly the same structure that derives from a theorem of Gleason to the Q. L. states when the logic on which they are defined is a standard one [11].

A bijection exists in fact between the set of normal positive linear functionals and the set of von Neumann operators of unit trace (D. B., I, § 6, p. 107).

The center $\mathscr{C}(\mathscr{A})$ may be thought of as the « classical part » of $\mathscr{A}$. It generates in fact with any observable an abelian algebra and abelian algebras characterize classical systems. Hence, the meaning of conditions (*a*) and (*b*) of proposition 1.5 is that it is possible to construct the algebra of observables of a physical system using a classical algebra and a sequence of observation procedures.

From proposition 1.5 it follows that a Hilbert space $\mathscr{H} = \int_\Lambda^\oplus \mathscr{H}(\lambda) d\nu(\lambda)$ exists, with $\mathscr{H}(\lambda) \neq \{ 0 \}$ ν-a. e. and supp $\nu = \Lambda$, such that

$$\mathscr{H} \sim \hat{\mathscr{H}}, \mathscr{A} \sim \int_\Lambda^\oplus \mathscr{A}(\lambda) d\nu(\lambda), \mathscr{C}(\mathscr{A}) \sim \mathrm{D}(\hat{\mathscr{H}}).$$

Since from proposition 1.1 it follows that $\mathscr{C}(\mathscr{A}) \sim \mathrm{L}^\infty(\Lambda, \nu)$, the Hausdorff locally compact space Λ may be interpreted as the phase space of the classical part of the physical system represented by $\mathscr{A}$.

DEFINITION 2.3. — The observables which are affiliated to $\mathscr{C}(\mathscr{A})$ and which are not multiples of identity are called superobservables. The W*-algebras $\mathscr{A}(\lambda)$ are called superselected systems.

The term « superobservable » introduced with definition 2.3 is not without justification. It is in fact easy to show that a selfadjoint operator in $\mathscr{H}$ affiliated to $\mathscr{C}(\mathscr{A})$ exists which is not a multiple of identity iff a non trivial subspace $\mathscr{H}'$ of $\mathscr{H}$ exists such that, for any x_1 of $\mathscr{H}'$ and x_2 orthogonal to $\mathscr{H}'$, « no physical measurement can distinguish between the state-vectors $x_1 + x_2$ and $e^{i\alpha_1}x_1 + e^{i\alpha_2}x_2$ » [*12*], namely $\omega_{x_1+x_2}(\mathrm{A}) = \omega_{e^{i\alpha_1}x_1+e^{i\alpha_2}x_2}(\mathrm{A})$, $\forall \mathrm{A} \in \mathscr{A}$. Hence superobservables exist iff a superselection rule acts in the physical system represented by $\mathscr{A}$.

While the algebra $\mathscr{C}(\mathscr{A})$ of superobservables is purely classical, since it is abelian, the superselected systems are purely quantal. In fact from proposition 1.5 it follows that they are factors, hence centerless. Therefore no superselected system can embody any classical part or superobservable. If $\mathscr{H}$ is separable, from remark 1.3 we get that any superobservable can be considered to be a function of just one superobservable.

The aim of the next axiom is to introduce in the algebra of observables complete sets of commuting observables.

AXIOM 2. — *The algebra $\mathscr{A}$ of observables of a physical system satisfies the conditions of proposition* 1.4.

From axiom 2 it follows that a W*-algebra $\mathscr{B}$ exists such that $\mathscr{B} = \mathscr{B}'$ and $\mathscr{B} \subset \mathscr{A}$.

DEFINITION 2.4. — Let $\mathscr{B}$ be a W*-algebra such that $\mathscr{B} = \mathscr{B}'$ and $\mathscr{B} \subset \mathscr{A}$.

The observables which are affiliated to $\mathscr{B}$ are said to constitute a complete set of commuting observables.

This definition agrees with Dirac's one [*13*], if proposition 1.3 is taken into account. In the same way as in the proof of proposition 1.4 (*d*), it can be shown that axiom 2 is equivalent to the following property for $\mathscr{A}$. For any observable T affiliated to $\mathscr{A}$ a complete set of commuting observables $\mathscr{B}$ exists such that T is affiliated to $\mathscr{B}$. The axiom 2 is then equivalent to the request (*iii*), page 57 of Dirac's book [*13*]. Another justification to assume axiom 2 will now be given.

An involutive algebra $\mathscr{A}$ in $\mathscr{L}(\mathscr{H})$ is said to be irreducible if no non-trivial subspace of $\mathscr{H}$ exists which is invariant with respect to all the operators of $\mathscr{A}$. It can be easily shown that a subspace $\mathscr{H}_1$ of $\mathscr{H}$ is $\mathscr{A}$-invariant iff $P_1 \in \mathscr{A}'$, where P_1 is the projection with range $\mathscr{H}_1$. As the commutant $\mathscr{A}'$ is a W*-algebra, $\mathscr{A}'$ is completely characterized by its projections. Hence $\mathscr{A}$ is irreducible iff $\mathscr{A}' = \{ k\mathbb{1} ; k \in \mathbb{C} \}$. When $\mathscr{A}$ is a W*-algebra, from $\mathscr{A}'' = \mathscr{A}$ it follows that $\mathscr{A}$ is irreducible iff $\mathscr{A} = \mathscr{L}(\mathscr{H})$. If $\mathscr{A}$ is reducible, then we can get two representations of $\mathscr{A}$. In one representation each element of $\mathscr{A}$ is mapped into its restriction to a non trivial invariant subspace $\mathscr{H}_1$, in the other one into its restriction to $\mathscr{H}_1^{\perp}$. If $\mathscr{A}$ represents a physical system, the reducibility of $\mathscr{A}$ amounts to the possibility of decomposing the physical system into subsystems. For this reason a quantum system represented by an irreducible algebra is often called simple [*14*]. To assume axiom 2 is then equivalent to assume each superselected system to be simple. In fact from proposition 1.6 it follows that a W*-algebra $\mathscr{A}$ satisfies the conditions of proposition 1.4 iff in its decomposition into factors $\mathscr{A} \sim \int_{\Lambda}^{\oplus} \mathscr{A}(\lambda) d\nu(\lambda)$ we get $\mathscr{A}(\lambda) = \mathscr{L}(\mathscr{H}(\lambda))$ ν-a. e. (1).

Once axiom 2 has been assumed, the superobservables can be characterized by means of the complete sets of observables. From remark 1.5 it follows in fact that the superobservables are exactly those observables which are commun to all the complete sets of observables.

Not all selfadjoint operators in the Hilbert space $\mathscr{H}$ correspond to observables. This class of physically significant selfadjoint operators depends upon the superobservables of the system. As a consequence of axiom 2 we get in fact that a selfadjoint operator in $\mathscr{H}$ is an observable iff it commutes with all the superobservables (2). This is easily seen as follows:

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(1) It should be remarked that a homomorphism exists from $\mathscr{A}$ to $\mathscr{A}(\lambda)$, $\forall \lambda \in \Lambda$, only if some conditions hold for $\mathscr{A}$ (D. A., II, § 2, prop. 6 and ex. 8). In this case $\mathscr{A}(\lambda)$ can be properly considered as a subsystem of $\mathscr{A}$, $\forall \lambda \in \Lambda$.

(2) Two selfadjoint operators are said to commute when their spectral projections commute.

A is an observable iff $A \eta \mathscr{A}$ (axiom 1),
iff $\forall \Delta \in \mathbb{B}_{\mathbb{R}}$, $P_\Delta^{(A)} \in \mathscr{A}$ (def. 2.1),
iff $\forall \Delta \in \mathbb{B}_{\mathbb{R}}$, $P_\Delta^{(A)} \in \mathscr{C}(\mathscr{A})'$ (axiom 2),
iff $\forall \Delta, \Delta' \in \mathbb{B}_{\mathbb{R}}$, $[P_\Delta^{(A)}, P_{\Delta'}^{(B)}] = 0$,
$\forall B \eta \mathscr{C}(A)$ (D. B., cor. 2 at p. 4),

where $\mathbb{B}_{\mathbb{R}}$ is the σ-algebra of the Borel sets of $\mathbb{R}$ and $\{ P_\Delta^{(A)} ; \Delta \in \mathbb{B}_{\mathbb{R}} \}$ is the set of spectral projections of A.

3. THE PURE STATES

In this and in the next section we shall not distinguish the algebra of observables $\mathscr{A}$ from the algebra $\int_\Lambda^\oplus \mathscr{L}(\mathscr{H}(\lambda))d\nu(\lambda)$ to which $\mathscr{A}$ is isomorphic. In the same way we shall identify $\mathscr{H}$ and $\int_\Lambda^\oplus \mathscr{H}(\lambda)d\nu(\lambda)$, $\mathscr{C}(\mathscr{A})$ and $D(\mathscr{H})$. If the system is classical then from proposition 1.3 we have $\mathscr{H} = L^2(\Lambda, \nu)$ and $\mathscr{L}(\mathscr{H}(\lambda)) = \mathscr{L}(\mathbb{C})$ ν-a. e. For convenience $\mathscr{H}$ will be assumed to be separable.

In this section we shall define a particular class of states on $\mathscr{A}$ which will be shown to be the counterpart in our picture of the usual pure states of the logic approach to quantum mechanics.

The following relation can be introduced in the set

$$\mathscr{H}' = \{ x \in \mathscr{H} ; \| x \| = 1 \} :$$

if $x, y \in \mathscr{H}'$, then $x R y$ iff $\exists k \in \mathbb{C}$, $| k | = 1$, such that $x \in ky$. The relation R is an equivalence relation and the quotient set $\mathscr{H}'/R$ is called the set of rays. The ray $\tilde{x}$ associated to $x \in \mathscr{H}'$ is then the subset of $\mathscr{H}'$

$$\tilde{x} = \{ y \in \mathscr{H}' ; y = kx, k \in \mathbb{C}, | k | = 1 \} .$$

DEFINITION 3.1. — Let $\mathscr{A}$ be a W*-algebra in $\mathscr{L}(\mathscr{H})$ and $\tilde{x}$ a ray in $\mathscr{H}$. The ray $\tilde{x}$ is called pure relative to $\mathscr{A}$ (shortly pure, when no confusion can occur) when the implication holds:

$$x \in \tilde{x}, \quad y \in \mathscr{H}, \quad (x, Ax) = (y, Ay), \quad \forall A \in \mathscr{A} \Rightarrow y \in \tilde{x} .$$

After definition 3.1, a ray is pure when it is thoroughly determined by physical observations. We shall now characterize pure rays.

PROPOSITION 3.1. — *Let* $x = \int_\Lambda^\oplus x(\lambda)d\nu(\lambda)$ *be a vector of* $\mathscr{H}'$. *Then the following are equivalent conditions:*

(a) *the ray* $\tilde{x}$ *is pure,*

(b) *no pair of Borel sets* E_1 *and* E_2 *in* Λ *exists such that:*

$$E_1 \cap E_2 = \emptyset, \quad \nu(E_i) \neq 0, \quad x(\lambda) \neq 0 \quad \nu\text{-a. e. } in\ E_i \quad (i = 1, 2),$$

(c) $\exists \lambda_0 \in \Lambda$ *such that:*

$$\nu(\{\lambda_0\}) \neq 0, \quad x(\lambda_0) \neq 0, \quad x(\lambda) = 0 \quad \nu\text{-a. e. } in\ \Lambda - \{\lambda_0\};$$

(d) $\forall A \in D(\mathscr{H})$, x *is an eigenvector of* A.

Proof:

(*a*) $\Rightarrow$ (*b*): let two Borel sets E_1 and E_2 in Λ exist such that

$$E_1 \cap E_2 = \emptyset, \quad \nu(E_i) \neq 0, \quad x(\lambda) \neq 0 \quad \nu\text{-a. e. in } E_i \quad (i = 1, 2).$$

If k_1, k_2 are two distinct complex numbers such that $|k_1| = |k_2| = 1$, define the function $f = k_1\chi_{E_1} + k_2\chi_{\Lambda - E_1}$, where χ_E is the characteristic function of a Borel set E. Trivially f is measurable and essentially bounded. Hence we can define the operator $V = \int_\Lambda^\oplus f(\lambda)\mathbb{1}_\lambda d\nu(\lambda)$ which is an element of $D(\mathscr{H})$. It is easily shown that V is unitary. Moreover we get $Vx \neq kx$, $\forall k \in \mathbb{C}$. In fact, if a $k \in \mathbb{C}$ would exist such that $Vx = kx$, we should get $k_1x(\lambda) = kx(\lambda)$ ν-a. e. in E_1 and $k_2x(\lambda) = kx(\lambda)$ ν-a. e. in E_2, whence $k_2 = k_1 = k$ would follow, since $x(\lambda) \neq 0$ ν-a. e. in both E_1 and E_2 and both E_1 and E_2 have ν-measure different from zero. Since $k_2 \neq k_1$ by assumption, we get $Vx \neq kx$, $\forall k \in \mathbb{C}$. On the other hand, since $D(\mathscr{H}) = \mathscr{C}(\mathscr{A})$ we get also $(Vx, AVx) = (x, Ax)$, $\forall A \in \mathscr{A}$. Then $\tilde{x}$ is not pure.

(*b*) $\Rightarrow$ (*c*): let $E_0 = \{\lambda \in \Lambda; x(\lambda) \neq 0\}$. The set E_0 is measurable, since the function $\lambda \to \|x(\lambda)\|$ is measurable, and $\nu(E_0) \neq 0$, since $\|x\| = 1$. As $\mathscr{H}$ is separable, Λ may be assumed to be the spectrum of a selfadjoint operator of $\mathscr{L}(\mathscr{H})$ (see remark 1.3). A finite interval $[m, M]$ then exists such that $\Lambda \subset [m, M]$. If

$$E_1^{(a)} = E_0 \cap [m, (M - m)/2) \quad \text{and} \quad E_1^{(b)} = E_0 \cap [(M - m)/2, M]$$

then either

$$\nu(E_1^{(a)}) = \nu(E_0) \neq 0 \quad \text{and} \quad \nu(E_1^{(b)}) = 0$$

or

$$\nu(E_1^{(b)}) = \nu(E_0) \neq 0 \quad \text{and} \quad \nu(E_1^{(a)}) = 0.$$

If E_1 is the one of $E_1^{(a)}$ and $E_1^{(b)}$ which has ν-measure different from zero, we can repeat in E_1 the procedure used in E_0 which has led to E_1, obtaining in this way a subset E_2 of E_1 with $\nu(E_2) = \nu(E_1) = \nu(E_0) \neq 0$. Hence we get a sequence $\{E_n\}$ of Borel sets of Λ such that $E_{n+1} \subset E_n$. Since we can assume ν to be finite, we have

$$\nu\left(\bigcap_n E_n\right) = \lim_{n\to\infty} \nu(E_n) = \nu(E_0) \neq 0.$$

Moreover $\bigcap_n E_n$ is a point. In fact $\bigcap_n E_n \neq \emptyset$, since $\nu\left(\bigcap_n E_n\right) \neq 0$, and $\bigcap_n E_n \subset E_k \subset I_k$ for each integer k, where I_k is an interval with lenght $(M - m)/2^k$. Setting $\{ \lambda_0 \} = \bigcap_n E_n$, then we have

$$\nu(\{ \lambda_0 \}) = \nu(E_0) \neq 0,$$

$x(\lambda_0) \neq 0$ since $\lambda_0 \in E_0$, $x(\lambda) = 0$ ν-a. e. in $\Lambda - \{ \lambda_0 \}$ since $x(\lambda) = 0$ for $\lambda \notin E_0$ and $\nu(E_0 - \{ \lambda_0 \}) = 0$.

$(c) \Rightarrow (a)$: let $y = \int_\Lambda^\oplus y(\lambda)d\nu(\lambda)$ be a vector of $\mathscr{H}$ such that

$$(y, Ay) = (x, Ax), \qquad \forall A \in \mathscr{A}.$$

Taking

$$A = \int_\Lambda^\oplus \chi_{\Lambda - \{\lambda_0\}}(\lambda)\mathbb{1}_\lambda d\nu(\lambda),$$

we get

$$\int_{\Lambda - \{\lambda_0\}} \| y(\lambda) \|^2 d\nu(\lambda) = 0,$$

whence

$$y(\lambda) = 0_\lambda \qquad \nu\text{-a. e. in } \Lambda - \{ \lambda_0 \}.$$

Hence

$$(y, Ay) = (x, Ax), \qquad \forall A \in \mathscr{A},$$

is equivalent to

$$(y(\lambda_0), A(\lambda_0)y(\lambda_0))_{\lambda_0} = (x(\lambda_0), A(\lambda_0)x(\lambda_0))_{\lambda_0}, \qquad \forall A(\lambda_0) \in \mathscr{L}(\mathscr{H}(\lambda_0)).$$

From this follows that a $k \in \mathbb{C}$ exists with $| k | = 1$ such that $y(\lambda_0) = kx(\lambda_0)$. Therefore $y = kx$ and this shows that $\tilde{x}$ is pure.

$(c) \Rightarrow (d)$: let A be any element of $D(\mathscr{H})$. Then $A = \int_\Lambda^\oplus f(\lambda)\mathbb{1}_\lambda d\nu(\lambda)$ and we get

$$Ax = \int_\Lambda^\oplus f(\lambda)x(\lambda)d\nu(\lambda) = f(\lambda_0)\int_\Lambda^\oplus x(\lambda)d\nu(\lambda) = f(\lambda_0)x.$$

$(d) \Rightarrow (b)$: from (d) we get that for any $A \in D(\mathscr{H})$ a $\lambda_A \in \mathbb{C}$ exists such that $Ax = \lambda_A x$. Since $A = \int_\Lambda^\oplus f_A(\lambda)\mathbb{1}_\lambda d\nu(\lambda)$, we have $(f_A(\lambda) - \lambda_A)x(\lambda) = 0$ ν-a. e. Let now two Borel sets E_1 and E_2 exist such that

$$E_1 \cap E_2 = \emptyset, \qquad \nu(E_i) \neq 0, \qquad x(\lambda) \neq 0 \qquad \nu\text{-a. e. in } E_i \qquad (i = 1, 2).$$

Then $(f_A(\lambda) - \lambda_A)x(\lambda) = 0$ ν-a. e. in both E_1 and E_2. Taking $f_A = \chi_{E_1}$ we should have $\lambda_A = 1$ along with $\lambda_A = 0$, which is impossible. Hence (b) must hold.

Coming now to states, we begin with the definition of pure states.

DEFINITION 3.2. — A state φ on the algebra of observables $\mathscr{A}$ is said to be pure if, for any two states φ_1 and φ_2, the equation $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$, with $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$, implies that $\varphi = \varphi_1 = \varphi_2$.

This definition is equivalent to the definition of pure states as extremal points of the set of states [*15*]. Before stating the theorem which characterizes pure states, we need a lemma.

LEMMA. — *Let φ_1 and φ_2 be two states on a W*-algebra $\mathscr{A}$ of operators on $\mathscr{H}$, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ their restrictions to the logic $\mathscr{L}_{\mathscr{A}}$ of the projections of $\mathscr{A}$. If $\tilde{\varphi}_1 = \tilde{\varphi}_2$, then $\varphi_1 = \varphi_2$.*

Proof. — For each selfadjoint element A of $\mathscr{A}$ we can write

$$(x, \mathrm{A}y) = \int_{\mathrm{Sp\,A}} z d(x, \mathrm{P}_z y), \qquad \forall x, y \in \mathscr{H},$$

where the spectral projections P_z belong to $\mathscr{A}$.

From normalcy of φ_i we get

$$\varphi_i = \sum_{n=1}^{\infty} \omega_{x_n^{(i)}} \qquad (i = 1, 2)$$

From $\tilde{\varphi}_1 = \tilde{\varphi}_2$ then we get

$$\begin{aligned}\varphi_1(\mathrm{A}) &= \sum_{n=1}^{\infty} (x_n^{(1)}, \mathrm{A}x_n^{(1)}) = \sum_{n=1}^{\infty} \int_{\mathrm{Sp\,A}} z d(x_n^{(1)}, \mathrm{P}_z x_n^{(1)}) \\ &= \int_{\mathrm{Sp\,A}} z d\left(\sum_{n=1}^{\infty} (x_n^{(1)}, \mathrm{P}_z x_n^{(1)})\right) \\ &= \int_{\mathrm{Sp\,A}} z d(\tilde{\varphi}_1(\mathrm{P}_z)) = \int_{\mathrm{Sp\,A}} z d(\tilde{\varphi}_2(\mathrm{P}_z)) \\ &= \ldots\ldots\ldots\ldots = \varphi_2(\mathrm{A}).\end{aligned}$$

Since each element of $\mathscr{A}$ can be written as a linear combination of selfadjoint elements of $\mathscr{A}$ and since φ_1 and φ_2 are linear, we can conclude that $\varphi_1 = \varphi_2$.

PROPOSITION 3.2. — *Let φ be a state on $\mathscr{A}$. The following conditions are equivalent:*

(a) *φ is a pure state,*

(b) *$\varphi = \omega_x$ with $\tilde{x}$ pure ray.*

Proof:

(*a*) $\Rightarrow$ (*b*) : if (*b*) does not hold, we have in fact two possibilities. Namely it could be either (1) $\varphi = \omega_x$, with $\tilde{x}$ non pure, or (2) $\varphi = \sum_{n=1}^{\infty} \omega_{x_n}$, where $x_n \neq 0$ for at least two indices. We shall prove that in both cases (*a*) cannot hold.

Case (1). From proposition 3.1 it follows that two Borel sets E_1 and E_2 exist such that

$$E_1 \cap E_2 = \emptyset, \qquad \nu(E_i) \neq 0, \qquad x(\lambda) \neq 0 \qquad \nu\text{-a. e. in } E_i \qquad (i = 1, 2)$$

We can then define the two vectors of $\mathcal{H}$

$$x_1 = \int_\Lambda^\oplus \chi_{E_1}(\lambda)x(\lambda)d\nu(\lambda) \qquad \text{and} \qquad x_2 = \int_\Lambda^\oplus \chi_{\Lambda - E_1}(\lambda)x(\lambda)d\nu(\lambda).$$

Since $x = x_1 + x_2$ and $(x_1, Ax_2) = 0$, $\forall A \in \mathcal{A}$, we get

$$\omega_x = \omega_{x_1} + \omega_{x_2}.$$

Since $x_1 \neq 0$ and $x_2 \neq 0$, we can define $y_1 = x_1/\|x_1\|$ and $y_2 = x_2/\|x_2\|$. Thus we have $\omega_x = \|x_1\|^2\omega_{y_1} + \|x_2\|^2\omega_{y_2}$. The two states ω_{y_1} and ω_{y_2} are different, because if we consider the operator

$$P = \int_\Lambda^\oplus \chi_{E_1}(\lambda)\mathbb{1}_\lambda d\nu(\lambda)$$

then we have

$$P \in \mathcal{A} \qquad \text{and} \qquad \omega_{y_1}(P) = \|y_1\|^2 = 1$$

along with $\omega_{y_2}(P) = 0$. Hence, since $\|x_1\|^2 + \|x_2\|^2 = \|x\|^2 = 1$, we get that φ is not a pure state.

Case (2). In $\varphi = \sum_n \omega_{x_n}$ we write only the terms with $x_n \neq 0$. Setting

$$y_n = \frac{x_n}{\|x_n\|} \qquad \text{and} \qquad \alpha_n = \|x_n\|^2,$$

we can write $\varphi = \sum_n \alpha_n\omega_{y_n}$. We can suppose $\omega_{y_1} \neq \sum_{n\neq 1} \frac{\alpha_n}{1-\alpha_1}\omega_{y_n}$, for if the equality holds then we get $\omega_{y_1} = \sum_n \alpha_n\omega_{y_n} = \varphi$ and hence we have in fact case (1). Taking now

$$\varphi_1 = \omega_{y_1} \qquad \text{and} \qquad \varphi_2 = \sum_{n\neq 1} \frac{\alpha_n}{1-\alpha_1}\omega_{y_n},$$

we have $\varphi = \alpha_1\varphi_1 + (1-\alpha_1)\varphi_2$. Since $\varphi_1 \neq \varphi_2$, $\alpha_1 = \|x_1\|^2 \neq 0$ and $\alpha_1 \neq 1$ ($\alpha_1 = 1$ would imply $\varphi = \omega_{y_1}$, which has been excluded) it follows that φ is not a pure state.

(*b*) $\Rightarrow$ (*a*) : let λ_0 be the point of Λ singled out by proposition 3.1 (*c*). The operator

$$P = \int_\Lambda^\oplus \chi_{\{\lambda_0\}}(\lambda)\mathbb{1}_\lambda d\nu(\lambda)$$

is a projection of $D(\mathcal{H})$. Suppose now that $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$, where φ_1 and φ_2 are states on $\mathcal{A}$ and $0 < \alpha < 1$. For each $A \in \mathcal{A}$, we get easily $\varphi((\mathbb{1} - P)A) = 0$, whence $\varphi_1((\mathbb{1} - P)A) = \varphi_2((\mathbb{1} - P)A) = 0$. Moreover, from normalcy of φ_i we get

$$\varphi_i = \sum_{n=1}^{\infty} \omega_{x_n^{(i)}} \qquad (i = 1, 2),$$

and thus

$$\varphi(PA) = \alpha\sum_{n=1}^{\infty}(x_n^{(1)}, PAx_n^{(1)}) + (1-\alpha)\sum_{n=1}^{\infty}(x_n^{(2)}, PAx_n^{(2)}), \qquad \forall A \in \mathcal{A},$$

which is equivalent to

$$(y, A(\lambda_0)y) = \alpha\sum_{n=1}^{\infty}\omega_{y_n^{(1)}}(A(\lambda_0)) + (1-\alpha)\sum_{n=1}^{\infty}\omega_{y_n^{(2)}}(A(\lambda_0)), \quad \forall A(\lambda_0) \in \mathcal{L}(\mathcal{H}(\lambda_0)),$$

where $y = x(\lambda_0)(\nu(\lambda_0))^{1/2}$, $y_n^{(i)} = x_n^{(i)}(\lambda_0)(\nu(\lambda_0))^{1/2}$.

Let $\tilde{\varphi}$ and $\tilde{\varphi}_i$ be the restrictions to the logic of projections $\mathcal{P}(\mathcal{H}(\lambda_0))$ of ω_y and $\sum_{n=1}^{\infty}\omega_{y_n^{(i)}}$, which are states on the W*-algebra $\mathcal{L}(\mathcal{H}(\lambda_0))$. From theorem 7.23 of [3], then it follows that $\tilde{\varphi} = \tilde{\varphi}_1 = \tilde{\varphi}_2$. Hence, by the lemma, we have

$$\omega_y = \sum_{n=1}^{\infty}\omega_{y_n^{(1)}} = \sum_{n=1}^{\infty}\omega_{y_n^{(2)}},$$

whence

$$\varphi(PA) = \varphi_1(PA) = \varphi_2(PA), \qquad \forall A \in \mathcal{A}.$$

Therefore we obtain

$$\begin{aligned}\varphi(A) &= \varphi(PA) + \varphi((\mathbb{1} - P)A)\\ &= \varphi_1(PA) + \varphi_1((\mathbb{1} - P)A) = \varphi_2(PA) + \varphi_2((\mathbb{1} - P)A)\\ &= \varphi_1(A) = \varphi_2(A), \qquad \forall A \in \mathcal{A}.\end{aligned}$$

This shows that φ is a pure state.

We can now ask what is the relation between the states of our picture and the Q. L. states. We have already seen in section 2 that the restriction of a state φ on the algebra of observables $\mathcal{A}$ to the logic $\mathcal{L}_{\mathcal{A}}$ of the projections of $\mathcal{A}$ is a Q. L. state. Moreover, we notice that the part (*a*) $\Rightarrow$ (*b*) of proposition 3.1 is analogous to a theorem which is proved at page 109 of [*16*] and that proposition 3.2 is the counterpart in our scheme of theorem 6.19 of [*3*]. Now we will show that any pure state of definition 3.2 is in fact nothing else than a Q. L. pure state. For the definition of Q. L. pure states, see page 116 of [*3*].

PROPOSITION 3.3. — *If* φ *is a state on the* W^**-algebra of observables* $\mathscr{A}$ *and* $\tilde{\varphi}$ *is its restriction to* $\mathscr{L}_{\mathscr{A}}$, *then* φ *is pure iff* $\tilde{\varphi}$ *is pure. Namely the following conditions are equivalent:*

(a) *if* φ_1 *and* φ_2 *are two states on* $\mathscr{A}$ *and* $0 < \alpha < 1$, *then*

$$\varphi(A) = \alpha\varphi_1(A) + (1 - \alpha)\varphi_2(A), \qquad \forall A \in \mathscr{A} \Rightarrow \varphi = \varphi_1 = \varphi_2,$$

(b) *if* ψ_1 *and* ψ_2 *are two Q. L. states on* $\mathscr{L}_{\mathscr{A}}$ *and* $0 < \alpha < 1$, *then*

$$\tilde{\varphi}(P) = \alpha\psi_1(P) + (1 - \alpha)\psi_2(P), \qquad \forall P \in \mathscr{L}_{\mathscr{A}} \Rightarrow \tilde{\varphi} = \psi_1 = \psi_2 .$$

Proof:

(*a*) $\Rightarrow$ (*b*): from proposition 3.2 it follows that $\varphi = \omega_x$, where for x the conditions of proposition 3.1 (*c*) hold. Suppose now that two Q. L. states ψ_1 and ψ_2 on $\mathscr{L}_{\mathscr{A}}$ exist such that

$$\tilde{\varphi}(P) = \alpha\psi_1(P) + (1 - \alpha)\psi_2(P), \qquad \forall P \in \mathscr{L}_{\mathscr{A}} .$$

We can define $P^{(0)} = \int_\Lambda^\oplus \chi_{\{\lambda_0\}}(\lambda)\mathbb{1}_\lambda d\nu(\lambda)$, which is an element of $\mathscr{L}_{\mathscr{A}}$. For each $P \in \mathscr{L}_{\mathscr{A}}$, from $\tilde{\varphi}((\mathbb{1} - P^{(0)})P) = 0$ we get

$$\psi_1((\mathbb{1} - P^{(0)})P) = \psi_2((\mathbb{1} - P^{(0)})P) = 0 .$$

Moreover

$$\tilde{\varphi}(P^{(0)}P) = \alpha\psi_1(P^{(0)}P) + (1 - \alpha)\psi_2(P^{(0)}P), \qquad \forall P \in \mathscr{L}_{\mathscr{A}},$$

is equivalent to

$$(x_0, P_0x_0) = \alpha\psi_1^{(0)}(P_0) + (1 - \alpha)\psi_2^{(0)}(P_0), \qquad \forall P_0 \in \mathscr{P}(\mathscr{H}(\lambda_0)),$$

where $x_0 = x(\lambda_0)(\nu(\lambda_0))^{1/2}$ and, for $i = 1, 2$, $\psi_i^{(0)}$ is the Q. L. state on $\mathscr{P}(\mathscr{H}(\lambda_0))$

$$\psi_i^{(0)} : \mathscr{P}(\mathscr{H}(\lambda_0)) \to [0, 1], \qquad \psi_i^{(0)}(P_0) = \psi_i(P),$$

if in

$$P = \int_\Lambda^\oplus P(\lambda)d\nu(\lambda)$$

we have

$$P(\lambda) = O_\lambda \quad \text{for} \quad \lambda \neq \lambda_0 \quad \text{and} \quad P(\lambda_0) = P_0 .$$

Hence, from theorem 7.23 of [*3*] we get

$$\tilde{\varphi}(P^{(0)}P) = \psi_1(P^{(0)}P) = \psi_2(P^{(0)}P), \qquad \forall P \in \mathscr{L}_{\mathscr{A}} .$$

Therefore, we can conclude that $\tilde{\varphi} = \psi_1 = \psi_2$.

(*b*) $\Rightarrow$ (*a*): suppose that two states φ_1 and φ_2 exist on $\mathscr{A}$ such that

$$\varphi(A) = \alpha\varphi_1(A) + (1 - \alpha)\varphi_2(A), \qquad \forall A \in \mathscr{A} .$$

Then, if $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the restrictions of φ_1 and φ_2 to $\mathcal{L}_{\mathcal{A}}$, we get

$$\tilde{\varphi}(\mathrm{P}) = \alpha\tilde{\varphi}_1(\mathrm{P}) + (1 - \alpha)\tilde{\varphi}_2(\mathrm{P}), \qquad \forall \mathrm{P} \in \mathcal{L}_{\mathcal{A}}.$$

Hence we have $\tilde{\varphi} = \tilde{\varphi}_1 = \tilde{\varphi}_2$ and, by the lemma, $\varphi = \varphi_1 = \varphi_2$.

From proposition 3.2 it follows that pure states on $\mathcal{A}$ exist iff pure rays exist. Hence, by proposition 3.1 (*c*), pure states exist iff points of Λ exist of ν-measure different from zero. This is the case iff the point spectrum of B is not empty, where B is the selfadjoint operator which generates $\mathcal{C}(\mathcal{A})$, according to remark 1.3. In fact, if λ_0 is an eigenvalue of B, then

$$\mathrm{B}x = \lambda_0 x \Rightarrow \int_\Lambda^\oplus (\lambda - \lambda_0)x(\lambda)d\nu(\lambda) = 0$$
$$\Rightarrow (\lambda - \lambda_0)x(\lambda) = 0 \qquad \nu\text{-a. e.} \Rightarrow \nu(\{\lambda_0\}) \neq 0.$$

Conversely, if λ_0 is such that $\nu(\{\lambda_0\}) \neq 0$, then we can construct a vector x for which the conditions of proposition 3.1 hold. Hence x is an eigenvector of B and λ_0 the corresponding eigenvalue.

If $\mathcal{A}$ represents a classical system then from proposition 3.2, proposition 3.1 and proposition 1.3 we get that pure states may be identified with the characteristic functions of the points of the phase space Λ of ν-measure different from zero, in the sense that each pure state can be written as

$$\frac{1}{\nu(\{\lambda_0\})}\int_\Lambda^\oplus \chi_{\{\lambda_0\}}(\lambda)d\nu(\lambda) \qquad \text{if} \qquad \nu(\{\lambda_0\}) \neq 0.$$

This is the counterpart in our framework of theorem 6.6 of [*3*], which has been generalized by Gudder [*17*].

4. THE REPRESENTATION OF THE ALGEBRA OF OBSERVABLES INDUCED BY A STATE

In this section a representation of the algebra of observables $\mathcal{A}$, that is of the algebra of the decomposable operators on the Hilbert space $\mathcal{H} = \int_\Lambda^\oplus \mathcal{H}(\lambda)d\nu(\lambda)$, will be constructed in correspondence with any state φ. Namely, we shall construct a new Hilbert space $\mathcal{H}_\varphi$ and therein a W*-algebra $\mathcal{A}_\varphi$ such that a homomorphism exists from $\mathcal{A}$ onto $\mathcal{A}_\varphi$.

Let φ be a state on $\mathcal{A}$. Then the set function

$$\mu_\varphi : \mathbb{B} \to [0, 1], \qquad \mu_\varphi(\mathrm{E}) = \varphi\left(\int_\Lambda^\oplus \chi_{\mathrm{E}}(\lambda)\mathbb{1}_\lambda d\nu(\lambda)\right)$$

is a measure on the Borel σ-algebra $\mathbb{B}$ such that $\mu_\varphi \ll \nu$ and $\mu_\varphi(\Lambda) = 1$. In fact from $\varphi = \sum_{n=1}^{\infty} \omega_{x_n}$ we get

$$\mu_\varphi(E) = \sum_{n=1}^{\infty} \left(x_n, \int_\Lambda^\oplus \chi_E(\lambda) \mathbb{1}_\lambda d\nu(\lambda) x_n \right)$$

$$= \sum_{n=1}^{\infty} \int_E \| x_n(\lambda) \|^2 d\nu(\lambda) = \int_E \sum_{n=1}^{\infty} \| x_n(\lambda) \|^2 d\nu(\lambda),$$

and

$$\lambda \to \sum_{n=1}^{\infty} \| x_n(\lambda) \|^2$$

is a ν-measurable non negative function such that

$$\int_\Lambda \sum_{n=1}^{\infty} \| x_n(\lambda) \|^2 d\nu(\lambda) = \sum_{n=1}^{\infty} \| x_n \|^2 = 1 .$$

We can now introduce the aforementioned representation of $\mathscr{A}$.

PROPOSITION 4.1. — *It is possible to define on* $\prod_{\lambda \in \Lambda} \mathscr{H}(\lambda)$ *a structure of* μ_φ*-measurable field of Hilbert spaces and hence the Hilbert space*

$$\mathscr{H}_\varphi = \int_\Lambda^\oplus \mathscr{H}(\lambda) d\mu_\varphi(\lambda) .$$

We can define the mapping

$$\Phi : \mathscr{A} \to \mathscr{L}(\mathscr{H}_\varphi), \qquad \Phi\left(\int_\Lambda^\oplus A(\lambda) d\nu(\lambda) \right) = \int_\Lambda^\oplus A(\lambda) d\mu_\varphi(\lambda) .$$

If $\mathscr{A}_\varphi$ *is the* W*-*algebra of the decomposable operators on* $\mathscr{H}_\varphi$, *then* Φ *is a norm-decreasing homomorphism of* $\mathscr{A}$ *onto* $\mathscr{A}_\varphi$.

The following conditions are equivalent:

(a) Φ *is faithful,*

(b) $\nu \simeq \mu_\varphi$,

(c) *a vector* $x \in \mathscr{H}$ *exists such that* x *is cyclic for* $\mathscr{A}$ *and* $\mu_\varphi \simeq \nu_{x,x}$.

Proof. — Let $\{ x_n \}$ be a foundamental sequence for the structure of ν-measurable field of Hilbert spaces which defines $\mathscr{H}$. Then $\{ x_n(\lambda) \}$ is a family of vectors which spans $\mathscr{H}(\lambda)$ ν-a. e. and the functions $\lambda \to (x_n(\lambda), x_m(\lambda))$ are ν-measurable, hence μ_φ-measurable since $\mu_\varphi \ll \nu$ [see appendix (*a*)].

Hence a structure of μ_φ-measurable fields of Hilbert spaces is uniquely defined on $\prod_{\lambda\in\Lambda}\mathscr{H}(\lambda)$ for which $\{x_n\}$ is a foundamental sequence (D. B., II, § 1, prop. 4). Therefore we can define

$$\mathscr{H}_\varphi = \int_\Lambda^\oplus \mathscr{H}(\lambda)d\mu_\varphi(\lambda).$$

Let $A = \int_\Lambda^\oplus A(\lambda)d\nu(\lambda)$ be an element of $\mathscr{A}$. Then the functions

$$\lambda \to (x_n(\lambda), A(\lambda)x_m(\lambda))$$

are ν-measurable, hence μ_φ-measurable, and $\lambda \to A(\lambda)$ is a μ_φ-measurable field of operators (D. B., II, § 2, prop. 1). Since $\lambda \to \|A(\lambda)\|$ is a ν-essentially bounded function, it is also μ_φ-essentially bounded. Therefore we can define the operator $\int_\Lambda^\oplus A(\lambda)d\mu_\varphi(\lambda)$, which is an element of the algebra $\mathscr{A}_\varphi$ of the decomposable operators on $\mathscr{H}_\varphi$. In this way we have shown that the mapping Φ is well defined and that its range is contained in $\mathscr{A}_\varphi$. It is trivial to see that Φ is in fact a homomorphism (D. B., II, § 2, prop. 3) and it is norm-decreasing since

$$\|\Phi(A)\| = \mu_\varphi\text{-ess. sup}\,\|A(\lambda)\| \leq \nu\text{-ess. sup}\,\|A(\lambda)\| = \|A\|.$$

We shall now show that the range of Φ coincides with $\mathscr{A}_\varphi$. Take an element B of $\mathscr{A}_\varphi$. Then a field of operators $\lambda \to B(\lambda)$ exists such that

$$\lambda \to f_{nm}(\lambda) = (x_n(\lambda), B(\lambda)x_m(\lambda))$$

is μ_φ-measurable, $\lambda \to \|B(\lambda)\|$ is bounded and

$$B = \int_\Lambda^\oplus B(\lambda)d\mu_\varphi(\lambda).$$

Let E_{nm} be a μ_φ-negligible element of $\mathbb{B}$ such that $\chi_{\Lambda - E_{nm}}f_{nm}$ is ν-measurable [see appendix (*b*)]. We define the field of operators

$$\lambda \to B'(\lambda) = \chi_{\Lambda - \bigcup_{n,m} E_{n,m}}(\lambda)\cdot B(\lambda).$$

This field of operators is ν-measurable, since the function

$$\lambda \to (x_{\bar n}(\lambda), B'(\lambda)x_{\bar m}(\lambda)) = \chi_{\Lambda - \bigcup_{\substack{n\neq\bar n\\ m\neq\bar m}} E_{nm}}(\lambda)(\chi_{\Lambda - E_{\bar n\bar m}}(\lambda)\cdot f_{\bar n\bar m}(\lambda))$$

is the product of two ν-measurable functions. Moreover the function $\lambda \to \|B'(\lambda)\|$ is ν-essentially bounded, simply because it is bounded. Then we can define $\int_\Lambda^\oplus B'(\lambda)d\nu(\lambda)$, which is an element of $\mathscr{A}$. We have now that $\Phi\left(\int_\Lambda^\oplus B'(\lambda)d\nu(\lambda)\right) = B$. In fact, as a countable union of μ_φ-negligible sets, $\bigcup_{n,m} E_{nm}$ is μ_φ-negligible. Hence the field of operators $\lambda \to B'(\lambda)$ is μ_φ-a. e. equal to the field $\lambda \to B(\lambda)$.

The equivalence of (*a*) and (*b*) can be shown in the following way:

$$\begin{aligned}\Phi \text{ is faithful } &\Rightarrow \{ A \in \mathscr{A}, \Phi(A) = 0 \Rightarrow A = 0 \} \\ &\Rightarrow \left\{ f\ \nu\text{-measurable}, \int_\Lambda |f|\, d\mu_\varphi = 0 \Rightarrow \int_\Lambda |f|\, d\nu = 0 \right\} \\ &\Rightarrow \left\{ \forall E \in \mathbb{B},\ \mu_\varphi(E) = \int_\Lambda \chi_E d\mu_\varphi = 0 \Rightarrow \nu(E) = \int_\Lambda \chi_E d\nu = 0 \right\} \\ &\Rightarrow \nu \ll \mu_\varphi \Rightarrow \nu \simeq \mu_\varphi ;\end{aligned}$$

conversely

$$\nu \simeq \mu_\varphi \Rightarrow \| \Phi(A) \| = \| A \|, \qquad \forall A \in \mathscr{A} \Rightarrow \Phi \text{ is faithful.}$$

The equivalence of (*b*) and (*c*) can be shown in the following way. If $\nu \simeq \mu_\varphi$, then μ_φ is a bounded basic measure for $\mathscr{C}(\mathscr{A})$. Hence in $\mathscr{H}$ a cyclic vector for $\mathscr{C}(\mathscr{A})'$ exists such that $\mu_\varphi \simeq \nu_{x,x}$ and from axiom 2 it follows that $\mathscr{C}(\mathscr{A})' = \mathscr{A}$. Conversely, if in $\mathscr{H}$ a cyclic vector for $\mathscr{A} = \mathscr{C}(\mathscr{A})'$ exists such that $\mu_\varphi \simeq \nu_{x,x}$, then $\nu_{x,x}$ is a basic measure for $\mathscr{C}(\mathscr{A})$. Hence $\nu_{x,x} \simeq \nu$ and we have $\nu \simeq \mu_\varphi$. We have used the relation between basic measures and cyclic vectors which can be found in remark 1.1.

It should be remarked that $\mathscr{A}$ can be directly interpreted as an $\mathscr{A}_\varphi$ iff a state φ exists such that $\nu = \mu_\varphi$. This is true iff $\nu(\Lambda) = 1$. In fact, if $\nu = \mu_\varphi$ then trivially $\nu(\Lambda) = 1$. Conversely, if $\nu(\Lambda) = 1$ we can take

$$x = \int_\Lambda^\oplus x(\lambda)d\nu(\lambda) \qquad \text{with} \qquad \| x(\lambda) \| = 1, \qquad \forall \lambda \in \Lambda .$$

For $\varphi = \omega_x$ we have then

$$\mu_\varphi(E) = \int_E \| x(\lambda) \|^2 d\nu(\lambda) = \nu(E), \qquad \forall E \in \mathbb{B} .$$

The center $\mathscr{C}(\mathscr{A})$ represents the classical part of the system which is represented by $\mathscr{A}$ and, from proposition 1.1, we know that $\mathscr{C}(\mathscr{A}) \sim L^\infty(\Lambda, \nu)$. If a state φ is given on $\mathscr{A}$, we can then define the classical state

$$\varphi'_c : L^\infty(\Lambda, \nu) \to [0, 1], \qquad \varphi'_c(f) = \varphi\left(\int_\Lambda^\oplus f(\lambda)\mathbb{1}_\lambda d\nu(\lambda) \right),$$

and we know from section 2 that a measure μ on Λ exists such that

$$\varphi'_c(f) = \int_\Lambda f\, d\mu \qquad \forall f \in L^\infty(\Lambda, \nu).$$

We have that $\mu = \mu_\varphi$ since

$$\mu(\mathrm{E}) = \varphi'_c(\chi_{\mathrm{E}}) = \sum_{n=1}^{\infty} \omega_{x_n}\left(\int_\Lambda^\oplus \chi_{\mathrm{E}}(\lambda) \mathbb{1}_\lambda d\nu(\lambda) \right)$$

$$= \int_{\mathrm{E}} \sum_{n=1}^{\infty} \| x_n(\lambda) \|^2 d\nu(\lambda) = \mu_\varphi(\mathrm{E}) \qquad \forall \mathrm{E} \in \mathbb{B}.$$

We shall now introduce a theorem which characterizes the irreducibility of the algebra $\mathscr{A}_\varphi$.

PROPOSITION 4.2. — *Given a state φ on $\mathscr{A}$, let $\mathscr{A}_\varphi$ be the W*-algebra constructed in proposition 4.1 and φ_c the restriction of φ to $\mathscr{C}(\mathscr{A})$. Obviously φ_c is a state on the W*-algebra $\mathscr{C}(\mathscr{A})$. The following conditions are equivalent:*

(a) *φ_c is a pure state,*
(b) *$\mathscr{A}_\varphi$ is irreducible,*
(c) *$\exists \lambda_0 \in \Lambda$ such that:*

$$\nu(\{\lambda_0\}) \neq 0, \qquad x_n(\lambda_0) \neq 0 \qquad \text{and} \qquad x_n(\lambda) = 0 \qquad \nu\text{-a. e.}$$

in $\Lambda - \{\lambda_0\}$ for any vector x_n occurring in $\varphi = \sum_{n=1}^{\infty} \omega_{x_n}$.

Proof:

(a) $\Rightarrow$ *(b)*: first, along the same lines as in the case (1) of the *(a)* $\Rightarrow$ *(b)* part of the proof of proposition 3.2, we can show that no pair of Borel sets E_1 and E_2 exists such that $\mathrm{E}_1 \cap \mathrm{E}_2 = \emptyset$ and $\mu_\varphi(\mathrm{E}_i) \neq 0$ $(i = 1, 2)$. Then, in the same way as in the *(b)* $\Rightarrow$ *(c)* part of the proof of proposition 3.1 with Λ in the place of E_0, we get that a point λ_0 exists in Λ for which $\mu_\varphi(\{\lambda_0\}) = 1$ and hence $\mu_\varphi(\Lambda - \{\lambda_0\}) = 0$. This is equivalent to $\mathscr{A}_\varphi \sim \mathscr{L}(\mathscr{H}(\lambda_0))$, which amounts to the irreducibility of $\mathscr{A}_\varphi$.

(b) $\Rightarrow$ *(c)*: since

$$\mathscr{A}_\varphi = \int_\Lambda^\oplus \mathscr{L}(\mathscr{H}(\lambda)) d\mu_\varphi(\lambda),$$

from irreducibility of $\mathscr{A}_\varphi$ it follows that a point λ_0 in Λ exists such that $\mathscr{A}_\varphi \sim \mathscr{L}(\mathscr{H}(\lambda_0))$. From

$$1 = \mu_\varphi(\{\lambda_0\}) = \sum_{n=1}^{\infty} \| x_n(\lambda_0) \|^2 \nu(\lambda_0)$$

then we get $\nu(\{\lambda_0\}) \neq 0$. From

$$0 = \mu_\varphi(\Lambda - \{\lambda_0\}) = \int_{\Lambda - \{\lambda_0\}} \sum_{n=1}^{\infty} \| x_n(\lambda) \|^2 d\nu(\lambda)$$

we get $x(\lambda) = 0$ ν-a. e. in $\Lambda - \{\lambda_0\}$ for any vector x_n.

$(c) \Rightarrow (a)$: suppose that $\varphi_c = \alpha\psi_1 + (1-\alpha)\psi_2$, where ψ_1 and ψ_2 are states on $\mathscr{C}(\mathscr{A})$ and $0 < \alpha < 1$. From normalcy of ψ_i we get

$$\psi_i = \sum_{n=1}^{\infty} \omega_{x_n^{(i)}} \qquad (i = 1, 2).$$

Then, for

$$\mathrm{P} = \int_{\Lambda}^{\oplus} \chi_{\Lambda - \{\lambda_0\}}(\lambda) \mathbb{1}_\lambda d\nu(\lambda),$$

the equalities

$$0 = \mu_\varphi(\Lambda - \{\lambda_0\}) = \varphi_c(\mathrm{P}) = \alpha\psi_1 + (1-\alpha)\psi_2(\mathrm{P})$$

hold, whence

$$\psi_i(\mathrm{P}) = \int_{\Lambda - \{\lambda_0\}} \sum_{n=1}^{\infty} \| x_n^{(i)}(\lambda) \|^2 d\nu(\lambda) = 0$$

and this implies $x_n^{(i)}(\lambda) = 0$ ν-a. e. in $\Lambda - \{\lambda_0\}$ for any vector $x_n^{(i)}$ $(i = 1, 2)$. Therefore for any element

$$\mathrm{A} = \int_{\Lambda}^{\oplus} f(\lambda) \mathbb{1}_\lambda d\nu(\lambda)$$

of $\mathscr{C}(\mathscr{A})$ the relation

$$\varphi_c(\mathrm{A}) = \alpha\psi_1(\mathrm{A}) + (1-\alpha)\psi_2(\mathrm{A})$$

reads

$$\sum_{n=1}^{\infty} f(\lambda_0) \| x_n(\lambda_0) \|^2 \nu(\{\lambda_0\}) = \alpha \sum_{n=1}^{\infty} f(\lambda_0) \| x_n^{(1)}(\lambda_0) \|^2 \nu(\{\lambda_0\}) + (1-\alpha) \sum_{n=1}^{\infty} f(\lambda_0) \| x_n^{(2)}(\lambda_0) \|^2 \nu(\{\lambda_0\}).$$

Since $\nu(\{\lambda_0\}) \neq 0$, we have

$$\sum_{n=1}^{\infty} \| x_n(\lambda_0) \|^2 = \alpha \sum_{n=1}^{\infty} \| x_n^{(1)}(\lambda_0) \|^2 + (1-\alpha) \sum_{n=1}^{\infty} \| x_n^{(2)}(\lambda_0) \|^2,$$

which implies

$$\sum_{n=1}^{\infty} \| x_n(\lambda_0) \|^2 = \sum_{n=1}^{\infty} \| x_n^{(1)}(\lambda_0) \|^2 = \sum_{n=1}^{\infty} \| x_n^{(2)}(\lambda_0) \|^2 .$$

From this it follows that $\varphi_c = \psi_1 = \psi_2$, namely that φ_c is a pure state.

We notice that φ_c can be pure even if φ is not pure. For purity of φ_c it is in fact sufficient that only one superselected sector is « detected » by φ, while what happens inside the superselected sectors has no relevance at all. We now state the last axiom of our picture.

AXIOM 3. — *If* $\mathcal{A} = \int_\Lambda^\oplus \mathcal{L}(\mathcal{H}(\lambda))d\nu(\lambda)$ *is a non abelian algebra of observables then for each* $\lambda \in \Lambda$ *a state* φ *exists such that* $\mathcal{A}_\varphi \sim \mathcal{L}(\mathcal{H}(\lambda))$.

This axiom reflects a usual assumption of elementary quantum mechanics, namely the « absence of superselection rules », which is equivalent to assume the possibility of giving a picture of the system in which all superselected sectors but one are disregarded. The axiom has been formulated for a non classical algebra since in the classical case all the structure is beared by Λ and hence the restriction to only one point of Λ does not make sense. We can now find some conditions which are equivalent to axiom 3.

PROPOSITION 4.3. — *The following conditions are equivalent:*

(a) *axiom* 3 *holds,*

(b) $\forall \lambda \in \Lambda,\ \nu(\{\lambda\}) \neq 0$,

(c) *each* $\lambda \in \Lambda$ *is an eigenvalue of* B, *if* B *is the selfadjoint operator which generates* $\mathcal{C}(\mathcal{A})$, *according to remark* 1.3.

Proof:

$(a) \Rightarrow (b)$: this follows directly from proposition 4.2.

$(b) \Rightarrow (c)$: for each $\lambda \in \Lambda$, we can construct the vector

$$x = \int_\Lambda^\oplus x(\lambda')d\nu(\lambda'),$$

with $x(\lambda') = 0$ when $\lambda' \neq \lambda$ and $x(\lambda) \neq 0$. This vector is non null since $\nu(\{\lambda\}) \neq 0$ and easily we get

$$\mathrm{B}x = \int_\Lambda^\oplus \lambda' \mathbb{1}_{\lambda'} d\nu(\lambda') \int_\Lambda^\oplus x(\lambda')d\nu(\lambda') = \lambda x .$$

$(c) \Rightarrow (a)$: for each $\lambda \in \Lambda$, a non null vector $x = \int_\Lambda^\oplus x(\lambda')d\nu(\lambda')$ exists such that $\mathrm{B}x = \lambda x$, which implies $(\lambda - \lambda')x(\lambda') = 0$ ν-a. e. and from this we get $\nu(\{\lambda\}) \neq 0$ and $x(\lambda') = 0$ ν-a. e. in $\Lambda - \{\lambda\}$. Hence, using proposition 4.2, we get $\mathcal{A}_\varphi \sim \mathcal{L}(\mathcal{H}(\lambda))$ for the state $\varphi = \omega_x$.

From proposition 4.3 it follows that, if axiom 3 is assumed, then Λ is at most countable. Since $\mathcal{H}$ is a separable Hilbert space, any selfadjoint operator has in fact an at most denumerable infinity of eigenvalues. This is a very important point and it suggests the axiom of the next section.

5. FROM QUANTUM LOGIC TO THE ALGEBRA OF OBSERVABLES

First we recall some useful definitions [*18*]. A complete, orthocomplemented and weakly modular lattice is called shortly croc. A homomorphism from a croc $\mathscr{L}$ into a croc $\mathscr{L}'$ is a mapping τ such that, for any family $\{ p_\alpha \}$ of elements of $\mathscr{L}$ and for any element p of $\mathscr{L}$:

$$(a) \qquad \tau\Big(\bigvee_\alpha p_\alpha\Big) = \bigvee_\alpha{}' \tau(p_\alpha),$$

$$(b) \qquad \tau\Big(\bigwedge_\alpha p_\alpha\Big) = \bigwedge_\alpha{}' \tau(p_\alpha),$$

$$(c) \qquad \tau(cp) = c'\tau(p) \wedge' \tau(\mathrm{I}),$$

where $\vee$, $\wedge$, c and $\vee'$, $\wedge'$, c' are the join, the meet, the orthocomplementation in $\mathscr{L}$ and in $\mathscr{L}'$ respectively and I is the unit element of $\mathscr{L}$. A homomorphism is one to one iff $\{ \emptyset \} = \operatorname{Ker} \tau = \{ p \in \mathscr{L} ; \tau(p) = \emptyset' \}$, where $\emptyset$ and $\emptyset'$ are the zero elements of $\mathscr{L}$ and $\mathscr{L}'$ respectively. A homomorphism is said to be an isomorphism if it is both one to one and onto. A croc $\mathscr{L}'$ is said to be a subcroc of $\mathscr{L}$ when it is a subset of $\mathscr{L}$ and the canonical injection is a homomorphism from $\mathscr{L}'$ into $\mathscr{L}$. The set $\mathscr{C}(\mathscr{L})$ of elements of $\mathscr{L}$ which are compatible with all the elements of $\mathscr{L}$ is called the center of $\mathscr{L}$ and it is a subcroc of $\mathscr{L}$. When a family $\{ \mathrm{I}^{(\lambda)} \}$ $(\lambda \in \Lambda)$ of elements of $\mathscr{C}(\mathscr{L})$ exists such that $\mathrm{I}^{(\lambda)} \wedge \mathrm{I}^{(\lambda')} = \emptyset$ for $\lambda \neq \lambda'$ and $\bigvee_{\lambda \in \Lambda} \mathrm{I}^{(\lambda)} = \mathrm{I}$, then $\mathscr{L}$ is said to be the direct union of the subcrocs

$$\mathscr{L}^{(\lambda)} = \{ p \in \mathscr{L} ; \emptyset \leq p \leq \mathrm{I}^{(\lambda)} \},$$

where $\leq$ is the order relation on $\mathscr{L}$. In this case $\mathscr{L}$ is written as

$$\mathscr{L} = \bigvee_{\lambda \in \Lambda}{}^{\oplus} \mathscr{L}^{(\lambda)}$$

and every element p of $\mathscr{L}$ can be written uniquely as a join of elements of $\mathscr{L}^{(\lambda)}$, which are called the components of p. Moreover, for each $\lambda \in \Lambda$, the mapping $\pi^{(\lambda)}$, which sends every element of $\mathscr{L}$ into its component in $\mathscr{L}^{(\lambda)}$ is called « projection » and it is a homomorphism from $\mathscr{L}$ onto $\mathscr{L}^{(\lambda)}$.

In the logic approach to quantum mechanics the set $\mathscr{L}$ of the « propositions » of a physical system can be assumed to be an atomic croc [*19*]. From a decomposition theorem and from Piron's representation theorem [*20*] $\mathscr{L}$ results to be a direct union of irreducible subcrocs, namely

$$\mathscr{L} = \bigvee_{\lambda \in \Lambda}{}^{\oplus} \mathscr{L}^{(\lambda)},$$

where Λ is an index set and $\mathscr{L}^{(\lambda)}$ can be taken to be the family of all the projections of a Hilbert space $\mathscr{H}(\lambda)$.

The classical part of the physical system which is represented by $\mathscr{L}$ is represented by $\mathscr{C}(\mathscr{L})$. The croc $\mathscr{C}(\mathscr{L})$ may be identified with the power set $\mathbb{P}(\Lambda)$, in which the set-theoretical union, intersection and complementation are assumed as the join, the meet and the orthocomplementation. In fact the mapping

$$\mathscr{V} : \mathbb{P}(\Lambda) \rightarrow \mathscr{C}(\mathscr{L}),\ \pi^{(\lambda)}(\mathscr{V}(\Delta)) = \begin{cases} 0_\lambda & \text{if } \lambda \notin \Delta, \\ \mathbb{1}_\lambda & \text{if } \lambda \in \Delta, \end{cases}$$

is an isomorphism of $\mathbb{P}(\Lambda)$ with $\mathscr{C}(\mathscr{L})$. Thus, since the set Λ may be considered as the phase space of a classical system [*21*], we make the sensible assumption that Λ is a Hausdorff topological space.

We recall now the notion of quantum logic observable (Q. L. observable) and related definitions [*3*]. First we notice that the Borel σ-algebra of the real line $\mathbb{B}_{\mathbb{R}}$ can be considered to be a distributive σ-complete lattice with orthocomplementation. We will denote by Hom $(\mathbb{B}_{\mathbb{R}}, \mathscr{L})$ the set of the σ-homomorphisms from $\mathbb{B}_{\mathbb{R}}$ into a croc $\mathscr{L}$, namely the set of the mappings τ from $\mathbb{B}_{\mathbb{R}}$ into $\mathscr{L}$ such that the equalities (*a*), (*b*) and (*c*) hold when α ranges in an at most countable set.

DEFINITION 5.1. — If $\mathscr{L}$ is a croc which represents a physical system, a Q. L. observable ω is an element of Hom $(\mathbb{B}_{\mathbb{R}}, \mathscr{L})$ such that $\omega(\mathbb{R}) = \mathrm{I}$.

A Q. L. observable ω is said to be discrete if an at most countable subset Δ of $\mathbb{R}$ exists such that $\omega(\Delta) = \mathrm{I}$.

A Q. L. observable ω is said to be bounded if a compact subset Δ of $\mathbb{R}$ exists such that $\omega(\Delta) = \mathrm{I}$.

A Q. L. observable ω is said to be constant if a real number k exists such that $\omega(\{ k \}) = \mathrm{I}$.

If the range Im ω of a Q. L. observable ω is contained in $\mathscr{C}(\mathscr{L})$ and ω is not constant, then ω is called Q. L. superobservable.

We can now formulate a physically plausible axiom.

AXIOM. — *For a non classical physical system, namely for a system represented by a non distributive croc, every* Q. L. *superobservable is discrete.*

From this axiom an important feature follows for the set Λ, as it is shown by the next theorem.

PROPOSITION 5.1. — *Let a physical system be represented by the croc* $\mathscr{L} = \bigvee_{\lambda\in\Lambda}^{\oplus} \mathscr{L}^{(\lambda)}$. *If every* Q. L. *superobservable is discrete then* Λ *is at most countable.*

Proof. — If Λ is not at most countable then, by the hypothesis of the continuum, a mapping f exists from Λ onto $\mathbb{R}$. Hence we can define $\omega : \mathbb{B}_{\mathbb{R}} \rightarrow \mathscr{L}, \omega = \mathscr{V} \circ f^{-1}$. Since $\mathscr{V}$ is an isomorphism of $\mathbb{P}(\Lambda)$ with $\mathscr{C}(\mathscr{L})$, it is trivial to show that ω is a Q. L. superobservable. Moreover, ω is not discrete, since the only element Δ of $\mathbb{B}_{\mathbb{R}}$ such that $\omega(\Delta) = \mathrm{I}$ is $\Delta = \mathbb{R}$. In fact, if $\Delta \neq \mathbb{R}$ then $\mathbb{R} - \Delta$ is not the empty set of $\mathbb{R}$. Hence $\omega(\mathbb{R} - \Delta)$ results to be different from $\emptyset$, whence $\omega(\Delta) \neq \mathrm{I}$.

The notion of Q. L. state has already been used in sections 2 and 3. Anyway, for the relevance it has in what follows, we recall now explicitly its definition.

DEFINITION 5.2. — If $\mathscr{L}$ is a croc which represents a physical system, a Q. L. state is a function φ defined on $\mathscr{L}$ and with range in the interval [0, 1] such that:

(a) $\varphi(\mathrm{I}) = 1$,

(b) if $\{ p_n \}$ is a sequence of elements of $\mathscr{L}$ such that $p_n \leq cp_{n'}$ for $n \neq n'$, then $\varphi\left(\bigvee_n p_n\right) = \sum_n \varphi(p_n)$.

We notice that, as a consequence of the axiom and of proposition 5.1, $\mathscr{L}$ can be written as $\mathscr{L} = \bigvee_n^{\oplus} \mathscr{L}^{(n)}$, where we denote by n the n-th index of the set Λ. For the same reason, from theorem 6.19 of [3] we get that pure states exist for a quantum system and that each of them is concentrated in a « superselected sector » $\mathscr{L}^{(n)}$.

We want now to see how a Q. L. state φ defines a Hilbert space suitable for a representation of the system. First of all, the Borel σ-algebra of Λ is the discrete σ-algebra $\mathbb{P}(\Lambda)$. In fact, since Λ is a Hausdorff space, each set which contains just one point is a Borel set. Then from proposition 5.1 it follows that each subset of Λ is a Borel set. Moreover, a Q. L. state φ defines on $\mathbb{P}(\Lambda)$ the measure

$$\mu_\varphi : \mathbb{P}(\Lambda) \to \mathbb{R}, \qquad \mu_\varphi(\mathrm{E}) = \varphi(\mathscr{V}(\mathrm{E})).$$

We make now the assumption that each Hilbert space $\mathscr{H}(n)$ is separable and complex. In fact, if the field of the Hilbert space has to contain the reals as a subfield, then we are left to choose among the reals, the complex numbers and the quaternions. Other motivations can be found to consider only these fields [22]. Moreover it has been shown that the real and the quaternionic quantum mechanics are essentially equivalent to the complex one [23]. Then we can define the direct integral of Hilbert spaces

$$\mathscr{H}_\varphi = \int_\Lambda^\oplus \mathscr{H}(n) d\mu_\varphi(n)$$

and, along with it, the algebras $\mathrm{D}(\mathscr{H}_\varphi)$ of diagonal operators and $\mathrm{R}(\mathscr{H}_\varphi)$ of decomposable operators. We shall now show that, through the Hilbert space $\mathscr{H}_\varphi$, we can define a representation of $\mathscr{L}$.

PROPOSITION 5.2. — *Let $\mathscr{L}_\varphi$ be the croc associated with the* W*-*algebra* $R(\mathscr{H}_\varphi)$, *namely* $\mathscr{L}_\varphi = R(\mathscr{H}_\varphi) \cap \mathscr{P}(\mathscr{H}_\varphi)$. *Let σ be the mapping*

$$\sigma : \mathscr{L} \to \mathscr{P}(\mathscr{H}_\varphi), \qquad \sigma(p) = \int_\Lambda^\oplus \pi^{(n)}(p) d\mu_\varphi(n) .$$

Then we have:

(1) *σ is a homomorphism from $\mathscr{L}$ into $\mathscr{P}(\mathscr{H}_\varphi)$ such that its range* Im σ *is $\mathscr{L}_\varphi$,*

(2) $\sigma(\mathscr{C}(\mathscr{L})) = \mathscr{C}(\mathscr{L}_\varphi)$,

(3) *σ is an isomorphism iff $\mathscr{C}(\mathscr{L})$ is isomorphic with $\mathscr{C}(\mathscr{L}_\varphi)$ through σ, iff $\mu_\varphi(\{ n \}) \neq 0$, $\forall n \in \Lambda$.*

Proof:

(1) Since $\pi^{(n)}$ is a homomorphism, $\forall n \in \Lambda$, for any family $\{ p_\alpha \}$ of elements of $\mathscr{L}$ we have

$$\sigma\Big(\bigwedge_\alpha p_\alpha\Big) = \int_\Lambda^\oplus \pi^{(n)}\Big(\bigwedge_\alpha p_\alpha\Big) d\mu_\varphi(n) = \int_\Lambda^\oplus \bigwedge_\alpha \pi^{(n)}(p_\alpha) d\mu_\varphi(n) ,$$

hence

$$\int_\Lambda^\oplus \bigwedge_\alpha \pi^{(n)}(p_\alpha) d\mu_\varphi(n)$$

is a projection. Moreover, for an element

$$x = \int_\Lambda^\oplus x(n) d\mu_\varphi(n)$$

we have

$$\Big(\int_\Lambda^\oplus \bigwedge_\alpha \pi^{(n)}(p_\alpha) d\mu_\varphi(n)\Big) x = x \quad \text{iff} \quad \pi^{(n)}(p_\alpha) x(n) = x(n), \quad \forall \alpha, \quad \forall n \in \Lambda - \Lambda_\varphi^0 ,$$

[where $\Lambda_\varphi^0 = \{ n \in \Lambda ; \mu_\varphi(\{ n \}) = 0 \}$] iff

$$\Big(\bigwedge_\alpha \int_\Lambda^\oplus \pi^{(n)}(p_\alpha) d\mu_\varphi(n)\Big) x = x .$$

Hence $\sigma\Big(\bigwedge_\alpha p_\alpha\Big) = \bigwedge_\alpha \sigma(p_\alpha)$. Since $\pi^{(n)}$ is a homomorphism and $\pi^{(n)}(I) = \mathbb{1}_n$, for any element p of $\mathscr{L}$ we have

$$\sigma(cp) = \int_\Lambda^\oplus \pi^{(n)}(cp) d\mu_\varphi(n) = \mathbb{1}_{\mathscr{H}_\varphi} - \int_\Lambda^\oplus \pi^{(n)}(p) d\mu_\varphi(n) = c(\sigma(p)) .$$

Now we can get easily

$$\sigma\Big(\bigvee_\alpha p_\alpha\Big) = \sigma\Big(c\Big(\bigwedge_\alpha cp_\alpha\Big)\Big) = \bigvee_\alpha \sigma(p_\alpha) .$$

This completes the proof that σ is a homomorphism. It is then trivial to show that $\operatorname{Im} \sigma = \mathscr{L}_\varphi$.

(2) Since a projection of a W*-algebra $\mathscr{A}$ commutes with any other projection of $\mathscr{A}$ iff it is an element of $\mathscr{C}(\mathscr{A})$, we get

$$\mathscr{C}(\mathscr{L}_\varphi) = \mathscr{C}(\mathrm{R}(\mathscr{H}_\varphi)) \cap \mathscr{P}(\mathscr{H}_\varphi).$$

Moreover, from the very definition of σ we get $\sigma(\mathscr{C}(\mathscr{L})) = \mathrm{D}(\mathscr{H}_\varphi) \cap \mathscr{P}(\mathscr{H}_\varphi)$. Hence we have $\sigma(\mathscr{C}(\mathscr{L})) = \mathscr{C}(\mathscr{L}_\varphi)$.

(3) Since σ is onto $\mathscr{L}_\varphi$, it is an isomorphism of $\mathscr{L}$ with $\mathscr{L}_\varphi$ iff it is one to one, namely iff $\operatorname{Ker} \sigma = \{ \emptyset \}$. It is now trivial to show that $\operatorname{Ker} \sigma = \{ \emptyset \}$ iff $\mu_\varphi(\{ n \}) \neq 0$, $\forall n \in \Lambda$. This condition in turn is easily shown to be equivalent to

$$\sigma(p) = 0_{\mathscr{H}_\varphi} \Rightarrow p = \emptyset \qquad \text{for} \qquad p \in \mathscr{C}(\mathscr{L}),$$

namely it is equivalent to the condition that $\mathscr{C}(\mathscr{L})$ be isomorphic with $\mathscr{C}(\mathscr{L}_\varphi)$ through σ.

Since σ is a homomorphism, $\mathscr{L}_\varphi$ can be thought of as a representation of $\mathscr{L}$ generated by the Q. L. state φ. Moreover from propositions 1.3 and 2.5 of [*18*] it follows that $\mathscr{L}_\varphi$ is a subcroc of $\mathscr{P}(\mathscr{H}_\varphi)$ isomorphic with a subcroc of $\mathscr{L}$. We notice also that property (2) of proposition 5.2 is not shared by any homomorphism τ between two crocs $\mathscr{L}$ and $\mathscr{L}'$, since generally $\tau(\mathscr{C}(\mathscr{L})) \subset \mathscr{C}(\tau(\mathscr{L}))$. Finally, from condition (3) of proposition 5.2 it follows that a Q. L. state φ generates a faithful representation of the physical system iff the representation of the classical part is faithful.

We will now show that $\mathscr{H}_\varphi$ supports not only a representation of the croc $\mathscr{L}$ related to the physical system, but also of Q. L. observables and Q. L. states. For the Q. L. observables we have in fact the following theorem.

PROPOSITION 5.3. — *Let φ be a* Q. L. *state for the physical system represented by the croc $\mathscr{L}$. Then, if σ is the homomorphism of proposition* 5.2, *for any* Q. L. *observable ω of the system $\sigma \circ \omega$ is a projection valued measure from $\mathbb{B}_{\mathbb{R}}$ into $\mathscr{P}(\mathscr{H}_\varphi)$ and a unique selfadjoint operator A_ω in $\mathscr{H}_\varphi$ exists such that its domain is*

$$\mathscr{D}(\mathrm{A}_\omega) = \left\{ x \in \mathscr{H}_\varphi ; \int_{\mathbb{R}} \lambda^2 d\nu_{x,x}(\lambda) < \infty \right\}$$

and

$$(x, \mathrm{A}_\omega y) = \int_{\mathbb{R}} \lambda d\nu_{x,y}(\lambda), \qquad \forall x, y \in \mathscr{H}_\varphi,$$

where $\nu_{x,y}$ is the complex measure

$$\nu_{x,y}(\Delta) = (x, \sigma \circ \omega(\Delta) y), \qquad \forall \Delta \in \mathbb{B}_{\mathbb{R}}.$$

The following relations hold:

(1) $\{ \mathrm{A}_\omega ; \omega$ Q. L. *observable* $\} = \{ \mathrm{A} ; \mathrm{A}$ *observable of* $\mathrm{R}(\mathscr{H}_\varphi) \}$,

(2) $\{ \mathrm{A}_\omega ; \omega$ Q. L. *superobservable or constant observable* $\}$
$= \{ \mathrm{A} ; \mathrm{A}$ *selfadjoint operator in* $\mathscr{H}_\varphi$ *such that* $\mathrm{A} \eta \mathrm{D}(\mathscr{H}_\varphi) \}$,

(3) $\{ A_\omega ; \omega$ Q. L. *superobservable* $\} \supset \{ A ; A$ *superobservable of* $R(\mathscr{H}_\varphi) \}$ *and the equality holds iff* σ *is an isomorphism.*

Proof. — For any Q. L. observable ω a unique selfadjoint operator A_ω exists by the spectral theorem [*24*].

(1) It is trivial that, for any Q. L. observable ω, A_ω is a selfadjoint operator such that $A_\omega \eta R(\mathscr{H}_\varphi)$. Conversely, for any selfadjoint operator A in $\mathscr{H}_\varphi$ such that $A \eta R(\mathscr{H}_\varphi)$, we shall construct a Q. L. observable ω such that $A = A_\omega$. Namely, if β is the spectral measure of A, we shall construct a Q. L. observable ω such that $\sigma \circ \omega = \beta$. First we notice that $\beta \in \mathrm{Hom}\,(\mathbb{B}_\mathbb{R}, \mathscr{L}_\varphi)$ and $\beta(\mathbb{R}) = \mathbb{1}_{\mathscr{H}_\varphi}$. Since

$$\beta(\Delta) = \int_\Lambda^\oplus \beta_\Delta^{(n)} d\mu_\varphi(n)$$

we can define, $\forall n \in \Lambda - \Lambda_\varphi^0$, the mapping

$$\beta^{(n)} : \mathbb{B}_\mathbb{R} \to \mathscr{P}(\mathscr{H}^{(n)}), \qquad \beta^{(n)}(\Delta) = \beta_\Delta^{(n)},$$

and it can be easily shown that $\beta^{(n)} \in \mathrm{Hom}\,(\mathbb{B}_\mathbb{R}, \mathscr{P}(\mathscr{H}^{(n)}))$ along the same lines as in part (1) of proposition 5.2. Moreover,

$$\beta^{(n)}(\mathbb{R}) = \mathbb{1}_n, \qquad \forall n \in \Lambda - \Lambda_\varphi^0,$$

holds since

$$\mathbb{1}_{\mathscr{H}_\varphi} = \beta(\mathbb{R}) = \int_\Lambda^\oplus \beta_\mathbb{R}^{(n)} d\mu_\varphi(n).$$

For each $n \in \Lambda_\varphi^0$ take an element $\gamma^{(n)}$ of $\mathrm{Hom}\,(\mathbb{B}_\mathbb{R}, \mathscr{P}(\mathscr{H}^{(n)}))$ such that $\gamma^{(n)}(\mathbb{R}) = \mathbb{1}_n$. We can now construct the mapping

$$\omega : \mathbb{B}_\mathbb{R} \to \mathscr{L}, \ \pi^{(n)}(\omega(\Delta)) = \begin{cases} \beta^{(n)}(\Delta) & \text{if} \quad n \in \Lambda - \Lambda_\varphi^0 \\ \gamma^{(n)}(\Delta) & \text{if} \quad n \in \Lambda_\varphi^0 . \end{cases}$$

Since $\beta^{(n)}$ and $\gamma^{(n)}$ are homomorphisms, it is trivial to show that $\omega \in \mathrm{Hom}\,(\mathbb{B}_\mathbb{R}, \mathscr{L})$. Moreover, $\pi^{(n)}(\omega(\mathbb{R})) = \mathbb{1}_n$, $\forall n \in \Lambda$, is equivalent to $\omega(\mathbb{R}) = I$. Hence ω is a Q. L. observable. Finally, for any $\Delta \in \mathbb{B}_\mathbb{R}$ we have

$$\sigma \circ \omega(\Delta) = \int_\Lambda^\oplus \pi^{(n)}(\omega(\Delta)) d\mu_\varphi(n) = \int_\Lambda^\oplus \beta_\Delta^{(n)} d\mu_\varphi(n) = \beta(\Delta),$$

since $\pi^{(n)}(\omega(\Delta)) = \beta_\Delta^{(n)}$ μ_φ-a. e.

(2) The proof runs in the same way as in part (1) with $\mathscr{L}$ replaced by $\mathscr{C}(\mathscr{L})$. To construct ω it is in fact enough to take $\gamma^{(n)}$ such that its range is contained in $\{ 0_n, \mathbb{1}_n \}$.

(3) If A is a superobservable of $R(\mathscr{H}_\varphi)$, then it is a selfadjoint operator in $\mathscr{H}_\varphi$ such that $A \eta D(\mathscr{H}_\varphi)$ and $A \neq k \mathbb{1}_{\mathscr{H}_\varphi}$, $\forall k \in \mathbb{R}$. In the same way as in part (2) we can construct a Q. L. observable ω with range in $\mathscr{C}(\mathscr{L})$, such that

$A = A_\omega$. The Q. L. observable ω is not constant, since if a real number k would exist such that $\omega(\{k\}) = I$, then $\sigma \circ \omega(\{k\}) = \mathbb{1}_{\mathscr{H}_\varphi}$, which implies $A = k\mathbb{1}_{\mathscr{H}_\varphi}$. Hence ω is a Q. L. superobservable.

Finally, if σ is an isomorphism, then we have

$$\{ A_\omega ; \omega \text{ Q. L. superobservable} \} \subset \{ A ; A \text{ superobservable of } R(\mathscr{H}_\varphi) \}$$

as, for a Q. L. superobservable ω, $A_\omega \eta D(\mathscr{H}_\varphi)$ and also $A_\omega \neq k\mathbb{1}_{\mathscr{H}_\varphi}$, $\forall k \in \mathbb{R}$. In fact, if a real number k could exist such that $A_\omega = k\mathbb{1}_{\mathscr{H}_\varphi}$, then $\sigma \circ \omega(\{k\}) = \mathbb{1}_{\mathscr{H}_\varphi}$, whence ω would be constant since σ is an isomorphism. Conversely, if A_ω has to be a superobservable of $R(\mathscr{H}_\varphi)$ for any Q. L. superobservable ω, then $\mu_\varphi(\{n\}) \neq 0$, $\forall n \in \Lambda$, must hold. For, if an index $\bar{n} \in \Lambda$ would exist such that $\mu_\varphi(\{\bar{n}\}) = 0$, we could construct a Q. L. superobservable ω such that $\omega(\mathbb{R} - \{k\}) = \mathbb{1}_{\bar{n}}$ and $\omega(\{k\}) = \bigvee_{n \neq \bar{n}} \mathbb{1}_n$ where k is a real number. Then A_ω would not result to be a superobservable of $R(\mathscr{H}_\varphi)$ since A_ω would be equal to $k\mathbb{1}_{\mathscr{H}_\varphi}$.

It should be noticed that if ω is bounded then also A_ω is bounded since in this case $\mathscr{D}(A_\omega) = \mathscr{H}_\varphi$.

We complete our discussion with the following theorem about the states.

PROPOSITION 5.4. — *If ψ is a* Q. L. *state such that $\mu_\psi \ll \mu_\varphi$, then ψ can be represented in $\mathscr{H}_\varphi$ as the restriction to $\mathscr{L}_\varphi$ of a state on the algebra of observables* $R(\mathscr{H}_\varphi)$, *namely there exists a sequence $\{x_i\} \subset \mathscr{H}_\varphi$ such that*

$$\sum_i \| x_i \|^2 = 1$$

and

$$\psi(p) = \sum_i \omega_{x_i}(\sigma(p)) \qquad \forall p \in \mathscr{L}.$$

This holds true in particular for the Q. L. *state φ.*

Proof. — For each $n \in \Lambda$ we can define the function

$$\psi_n : \mathscr{L}^{(n)} \to [0, 1], \qquad \psi_n(\pi) = \begin{cases} 0 & \text{if } \psi(\mathbb{1}_n) = 0, \\ \psi(\pi)(\psi(\mathbb{1}_n))^{-1} & \text{if } \psi(\mathbb{1}_n) \neq 0. \end{cases}$$

If it is not the null function, then ψ_n is a Q. L. state on $\mathscr{L}^{(n)} = \mathscr{P}(\mathscr{H}^{(n)})$. Hence, by Gleason's theorem, for each $n \in \Lambda$ a sequence $\{ y_i(n) \} \subset \mathscr{H}^{(n)}$ exists such that

$$\psi_n(\pi) = \sum_i (y_i(n), \pi y_i(n)), \qquad \forall \pi \in \mathscr{L}^{(n)}$$

(if ψ_n is the null function, then $y_i(n)$ results simply to be the zero vector of $\mathscr{H}^{(n)}$). We can now define a sequence $\{ x_i \}$ of vectors in $\mathscr{H}_\varphi$ taking

$$x_i = \int_\Lambda^\oplus x_i(n) d\mu_\varphi(n),$$

where $x_i(n) = y_i(n)[(d\mu_\psi/d\mu_\varphi)(n)]^{1/2}$. Then, for each element p of $\mathscr{L}$, we have

$$\begin{aligned} \psi(p) &= \psi\Big(\bigvee_n \pi^{(n)}(p)\Big) = \sum_n \psi(\pi^{(n)}(p)) \\ &= \sum_n \psi_n(\pi^{(n)}(p))\psi(\mathbb{1}_n) = \int_\Lambda \psi_n(\pi^{(n)}(p)) d\mu_\psi(n) \\ &= \int_\Lambda \sum_i (y_i(n), \pi^{(n)}(p) y_i(n)) \frac{d\mu_\psi}{d\mu_\varphi}(n) d\mu_\varphi(n) \\ &= \sum_i \int_\Lambda (x_i(n), \pi^{(n)}(p) x_i(n)) d\mu_\varphi(n) \\ &= \sum_i (x_i, \sigma(p) x_i) = \sum_i \omega_{x_i}(\sigma)p)). \end{aligned}$$

From proposition 5.4 and from part (3) of proposition 5.2 it follows that, if the homomorphism σ from $\mathscr{L}$ onto $\mathscr{L}_\varphi$ is an isomorphism, then any Q. L. state can be represented in $\mathscr{H}_\varphi$ as the restriction of a state on the algebra of observables $R(\mathscr{H}_\varphi)$. Therefore in this case the quantum logic description has a complete algebraic representation since $\mathscr{L}$ is isomorphic with $\mathscr{L}_\varphi$ and a bijection exists between the set of Q. L. observables and observables, Q. L. superobservables and superobservables, Q. L. states and states. In this sense it can be said that the algebraic scheme of sections 2, 3 and 4 has been drawn from the quantum logic approach to quantum mechanics.

APPENDIX

Let Λ be a topological space, μ and ν two measures defined on the Borel σ-algebra $\mathbb{B}$ of Λ. If $\mu \ll \nu$ then we have:

(*a*) every ν-measurable function $f : \Lambda \to \mathbb{C}$ is μ-measurable,

(*b*) for every μ-measurable function $f : \Lambda \to \mathbb{C}$ a μ-negligible element E exists in $\mathbb{B}$ such that $\chi_{\Lambda - E} f$ is a ν-measurable function.

Proof:

(*a*) If $\mathbb{B}_\mu$ is the μ-completion and $\mathbb{B}_\nu$ the ν-completion of $\mathbb{B}$, then $\mathbb{B}_\nu \subset \mathbb{B}_\mu$. In fact from $A \in \mathbb{B}_\nu$ it follows that $A_1, A_2 \in \mathbb{B}$ exist such that $A_1 \subset A \subset A_2$ and $\nu(A_2 - A_1) = 0$, whence $\mu(A_2 - A_1) = 0$ and $A \in \mathbb{B}_\mu$. Every ν-measurable function $f : \Lambda \to \mathbb{C}$ results then trivially to be μ-measurable.

(*b*) Let us consider the family $\mathcal{F} = \{ Q_h^{(k,k')} ; k, k' \in N, h \in N^+ \}$, where N and N^+ are the sets of integers and positive integers respectively and

$$Q_h^{(k,k')} = \left\{ z \in \mathbb{C} ; \frac{k}{h} < \operatorname{Re} z \leqslant \frac{k+1}{h}, \frac{k'}{h} < \operatorname{Im} z \leqslant \frac{k'+1}{h} \right\}.$$

It can be easily seen that this family generates the Borel σ-algebra $\mathbb{B}_\mathbb{C}$ of $\mathbb{C}$. From μ-measurability of f then we get that for any $Q_h^{(k,k')} \in \mathcal{F}$, $f^{-1}(Q_h^{(k,k')}) = A_h^{(k,k')} \cup E_h^{(k,k')}$, where $A_h^{(k,k')} \in \mathbb{B}$, $A_h^{(k,k')} \cap E_h^{(k,k')} = \emptyset$ and $E_h^{(k,k')}$ is μ-negligible. The set

$$E = \bigcup_{(k,k'),h} E_h^{(k,k')}$$

is an element of $\mathbb{B}$. In fact for every h we have

$$\Lambda = f^{-1}(\mathbb{C}) = f^{-1}\left(\bigcup_{(k,k')} Q_h^{(k,k')} \right) = \left(\bigcup_{(k,k')} A_h^{(k,k')} \right) \cup \left(\bigcup_{(k,k')} E_h^{(k,k')} \right);$$

moreover

$$\left(\bigcup_{(k,k')} A_h^{(k,k')} \right) \cap \left(\bigcup_{(k,k')} E_h^{(k,k')} \right) = \emptyset$$

holds because $A_h^{(k,k')} \cap E_h^{(k,k')} = \emptyset$ and from $Q_h^{(k_1,k'_1)} \cap Q_h^{(k_2,k'_2)} = \emptyset$ when $(k_1, k'_1) \neq (k_2, k'_2)$ it follows that

$$f^{-1}(Q_h^{(k_1,k'_1)}) \cap f^{-1}(Q_h^{(k_2,k'_2)}) = \emptyset .$$

Hence we get $\bigcup_{(k,k')} E_h^{(k,k')} = \Lambda - \bigcup_{(k,k')} A_h^{(k,k')}$, whence $\bigcup_{(k,k')} E_h^{(k,k')} \in \mathbb{B}$. Then

$$E = \bigcup_h \left(\bigcup_{(k,k')} E_h^{(k,k')} \right)$$

is an element of $\mathbb{B}$. Moreover E is μ-negligible since it is a countable union of μ-negligible sets. Define now the function $f' = \chi_{\Lambda - E} f$. Obviously $f' = f$ μ-a. e. Furthermore we have

$$f'^{-1}(Q_h^{(k,k')}) = \begin{cases} A_h^{(k,k')} & \text{if} \quad 0 \notin Q_h^{(k,k')}, \\ A_h^{(k,k')} \cup E & \text{if} \quad 0 \in Q_h^{(k,k')}. \end{cases}$$

In this way we have $f'^{-1}(Q_h^{(k,k')}) \in \mathbb{B}$ for any $Q_h^{(k,k')} \in \mathcal{F}$ and hence the ν-measurability of f', since the family $\mathcal{F}$ generates $\mathbb{B}_\mathbb{C}$ and $\mathbb{B} \subset \mathbb{B}_\nu$.

REFERENCES

[1] I. E. SEGAL, *Ann. Math.*, t. **48**, 1947, p. 930.
[2] J. M. JAUCH, *Helv. Phys. Acta*, t. **33**, 1960, p. 711 ; J. M. JAUCH and B. MISRA, *Ibid.*, t. **34**, 1961, p. 699.
[3] V. S. VARADARAJAN, *Geometry of Quantum Theory*, vol. I, Princeton, 1968.
[4] G. LUDWIG, *Z. Phys.*, t. **181**, 1964, p. 233.
[5] E. CHEN, *J. Math. Phys.*, t. **12**, 1971, p. 2364.
[6] M. GUENIN, *J. Math. Phys.*, t. **7**, 1966, p. 271.
[7] C. PIRON, *Helv. Phys. Acta*, t. **42**, 1969, p. 330.
[8] J. DIXMIER, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*, Paris, 1957 (première édition); 1969 (deuxième édition).
[9] K. MAURIN, *General Eigenfunction Expansions and Unitary Representations of Topological Groups*, Warszawa, 1968.
[10] J.-P. ECKMANN et M. GUENIN, *Méthodes Algébriques en Mécanique Statistique*, Berlin, 1969; R. GILES, *J. Math. Phys.*, t. **11**, 1970, p. 2139.
[11] A. M. GLEASON, *J. Rat. Mech. Anal.*, t. **6**, 1957, p. 885.
[12] G. C. WICK, A. S. WIGHTMAN and E. P. WIGNER, *Phys. Rev.*, t. **88**, 1952, p. 101.
[13] P. A. M. DIRAC, *The Principles of Quantum Mechanics*, 4th ed., Oxford, 1958.
[14] G. G. EMCH, *Rigourous Results in Non-Equilibrium Statistical Mechanics (Lectures in Theoretical Physics*, VIII *a*, Boulder, 1966).
[15] J. DIXMIER, *Les C*-algèbres et leurs représentations*, 2.5.5, Paris, 1964.
[16] J. M. JAUCH, *Foundations of Quantum Mechanics*, Reading, Massachusetts, 1968.
[17] S. P. GUDDER, *J. Math. Phys.*, t. **11**, 1970, p. 1037.
[18] F. GALLONE and A. MANIÀ, *Ann. Inst. H. Poincaré*, t. **15 A**, 1971, p. 37.
[19] C. PIRON, *Helv. Phys. Acta*, t. **37**, 1964, p. 439; Jauch's book quoted in ref. [16]; J. M. JAUCH and C. PIRON, *Ibid.*, t. **42**, 1969, p. 842.
[20] C. PIRON, *Helv. Phys. Acta*, t. **37**, 1964, p. 439; I. AMEMIYA and H. ARAKI, *Publ. Research Inst. Math. Kyoto University*, t. **A2**, 1967, p. 423.
[21] C. PIRON, *Lecture Notes of Varenna Summer School*, 1970.
[22] S. GUDDER and C. PIRON, *J. Math. Phys.*, t. **12**, 1971, p. 1583; R. CIRELLI and P. COTTA-RAMUSINO, *On the isomorphism of a « quantum logic » with the logic of the projections in a Hilbert space*, Preprint IFU M 135/FT, University of Milano.
[23] E. C. G. STUECKELBERG, *Helv. Phys. Acta*, t. **33**, 1960, p. 727; G. EMCH, *Ibid.*, t. **36**, 1963, p. 739, p. 770.
[24] V. I. SMIRNOV, *A Course of higher mathematics*, vol. V, § 192, Oxford, 1964.

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