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Section A :
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Bianchi universes admitting full groups of motions

by

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ABSTRACT. — With the use of the theory of topological groups, spatially homogeneous space-times of all the Bianchi types are investigated and their topology is determined.

RÉSUMÉ. — En utilisant la théorie des groupes topologiques on a étudié les espaces-temps spatialement homogènes de tous les types de Bianchi et on a déterminé leur topologie.

1. INTRODUCTION

At present global investigations of gravitational fields are given much attention (*see*, e. g., review of topics on space-time given by A. Lichnerowicz [1]). Strict mathematical analysis of problems in this field is very difficult, but there is an important case in which global investigation can be carried out to the last. This is the case of homogeneous gravitational fields, i. e. ones admitting transitive groups of motions. Local part of this problem has been solved by many authors (*see*, e. g., [2], chap. 4-8) and at now there is a lack of only global information about homogeneous gravitational fields.

We shall give here global investigation of gravitational fields which are homogeneous only spatially. These fields are known under the name of " Bianchi universes ". They admit groups of motions acting transitively on their space sections. If, as usually is accepted in relativistic cosmology, the whole space-time has the topology $V_4 = V_3 \times \mathbf{R}^1$, i. e. it is the topological product of the stright line $\mathbf{R}^1$ and a three-dimensional space V_3, then the topology of V_4 is completely determined by the topology of the space V_3. This is why that topology of space sections V_3 is to be investigated.

Below we give complete examination of all the Bianchi universes admitting full groups of motions.

2. GENERALITIES

2.1. Let V_4 be a Bianchi universe, V a space section of V_4, G a group of motions on V_4. Suppose that G acts transitively on V; then V may be represented as homogeneous space of G (*see*, e. g., [3], [4]) : V = G/H. H being a closed subgroup of G with the dimensionality

$$\dim H = \dim G - \dim V.$$

The usual supposition is that G is three-dimensional as well as V. Hence it follows that dim H = 0, that is H is a discrete closed subgroup of G.

The Lie group G usually in general relativity is given in terms of its Lie algebra L. There can be a number of Lie groups with the given Lie algebra L, but without loss of generality we can assume that G is simply connected (and also connected : we shall consider only connected groups G). Indeed, if G is multiply connected then G may be represented as factor-group of the simply connected group $(\tilde{G}, f)$ which covers G (*see* [4], chap. 9), $f: \tilde{G} \to G$ is covering mapping (topological homomorphism of $\tilde{G}$ onto G). It follows from this that V may be represented as homogeneous space of $\tilde{G}$: $V = G/H = \tilde{G}/\tilde{H}$, which proves our assertion. Here $\tilde{H} = f^{-1}(H)$ also is discrete.

Using of simply connected (instead of arbitrary) Lie groups is a great advantage, since every simply connected (and connected : see above) Lie group is uniquely determined by its Lie algebra.

2.2. So to classify spatially homogeneous space-times admitting a Lie algebra L of Killing vectors, one must first of all calculate the simply connected group G = G (L) with the Lie algebra L.

This can be done as follows ([4], § 56). Let

$$(2.1) \qquad \{ X_1, X_2, X_3 \}, \quad [X_i\, X_j] = c_{ij}^k\, X_k$$

be a basis and structural equations of L (by repeated indices the summation is meant over the range 1, 2, 3). Here c_{ij}^k are structural constants of L. Consider now the following system of equations

$$(2.2) \qquad \frac{dw_k^i}{dt} = \delta_k^i + c_{mn}^i\, a^m\, w_k^n, \qquad w_k^i\,(0, a) = 0$$

for unknown functions $w_k^i\,(t, a)$. The solution of this system exists and is unique for all t and a^i. Then we set $v_k^i\,(x) = w_k^i\,(1, x)$ and solve the following system :

$$v^i_j(f)\frac{\partial f^j}{\partial x^k} = v^i_k(x), \qquad f(e, y) = y \tag{2.3}$$

for unknown functions $f^i(x,y)$. There exists a neighbourhood G_0 of the origin $e = (0, 0, 0)$ in three-dimensional Euclidean space $\mathbf{R}^3$ such that the solution of the system (2.3) exists and is unique at all x, y from G_0. The topological space $G_0 \subset \mathbf{R}^3$ with the composition law according to

$$\begin{cases} (x, y) \mapsto xy = f(x, y), & x \in G_0, \quad y \in G_0, \\ (xy)^i = f^i(x^1, x^2, x^3, y^1, y^2, y^3), & 1 \leq i \leq 3 \end{cases} \tag{2.4}$$

forms a local Lie group with the Lie algebra L, coordinates x^i of $x \in G_0$ being analytic and canonical. This local group G_0 should be completed, if necessary, to obtain the connected and simply connected Lie group $G \supset G_0$.

2.3. If G is determined then one must calculate homogeneous spaces G/H representing V. For this purpose in principle any discrete closed subgroup $H \subset G$ is admissible. G/H consists of all the left cosets g H, g from G, and G acts on $V = \{ g\,H \}_{g \in G}$ as follows :

$$(g, x) \mapsto g(x) = g(a\,H) = (ga)\,H, \qquad g \in G, \qquad x = a\,H \in V. \tag{2.5}$$

The trivial subgroup $H_0 = \{ e \} \subset G$ consisting of the only unit element $e \in G$ obviously is discrete and closed. The corresponding homogeneous space $V_0 = G/H_0$ will play fundamental role in the sequel. It can be identified with G, then equation (2.5) assumes the form $g(x) = gx$. Every homogeneous space $V = G/H$ can be represented as factor-space of the space V_0 : $V = G/H = V_0/H$, H here being considered as a discrete group of isometries acting on V_0.

2.4. Introduce now Riemannian structure in homogeneous spaces G/H. V_0 always admits positively defined Riemannian metrics which are invariant under the action of G. To obtain them, one may take in the tangent space T_{x_0} of a point $x_0 \in V_0$ arbitrary positively defined scalar product $g_{x_0}(A, B)$, A and B from T_{x_0}, and then with the use of transformations from G transfer this product in the tangent space T_x of any point $x \in V_0$:

$$g_x(A, B) = g_{x_0}(f^*(A), f^*(B)), \tag{2.6}$$

where A, B are from T_x, $f \in G$ is the transformation with the property $f(x) = x_0$, the mapping $f^* : T_x \to T_{x_0}$ of the tangent spaces is induced by f (*see* [3]). When $x_0 = e$ we obtain : $f(x) = fx = e$, $f = x^{-1}$.

Let $(x^i) = (x^1, x^2, x^3)$ be a chart in $x \in V_0$. Then vectors $\partial_i = \partial/\partial x^i$, $1 \leq i \leq 3$ form a basis of T_x and the scalar product g_x can be written in coordinate form as follows :

$$(2.7)\quad \begin{cases} g_x(\mathrm{A}, \mathrm{B}) = g_x(\mathrm{A}^i \partial_i, \mathrm{B}^i \partial_i) = g_{ij}(x)\,\mathrm{A}^i \mathrm{B}^j, \\ \mathrm{A} = \mathrm{A}^i \partial_i \in \mathrm{T}_x, \qquad \mathrm{B} = \mathrm{B}^i \partial_i \in \mathrm{T}_x, \qquad g_{ij}(x) = g_x(\partial_i, \partial_j). \end{cases}$$

Equation (2.6) then assumes the form :

$$(2.8)\qquad g_{ij}(x) = g_{mn}\left[\frac{\partial g^m(x)}{\partial x^i}\,\frac{\partial g^n(x)}{\partial x^j}\right]_{g=x^{-1}}, \qquad g_{mn} = g_{mn}(e).$$

2.5. The Riemannian structure in V_0, determined by an invariant scalar product g_x, can be transferred on $V = V_0/H$ when g_x agrees with the equivalence relation " x and y lie in the same left coset from G/H " associated with H, i. e. if and only if the conditions $x \equiv x'$, $\mathrm{A} \equiv \mathrm{A}'$, $\mathrm{B} \equiv \mathrm{B}' \pmod{\mathrm{H}}$ (where x, $x \in V_0$; A, $\mathrm{B} \in \mathrm{T}_x$; A′, $\mathrm{B}' \in \mathrm{T}_{x'}$) imply that

$$(2.9)\qquad g_x(\mathrm{A}, \mathrm{B}) = g_{x'}(\mathrm{A}', \mathrm{B}').$$

To represent this criterion in a more convenient form, let us assume that A, B, (resp. A′, B′) are infinitesimal vectors with their origin in x (resp. x'). The condition $x \equiv x' \pmod{\mathrm{H}}$ means that $x\mathrm{H} = x'\mathrm{H}$ or $x' = xh$, $h \in \mathrm{H}$. Since A, B are infinitesimal with the origin in $x \in V_0$, their " ends " can be written in the form xa, xb respectively, where a, b are near e in G. Obviously, A′ and B′ have their ends in points $(xa)\,h$, $(xb)\,h$ respectively. Let us transfer points x, x' (with their small neighbourhoods) in the origin e. If the transformations x^{-1}, $(xh)^{-1}$, from G, respectively, are used for this purpose, then scalar products $g_x(\mathrm{A}, \mathrm{B})$ and $g_{x'}(\mathrm{A}', \mathrm{B}')$ will be unchanged [since transformations from G are isometries of the Riemannian space (V_0, g_x)]. The new tangent vectors A, B, A′ and B′ have their origin in $e \in V_0$ and so they may be identified with their ends :

$$\mathrm{A} = x^{-1}(xa) = a, \qquad \mathrm{B} = x^{-1}(xb) = b,$$
$$\mathrm{A}' = (xh)^{-1}(xah) = h^{-1}ah, \qquad \mathrm{B}' = (xh)^{-1}(xbh) = h^{-1}bh.$$

Therefore, the scalar products will be as follows :

$$g_x(\mathrm{A}, \mathrm{B}) = g_e(a, b), \qquad g_{x'}(\mathrm{A}', \mathrm{B}') = g_e(h^{-1}ah, h^{-1}bh)$$

and the criterion (2.9) assumes the form

$$(2.10)\qquad g_e(h^{-1}ah, h^{-1}bh) = g_e(a, b)$$

for all infinitesimal a, $b \in \mathrm{G}$ and for all $h \in \mathrm{H}$.

2.6. Equation (2.10) may be rewritten in the more correct form

$$(2.11)\qquad g_e(f_h(\mathrm{A}), f_h(\mathrm{B})) = g_e(\mathrm{A}, \mathrm{B}),$$

where A, B are any tangent vectors from T_e, f_h is the isomorphism of the tangent space T_e induced by the inner automorphism $g \mapsto h^{-1} gh$ of G.

This criterion puts heavy restrictions on elements $h \in H$ (¹). Indeed, if T_e in natural manner is identified with L, then linear transformations f_h ($h \in H$) form linear representation $\mathrm{Ad}_H = \{ f_h : L \to L \}_{h \in H}$ of the isotropy group H. Ad_H leaves the positively defined scalar product g_e invariant and so it is a subgroup of the group $O(3, \mathbf{R})$ of all the automorphisms of the metric space (T_e, g_e); in particular, $\mathrm{Ad}_H \cong HZ/H$ (Z is the centre of G, $\cong$ denotes isomorphism) is compact and so it lies in a maximal compact subgroup of $\mathrm{Ad}_H \cong G/Z$.

For example, if G is solvable and Z is connected, then G/Z is homeomorphic to the space $\mathbf{R}^n$, $n = \dim G - \dim Z$, and obviously it has no compact non-trivial subgroups (since G/Z is solvable and simply connected). Hence it follows that $HZ = Z$, i. e. $H \subset Z$. Of course, the criterion (2.11) is trivially satisfied if $h \in Z$.

3. THE BIANCHI TYPE I-VI UNIVERSES

3.1. The Lie algebra L of the Bianchi type I has the structure [5] :

$$[X_1 X_2] = [X_1 X_3] = [X_2 X_3] = 0. \tag{3.1}$$

Obviously L is commutative, and so G. Hence the criterion (2.11) is satisfied for all $h \in G$ and H can be any discrete closed subgroup of G.

The simply connected group G is known to be the 3-dimensional vector group, i. e. the set of all the 3-tuples $x = (x^1, x^2, x^3)$, x^i from $\mathbf{R}$, with the composition law

$$(x^1, x^2, x^3)(y^1, y^2, y^3) = (x^1 + y^1, x^2 + y^2, x^3 + y^3).$$

Discrete closed subgroups of G are k-dimensional ($k = 0, 1, 2, 3$) discrete lattices, that is $\mathbf{Z}$-linear combinations ($\mathbf{Z}$ is the ring of integers) of k $\mathbf{R}$-linearly independent vectors. Automorphisms of G are arbitrary linear non-degenerate transformations of coordinates x^i. With the use of these transformations any k-dimensional lattice $N_k \subset G$ can be reduced to the form

$$N_k = \{ (n_1, \ldots, n_k, 0, \ldots, 0) \}_{n_i \in \mathbf{Z}}.$$

Then G/N_k consists of the elements :

$$(x^1, \ldots, x^k, y^1, \ldots, y^{3-k}), \qquad x^i \in T = R/Z, \qquad y^j \in R.$$

(¹) These remarks are due to D. V. Alexeevsky.

and so it is homeomorphic to the topological product $\mathbf{R}^{3-k}\,\mathbf{T}^k$ of k circles $\mathbf{T}^1$ and $3-k$ stright lines $\mathbf{R}^1$.

3.2. The Lie algebra L of the Bianchi type II has the structure [5] :

$$[X_1\,X_2] = [X_1\,X_3] = 0, \qquad [X_2\,X_3] = X_1. \tag{3.2}$$

Calculate at first $v^i_k\,(x)$:

$$v^1_1 = v^2_2 = v^3_3 = 1, \qquad v^1_2 = -\frac{x^3}{2}, \qquad v^1_3 = \frac{x^2}{2}, \qquad \text{other} \quad v^i_k = 0$$

and then determine the composition law in G :

$$\begin{cases} (xy)^1 = x^1 + y^1 + \dfrac{1}{2}(x^2\,y^3 - x^3\,y^2), \\ (xy)^2 = x^2 + y^2, \\ (xy)^3 = x^3 + y^3. \end{cases} \tag{3.3}$$

G is homeomorphic to $\mathbf{R}^3$ (as well as every 3-dimensional simply connected solvable group : *see* [4], § 59).

It is easy to see that the centre Z of G consists of the 3-tuples $(x^1, 0, 0)$, $x^1 \in \mathbf{R}$, and is isomorphic to $\mathbf{R}$:

$$(x^1, 0, 0)\,(y^1, 0, 0) = (x^1 + y^1, 0, 0).$$

The factor-group G/Z is commutative and simply connected, so that it has not non-trivial compact subgroups. Hence (*see* § 2.6) $H \subset Z$. It is known that every discrete closed subgroup of $\mathbf{R}$ has the only generator; therefore, $Z \cong \mathbf{R}$ implies that H has the only generator $(c, 0, 0)$. If $c = 0$ than $H = H_0$, $V = V_0 \sim \mathbf{R}^3$, the tilda denotes homeomorphism; otherwise the element $(c, 0, 0)^{-1} = (-c, 0, 0)$ also generates H and so we can think that $c > 0$. Then we observe that the transformation

$$x^1 \to \frac{x^1}{c}, \quad x^2 \to \frac{x^2}{c^{1/2}}, \quad x^3 \to \frac{x^3}{c^{1/2}} \tag{3.4}$$

does not change the composition law (3.3) and so it is an automorphism of G. With the aid of this automorphism $(c, 0, 0)$ can be reduced to $(1, 0, 0)$ and then H will be the set of elements $(n, 0, 0)$, $n \in \mathbf{Z}$. It is immediate that $V_0/H = G/H$ consists of the cosets

$$g\,H = \{\,(g^1 + n, g^2, g^3)\}_{n \in Z}$$

and is homeomorphic to $\mathbf{R}^1\,\mathbf{R}^1\,\mathbf{T}^1 = \mathbf{R}^2\,\mathbf{T}^1$.

The same result can be obtained by means of straighforward calculation. We have

$$(3.5)\quad \begin{cases} g_e(a, b) = g_{ij}\, a^i\, b^j, \\ g_e(h^{-1}\, ah, h^{-1}\, bh) = g_{ij}\, a^i\, b^j + g_{11}\,(a^2\, h^3 - a^3\, h^2)\,(b^2\, h^3 - b^3\, h^2) \\ \qquad + g_{1i}\, a^i\,(b^2\, h^3 - b^3\, h^2) + g_{1i}\, b^i\,(a^2\, h^3 - a^3\, h^2), \end{cases}$$

and now the criterion (2.11) implies $g_{11}\,(b^2\, h^3 - b^3\, h^2) = 0$ at $a^1 \neq 0$, $a^2 = a^3 = 0$. g_{ik} is positively defined hence $g_{11} \neq 0$ and we obtain : $b^2\, h^3 - b^3\, h^2 = 0$. Here b^2 and b^3 are arbitrary, so that $h^2 = h^3 = 0$ and h has the form $h = (h^1, 0, 0)$, and so on, as above.

3.3. The Lie algebra L of the Bianchi type III has the structure

$$(3.6)\qquad [X_1\, X_2] = X_1, \qquad [X_1\, X_3] = [X_2\, X_3] = 0$$

and obviously it is the direct sum :

$$(3.7)\qquad L = L_2 \oplus L_1, \qquad L_2 = \{ X_1, X_2 \}, \qquad L_1 = \{ X_3 \}.$$

The corresponding simply connected group G is the direct product :

$$(3.8)\qquad G = G_2 \times G_1, \qquad G_2 = G\,(L_2), \qquad G_1 = G\,(L_1).$$

The centre of G is the direct product of the centres of the factors :

$$(3.9)\qquad Z\,(G) = Z\,(G_2) \times Z\,(G_1).$$

Obviously, G_1 is commutative and so $Z\,(G_1) = G_1$. Further, G_2 has the structure $[X_1\, X_2] = X_1$ and the following composition law (*see* § 2.2) :

$$(xy)^1 = \frac{x^1\, E\,(x^2) + y^1\, E\,(y^2)\, e^{-x^2}}{E\,(x^2 + y^2)},$$
$$(xy)^2 = x^2 + y^2,$$
$$E\,(x) = \frac{1 - e^{-x}}{x},$$

which by means of the transformation $x^1\, E\,(x^2) \to x^1$, $x^2 \to - x^2$ may be simplified as follows :

$$(3.10)\qquad (xy)^1 = x^1 + y^1\, e^{x^2}, \qquad (xy)^2 = x^2 + y^2.$$

It is immediately verified that the centre of G_2 is trivial : $Z\,(G_2) = \{ e \}$, hence the centre of G, $Z\,(G) = Z\,(G_1) = G_1$ coincides with G_1. Clearly, the group $G/Z \cong G_2$ has not compact non-trivial subgroups and so $H \subset Z = G_1$. G_1/H will be isomorphic to $\mathbf{R}$ or $\mathbf{T}$, hence

$$V = \frac{V_0}{H} = G_2 \times \frac{G_1}{H},$$

will be homeomorphic to $\mathbf{R}^3$ or $\mathbf{R}^2\, \mathbf{T}^1$.

3.4. Consider then the type IV, V, or VI Bianchi universes. Lie algebras are :

$$(3.11)\quad \begin{cases} \text{IV.}\ \ [X_1\, X_2] = 0, & [X_1\, X_3] = X_1, \quad [X_2\, X_3] = X_1 + X_2; \\ \text{V.}\ \ [X_1\, X_2] = 0, & [X_1\, X_3] = X_1, \quad [X_2\, X_3] = X_2; \\ \text{VI.}\ \ [X_1\, X_2] = 0, & [X_1\, X_3] = X_1, \quad [X_2\, X_3] = c\, X_2, \\ & c \neq 0, \quad c \neq 1. \end{cases}$$

The composition law in G_3 (IV) is a follows :

$$(3.12)\quad \begin{cases} (xy)^1 = \dfrac{x^1\,E(x^3) + [x^2 + y^1\,E(y^3) + y^2\,e^{-y^3} - x^3\,y^2\,E(y^3)]\,e^{-x^3}}{E(x^3 + y^3)} \\ \qquad - \dfrac{[x^2\,E(x^3) + y^2\,E(y^3)\,e^{-x^3}]\,e^{-(x^3+y^3)}}{E^2(x^3 + y^3)}, \\ (xy)^2 = \dfrac{x^2\,E(x^3) + y^2\,E(y^3)\,e^{-x^3}}{E(x^3 + y^3)}, \\ (xy)^3 = x^3 + y^3. \end{cases}$$

In G_3 (V) :

$$(3.13)\quad \begin{cases} (xy)^i = \dfrac{x^i\,E(x^3) + y^i\,E(y^3)\,e^{-x^3}}{E(x^3 + y^3)}, & i = 1, 2, \\ (xy)^3 = x^3 + y^3. \end{cases}$$

In G_3 (VI) :

$$(3.14)\quad \begin{cases} (xy)^i = \dfrac{x^i\,E(c^i\,x^3) + y^i\,E(c^i\,y^3)\,e^{-c^i x^3}}{E(c^i\,x^3 + c^i\,y^3)}, & c^1 = 1, \quad c^2 = c, \\ (xy)^3 = x^3 + y^3. \end{cases}$$

In all these cases we obtain, solving Equation (2.11) : $h = (0, 0, 0)$; hence it follows : $H = H_0$, $V = V_0 \sim G \sim \mathbf{R}^3$.

The same result may be obtained if one observes that the centres Z of these groups G are trivial : $Z = \{ e \}$. Hence it follows :

$$\mathrm{Ad}_G \cong \frac{G}{Z} \cong G.$$

Secondly, the composition laws (3.12)-(3.14) in canonical coordinates are well defined for all x^i and y^i; hence, the exponential mapping $\exp : L \to G$ in each case is one-to-one; it follows from this that every discrete one-generator subgroup of G lies on the trajectory of a one-parameter subgroup of G and so it is non-compact. Therefore, the only compact subgroup of G is the trivial one and we again obtain : $V \sim \mathbf{R}^3$.

4. THE BIANCHI TYPE VII UNIVERSES

4.1. The Lie algebra L of the Bianchi type VII has the structure [5] :

$$\left\{ \begin{array}{ll} [X_1 X_2] = 0, & [X_1 X_3] = X_2, \\ [X_2 X_3] = - X_1 + e X_2, & |e| < 2. \end{array} \right. \tag{4.1}$$

By means of the transformation

$$k\left(X_1 - \frac{e}{2} X_2\right) \to X_1, \qquad X_2 \to X_2, \qquad X_3 \to X_3, \qquad k = \left(1 - \frac{e^2}{4}\right)^{-1/2}$$

the structural equations (4.1) can be reduced to the form

$$[X_1 X_2] = 0, \qquad [X_1 X_3] = c X_1 + X_2, \qquad [X_2 X_3] = - X_1 + c X_2, \tag{4.2}$$

where the new parameter $c = - e (4 - e^2)^{-1/2}$ may take all real values.

In this case Equation (2.3) gives very complicated composition law and we shall use another method to obtain a composition law in G. Obviously, the centre of L is equal to 0, hence L has isomorphic adjoint representation $X = x^i X_i \mapsto a(x)$ in terms of matrices $a(x) = (a_i^k)$ determined by the relation $[XX_i] = a_i^k X_k$. In our case the adjoint representation is as follows :

$$a(x) = \begin{bmatrix} - cx^3 & x^3 & cx^1 - x^2 \\ - x^3 & - cx^3 & x^1 + cx^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is immediate that the linear Lie algebra of matrices $a(x)$ is isomorphic to the linear Lie algebra of complex matrices

$$\bar{a}(x) = \begin{bmatrix} - \bar{c}x^3 & \bar{c}(x^1 + ix^2) \\ 0 & 0 \end{bmatrix}, \qquad \bar{c} = c + i.$$

The exponential mapping

$$\bar{a}(x) \mapsto A(x) = \exp \bar{a}(x),$$

$$A(x) = \begin{bmatrix} e^{-\bar{c}x^3} & \bar{c}(x^1 + ix^2) E(\bar{c}x^3) \\ 0 & 1 \end{bmatrix}$$

can be used to obtain the matrix representation of G. By means of the transformation $\bar{c}(x^1 + ix^2) E(\bar{c}x^3) \to x^1 + ix^2$, $x^3 \to - x^3$ the more simple matrix representation can be obtained :

$$A(x) = \begin{bmatrix} e^{-\bar{c}x^3} & x^1 + ix^2 \\ 0 & 1 \end{bmatrix} \tag{4.3}$$

The relation $A(xy) = A(x)A(y)$ then gives the composition law in G :

$$(4.4)\qquad \begin{cases} (xy)^1 = x^1 + (y^1 \cos x^3 - y^2 \sin x^3)\, e^{cx^3}, \\ (xy)^2 = x^2 + (y^1 \sin x^3 + y^2 \cos x^3)\, e^{cx^3}, \\ (xy)^3 = x^3 + y^3. \end{cases}$$

G is solvable and so it is homeomorphic to $\mathbf{R}^3$.

4.2. Consider at first the case $c \neq 0$. The criterion (2.11) assumes the form

$$(4.5)\qquad \begin{cases} g_{ik}\, a^i\, b^k = g_{ik}\, \bar{a}^i\, \bar{b}^k, \\ \bar{a}^1 = \{[\, a^1 + (ch^1 - h^2)\, a^3] \cos h^3 \\ \qquad + [a^2 + (h^1 + ch^2)\, a^3] \sin h^3 \}\, e^{-ch^3}, \\ \bar{a}^2 = \{ - [a^1 + (ch^1 - h^2)\, a^3] \sin h^3 \\ \qquad + [a^2 + (h^1 + ch^2)\, a^3] \cos h^3 \}\, e^{-ch^3}, \\ \bar{a}^3 = a^3, \text{ and similarly } \bar{b}^i. \end{cases}$$

Set $a^1 = b^1$, $a^2 = b^2$, $a^3 = b^3 = 0$; then Equation (4.5) reads as follows :

$$(4.6)\qquad \begin{cases} g_{pq}\, a^p\, a^q = g_{pq}\, \bar{a}^p\, \bar{a}^q \qquad p, q = 1, 2), \\ \bar{a}^p\,(h^3) = \tilde{a}^p\, e^{-ch^3}, \qquad \tilde{a}^1 = a^1 \cos h^3 + a^2 \sin h^3, \\ \tilde{a} = - a^1 \sin h^3 + a^2 \cos h^3. \end{cases}$$

Of course, the criterion (2.11) also should be satisfied for all h^n, $n \in Z$ is any. Hence, h^3 in (4.6) may be replaced by nh^3 :

$$(4.7)\qquad g_{pq}\, a^p\, a^q = e^{-2cnh^3}\, g_{pq}\, \tilde{a}^p\, \tilde{a}^q.$$

Let $(a^1)^2 + (a^2)^2 = 1$; then

$$g_1 \leqq g_{pq}\, a^p\, a^q \leqq g_2, \qquad g_1 \leqq g_{pq}\, \tilde{a}^p\, \tilde{a}^p \leqq g_2,$$

where g_1 and g_2 are respectively maximal and minimal eigenvalues of the matrix (g_{pq}), $g_1 > 0$ and $g_2 > 0$. Hence it follows from (4.7) that

$$\frac{g_1}{g_2} \leqq e^{2cnh^3} \leqq \frac{g_2}{g_1}.$$

Since $c \neq 0$ and $n \in Z$ is arbitrary, this implies that $h^3 = 0$. Then Equation (4.7) assumes the form (at $a = b$) :

$$(4.8)\qquad \begin{cases} g_{ik}\, a^i\, a^k = g_{ik}\, \bar{a}^i\, a^k, \\ \bar{a}^1 = a^1 + (ch^1 - h^2)\, a^3, \qquad \bar{a}^2 = a^2 + (h^1 + ch^2)\, a^3, \qquad \bar{a}^3 = a^3. \end{cases}$$

Differentiate Equation (4.8) on a^1 and a^2 :

$$g_{11}\,(ch^1 - h^2) + g_{12}\,(h^1 + ch^2) = 0,$$
$$g_{21}\,(ch^1 - h^2) + g_{22}\,(h^1 + ch^2) = 0.$$

Here

$$\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \neq 0,$$

and consequently $ch^1 - h^2 = h^1 + ch^2 = 0$, which gives $h^1 = h^2 = 0$.

Therefore, we have obtained the following unique solution of Equation (2.11) : $h = e$; hence, $\mathrm{H} = \mathrm{H}_0$, $\mathrm{V} \sim \mathrm{G} \sim \mathbf{R}^3$.

4.3. Consider secondly the case $c = 0$. The composition law (4.4) now assumes the form

$$(4.9) \qquad \begin{cases} (xy)^1 = x^1 + y^1 \cos x^3 - y^2 \sin x^3, \\ (xy)^2 = x^2 + y^1 \sin x^3 + y^2 \cos x^3, \\ (xy)^3 = x^3 + y^3. \end{cases}$$

Equation (4.7) reads as follows :

$$(4.10) \qquad \begin{cases} g_{ik}\, a^i\, b^k = g_{ik}\, \bar{a}^i\, \bar{b}^k, \\ \bar{a}^1 = \quad (a^1 - h^2 a^3) \cos h^3 + (a^2 + h^1 a^3) \sin h^3, \\ a^2 = -(a^1 - h^2 a^3) \sin h^3 + (a^2 + h^1 a^3) \cos h^3, \\ a^3 = a^3, \quad \text{and similarly } \bar{b}^i. \end{cases}$$

The centre of G consists of the elements $(0, 0, 2\pi n)$, $n \in \mathbf{Z}$. G/Z has the composition law which differs from (4.9) only for third coordinates : $(xy)^3 \equiv x^3 + y^3 \pmod{2\pi}$. This immediately implies that G/Z possesses the maximal compact subgroup consisting of the elements $(0, 0, x^3)$, $x^2 \in \mathbf{R}/2\pi\mathbf{Z}$. Hence, solution of Equation (4.10) must be of the form $h = (0, 0, h^3)$. In view of this Equation (4.10) may be simplified :

$$(4.11) \qquad \begin{cases} g_{ik}\, a^i\, b^k = g_{ik}\, \bar{a}^i\, \bar{b}^k, \\ \bar{a}^1 = a^1 \cos h^3 + a^2 \sin h^3, \qquad \bar{a}^2 = -a^1 \sin h^3 + a^2 \cos h^3, \\ \bar{a}^3 = a^3, \quad \text{and similarly } \bar{b}^i. \end{cases}$$

Equate coefficients at $a^i\, b^k$ on left and on right hands of this equation :

$$\begin{gathered} g_{13}(1 - \cos h^3) + g_{23} \sin h^3 = 0, \\ -g_{13} \sin h^3 + g_{23}(1 - \cos h^3) = 0, \\ [(g_{11} - g_{22}) \sin h^3 + 2\, g_{12} \cos h^3] \sin h^3 = 0, \\ [(g_{11} - g_{22}) \cos h^3 - 2\, g_{12} \sin h^3] \sin h^3 = 0. \end{gathered}$$

It follows from this :

(i) if $g_{13} \neq 0$, or $g_{23} \neq 0$, then $\sin h^3 = 1 - \cos h^3 = 0$, which gives $h^3 = 2\pi n$, $n \in \mathbf{Z}$;

(ii) if $g_{13} = g_{23} = 0$, but $g_{11} \neq g_{22}$, or $g_{12} \neq 0$, then $\sin h^3 = 0$, i. e. $h^3 = \pi n$, $n \in \mathbf{Z}$;

(iii) if $g_{11} = g_{22}$, $g_{ik} = 0$ $(i \neq k)$, then h^3 may be any.

In all cases discrete non-trivial subgroup $H \subset G$ lies on the axis x^3 and so it is generated by a single element of the form $(0, 0, q)$, $q \neq 0$. The corresponding homogeneous space consists of the points (x^1, x^2, x^3), $x^1 \in \mathbf{R}$, $x^2 \in \mathbf{R}$, $x^3 \in \mathbf{R}/q\,\mathbf{Z}$, and so it is homeomorphic to $\mathbf{R}^2\, \mathbf{T}^1$.

5. THE BIANCHI TYPE VIII UNIVERSES

5.1. The Bianchi type VIII algebra L has the structure :

$$[X_1\, X_2] = X_1, \qquad [X_1\, X_3] = 2\,X_2, \qquad [X_2\, X_3] = X_3. \tag{5.1}$$

There is a number of Lie groups with the Lie algebra L. We shall proceed from the group C of unimodular real (2×2)-matrices :

$$A\,(x, y, u, v) = \begin{bmatrix} x+u & v+y \\ v-y & x-u \end{bmatrix},$$
$$\det A = x^2 + y^2 - u^2 - v^2 = 1.$$

Introduce in C new coordinates $x^1 = u$, $x^2 = v$, $x^3 = w$, where w is determined by the relations

$$x = (1 + u^2 + v^2)^{1/2} \cos w, \qquad y = (1 + u^2 + v^2)^{1/2} \sin w, \qquad 0 \leqq w < 2\pi.$$

Here $x^1 \in \mathbf{R}$, $x^2 \in \mathbf{R}$, $x^3 \in \mathbf{R}/2\,\pi\,\mathbf{Z}$, hence C is homeomorphic to

$$C \sim \mathbf{R}^1\, \mathbf{R}^1\, \mathbf{T}^1 = \mathbf{R}^2\, \mathbf{T}^1,$$

and so it is multiply connected.

The simply connected group (G, f) which covers C may be taken as the semi-direct product of C and its fundamental group

$$\pi\,(C) = \pi\,(\mathbf{R}^2\, \mathbf{T}^1) = \pi\,(\mathbf{T}^1) = \mathbf{Z},$$

i. e. as the set of all the pairs (n, A), $n \in \mathbf{Z}$, $A \in C$, with the composition law :

$$(n_1, A_1)\,(n_2, A_2) = (n_1 + n_2 + d\,(A_1, A_2),\ A_1\, A_2),$$

where the integer-valued function $d\,(A_1, A_2)$ can be defined in terms of products of continuous paths in C (*see*, for details, [4]). The covering mapping f is defined as projection $(n, A) \mapsto A$ of G onto C. Elements of G may be numbered by three parameters :

$$(n, A\,(x, y, u, v)) : \quad x^1 = u, \qquad x^2 = v, \qquad x^3 = w + 2\,\pi\, n.$$

5.2. The centre Z of G consists of the elements $(0, 0, \pi\, n)$, $n \in \mathbf{Z}$. Hence, h in Equation (2.11) must be of the form $h = (0, 0, h^3)$.

Before solving Equation (2.11) we note that automorphisms of L are arbitrary linear transformations which leave the form

$$s(x, x) = s_{ik} x^i x^k, \qquad s_{ik} = \operatorname{diag}(1, 1, -1)$$

invariant. Using these automorphisms one may simplify the scalar product in e : $g_e(x, x) = g_{ik} x^i x^k$. Indeed, we have two quadratic forms, one of them (the second form) being positively defined; hence, these forms by means of the same transformation can be reduced to diagonal form

$$s(x, x) = (x^1)^2 + (x^2)^2 - (x^3)^2, \qquad g_e(x, x) = \sum_i g_i (x^i)^2.$$

Therefore, one may think that the matrix (g_{ik}) is diagonal :

$$g_{ik} = \operatorname{diag}(g_1, g_2, g_3). \tag{5.2}$$

Then Equation (2.11) implies :

(i) if $g_1 \neq g_2$, then h^3 in (2.11) must be of the form $h^3 = \pi n$, $n \in \mathbf{Z}$;

(ii) if $g_1 = g_2$, then h^3 may be any.

Homogeneous spaces $V = G/H$ are as follows (*cf.* § 4) : $V_0 \sim \mathbf{R}^3$, $V_0/H \sim \mathbf{R}^2\, \mathbf{T}^1$ (H is non-trivial).

6. THE BIANCHI TYPE IX UNIVERSES

The Bianchi type IX Lie algebra L has the structure :

$$[X_i\, X_j] = X_k \qquad (i, j, k = 123, 231, 312). \tag{6.1}$$

Automorphisms of L are all orthogonal transformations in the linear space $L = T_e$. With the use of these transformations the scalar product g_e in T_e may be reduced to the diagonal form (5.2).

The group G of all the unimodular quaternions $a = a^0 + a^k j_k$ stands for the simply connected group with the Lie algebra L (*see* [4]). Here j_k $(k = 1, 2, 3)$ are quaternionic units. The numbers a^k can be considered as coordinates of the quaternion $a \in G$ (if $a \neq -1$).

G is homeomorphic to the sphere $\mathbf{S}^3$ and so it is compact. Hence it follows that H may in principle be any closed discrete subgroup of G. More exactly, one obtains solving Equation (2.11) :

(i) if $g_1 \neq g_2 \neq g_3$, then $h \in \{\pm 1, \pm j_1, \pm j_2, \pm j_3\}$;

(ii) if $g_1 \neq g_2 = g_3$, then h has the form $\cos v + j_1 \sin v$ or $j_2 \cos w + j_3 \sin w$, $0 \leq v, w < 2\pi$; and similarly at $g_1 = g_2 \neq g_3$ or $g_1 = g_3 \neq g_2$;

(iii) if $g_1 = g_2 = g_3$, then h may be any from G, and so H is any closed discrete subgroup of G. All the corresponding homogeneous spaces (spherical space forms) are listed in [6], § 2.7.

7. REVIEW OF RESULTS. DISCUSSION

Give list of all the results obtained. Possible topologies of the Bianchi universes admitting full groups of motions are as follows :

$$\left\{\begin{array}{c} \mathrm{I}: \quad \mathbf{R}^3,\ \mathbf{R}^2\,\mathbf{T}^1,\ \mathbf{R}^1\,\mathbf{T}^2,\ \mathbf{T}^3; \\ \mathrm{II,\ III,\ VII}\,(c=0),\ \mathrm{VIII}: \quad \mathbf{R}^3,\ \mathbf{R}^2\,\mathbf{T}^1; \\ \mathrm{IV,\ V,\ VI,\ VII}\,(c \neq 0): \quad \mathbf{R}^3, \\ \mathrm{IX}: \quad \text{all spherical space forms.} \end{array}\right. \tag{7.1}$$

Topology of the types I-VIII Bianchi universes is very simple. For such of these types, there are the universes homeomorphic to the Euclidean space $\mathbf{R}^3$. Probably, only such spaces may have physical meaning, because spaces $\mathbf{R}^a\,\mathbf{T}^b$ ($b \geq 1$, $a + b = 3$) contains the factor $\mathbf{T}^1$ which indicates the presence of loops in these spaces. The Bianchi type IX universes can have very complicated topology, but there are no physical grounds to consider multiply connected spherical space forms $\mathbf{S}^3/\mathrm{H}$; probably, only the space $\mathbf{S}^3$ may have physical meaning.

Fundamental groups of the spaces (7.1) are as follows :

$$\pi\,(\mathbf{R}^a\,\mathbf{T}^b) \cong \mathbf{Z}^b, \qquad \pi\,(\mathbf{S}^3/\mathrm{H}) \cong \mathrm{H}. \tag{7.2}$$

All the spaces (7.1) are orientable (*see* [7] about the physical meaning of this condition).

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