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Section A :
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On integrability of discrete representations of Lie algebra $u(p, q)$

by

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ABSTRACT. — It is proved that every representation of the discrete series of hermitian representations of Lie algebra $u(p, q)$ constructed by the Gel'fand-Graev method is differential of a unitary one-valued representation of Lie group $U(p, q)$.

1. INTRODUCTION

In 1965 Gel'fand and Graev [1] described a method for constructing discrete series of hermitian irreducible representations of Lie algebra $u(p, q)$, i. e. series of irreducible hermitian representations of $u(p, q)$ characterized by a finite number of integers. The question of integrability of these representations to the corresponding connected simply-connected (universal covering) Lie group of $u(p, q)$ was not discussed. Recently theorems concerning integrability criteria of representations of finite dimensional real Lie algebra appear ([2], [3]) which complete the study of Nelson [4] and give us powerful tools for proving integrability of discrete representations of $u(p, q)$.

In section 2 a brief description of the discrete series of (skew-symmetric) irreducible representations of Lie algebra $u(p, q)$ is given. Section 3 contains the proof that the discrete representations of $u(p, q)$ are integrable.

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2. DISCRETE SERIES OF REPRESENTATIONS OF $u(p, q)$

According to Gel'fand and Graev [1] a basis for the (real) Lie algebra $u(p, q)$, $p + q = n$, $p \geq q$, is given by

$$(1)\quad \begin{cases} M_{kk} = i\,A_{kk} \qquad (k = 1, 2, \ldots, n). \\ M_{jk} = i\,(A_{jk} + A_{kj}), \qquad \tilde{M}_{jk} = (A_{jk} - A_{kj}) \\ \qquad (j < k \leq p \text{ or } p < j < k), \\ N_{jk} = i\,(A_{jk} - A_{kj}), \qquad \tilde{N}_{jk} = (A_{jk} + A_{kj}) \qquad (j \leq p < k) \end{cases}$$

the commutation relations of which follow from the commutation relations of A_{jk} :

$$(2)\qquad [A_{ij}, A_{km}] = \delta_{jk}\,A_{im} - \delta_{mi}\,A_{kj}.$$

Irreducible representations of $u(p, q)$ by skew-symmetric operators are described by all inequivalent systems of operators satisfying (2) and the condition of skew-symmetricity

$$(3)\qquad \begin{cases} A^{+}_{jk} = A_{jk} \quad \text{for } j \leq p,\ k < p \text{ and } j > p,\ k > p; \\ A^{+}_{jk} = -A_{kj} \quad \text{for } j \leq p,\ k > p \text{ and } j > p,\ k \leq p \end{cases} \text{(1)}.$$

The discrete irreducible representation of $u(p, q)$, $p \geq q$, by skew symmetric operators in a Hilbert space $\mathcal{H}$ is characterized by $n = p + q$ integers $m_n = (m_{1n}, m_{2n}, \ldots, m_{nn})$, $m_{1n} \geq m_{2n} \geq \ldots \geq m_{nn}$ and by the decomposition $p = \alpha + \beta$, α, β being non-negative integers.

Any state in $\mathcal{H}$ may be written as a linear combination of basis states $|\,m\rangle$ which are mutually orthonormal and labeled by integers $m_{j,k}$, $\leq k$, satisfying the following inequalities [1] :

$$(4)\quad \begin{cases} \text{(i)} \quad m_{j,k+1} \geq m_{jk} \geq m_{j+1,k+1} \\ \qquad (j = 1, 2, \ldots, k; \\ \qquad k = 1, 2, \ldots, p-1 \text{ or } j = \alpha+1, \alpha+2, \ldots, k-\beta; \\ \qquad k = p+1, p+2, \ldots, n-1), \\ \text{(ii)} \quad m_{1k} \geq m_{1,k+1} + 1 \geq m_{2k} \geq m_{2,k+1} + 1 \geq \ldots \\ \qquad \geq m_{\alpha,k} \geq m_{\alpha,k+1} + 1 \\ \qquad (k = p, p+1, \ldots, n-1), \\ \text{(iii)} \quad m_{k-\beta+2,k+1} - 1 \geq m_{k-\beta+1,k} \geq m_{k-\beta+3,k+1} - 1 \geq \ldots \\ \qquad \geq m_{k+1,k+1} - 1 \geq m_{kk} \\ \qquad (k = p, p+1, \ldots, n-1). \end{cases}$$

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(1) Generators and their representations will be denoted by the same letters.

The basis states $|m\rangle$ may be expressed as Gel'fand-Zetlin patterns which are a geometrical transcription of the above inequalities (for more detail *see* [1]).

The action of generators of $u(p, q)$ in $\mathcal{H}$ can easily be calculated by specifying the action of A_{jk} on the basis $|m\rangle$ in $\mathcal{H}$. In fact, it is sufficient to specify the action of A_{kk}, $A_{k-1,k}$ and $A_{k,k-1}$ $(k = 1, \ldots, n)$, since the action of the other A_{jk} can be calculated by using commutation relations (2).

The action of A_{jk} on the basis in $\mathcal{H}$ is given by [1] :

$$(5)\quad \begin{cases} A_{kk} \quad |m\rangle = \left[\sum_{i=1}^{k} m_{ik} - \sum_{i=1}^{k-1} m_{ik-1}\right] |m\rangle, \\ A_{k,k-1} |m\rangle = \sum_{j=1}^{k-1} a^{j}_{k-1}(m)\, |m^{j}_{k-1} - 1\rangle, \\ A_{k-1,k} |m\rangle = \sum_{j=1}^{k-1} b^{j}_{k-1}(m)\, |m^{j}_{k-1} + 1\rangle, \end{cases}$$

where $k = 1, 2, \ldots, n$ and

$$(6)\quad \begin{cases} a^{j}_{k-1}(m) = \left[- \dfrac{\left\{ \prod_{i=1}^{k} (m_{ik} - m_{j,k-1} - i + j + 1) \times \prod_{i=1}^{k-2} (m_{i,k-2} - m_{j,k-1} - i + j) \right\}}{\left\{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} (m_{i,k-1} - m_{j,k-1} - i + j + 1) \times (m_{i,k-1} - m_{j,k-1} - i + j) \right\}} \right]^{1/2}, \\[2ex] b^{j}_{k-1}(m) = \left[- \dfrac{\left\{ \prod_{i=1}^{k} (m_{ik} - m_{j,k-1} - i + j) \times \prod_{i=1}^{k-2} (m_{i,k-2} - m_{j,k-1} - i + j - 1) \right\}}{\left\{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} (m_{i,k-1} - m_{j,k-1} - i + j) \times (m_{i,k-1} - m_{j,k-1} - i + j - 1) \right\}} \right]^{1/2}. \end{cases}$$

$| m^j_{k-1} - 1 \rangle$ and $| m^j_{k-1} + 1 \rangle$ are Gel'fand-Zetlin patterns which are obtained from $| m \rangle$ by changing there $m_{j,k-1}$ into $m_{j,k-1} - 1$ and $m_{j,k-1} + 1$ respectively.

Moreover, in order to define the action of A_{jk} uniquely we take

$$\arg a^j_{k-1} = \arg b^j_{k-1} = \begin{cases} 0 & (k \neq p+1), \\ \dfrac{\pi}{2} & (k = p+1). \end{cases}$$

3. INTEGRABILITY OF DISCRETE REPRESENTATIONS OF $u(p, q)$

First we state a result (Corollary 2) proved by Simon [3] : Let T be a representation of a real finite dimensional Lie algebra g defined on a dense domain D in a Hilbert space H, invariant under $T(g)$, by skew symmetric operators. Suppose that there exists a set of generators $\{ x_1, \ldots, x_s \}$ of g (²) such that D is a domain of analytic vectors for the operators $X_i = T(x_i)$ $(1 \leq i \leq s)$ then T is the differential (on D) of a unitary representation of the connected simply connected real Lie group G (the Lie algebra of which is g) on Hilbert space H.

Since the action of skew symmetric generators of $u(p, q)$ on an arbitrary basis vector $| m \rangle$ of $\mathcal{H}$ can be calculated by using (5) the results of Simon may be applied provided that D is considered as all finite linear combination of $| m \rangle$ and for each generator x_i $(i = 1, \ldots, s)$ from the set of generators of $u(p, q)$ any vector $| m \rangle$ is an analytic vector, i. e. for each vector $| m \rangle$ there exists $t < 0$ such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} t^n \| (X_i)^n | m \rangle \| < +\infty \qquad (i = 1, 2, \ldots, s).$$

This is equivalent to show that for each x_i and for each $| m \rangle$ there exists a constant $C > 0$ such that

$$\| (X_i)^n | m \rangle \| \leq n! \, C^n. \tag{7}$$

First let remark that the set of generators x_i of $u(p, q)$ is formed by generators M_{11}, $M_{k-1,k}$ $(k = 2, 3, \ldots, p;\ k = p+2, p+3, \ldots, p+q)$ and $N_{p,p+1}$ defined in (1) (³).

(²) A set of generators of g is a set of vectors $\{ x_1, \ldots, x_s \}$ in g such that g is generated by linear combinations of the vectors $x_1, x_2, \ldots, x_s, [x_{i_1}, x_{i_2}], [x_{i_3}, [x_{i_4}, x_{i_5}]], \ldots$ when $1 \leq i_1, i_2, \ldots \leq s$.

(³) Really, taking commutator $[M_{12}, M_{11}]$ we get $\tilde{M}_{12}$ and taking $[M_{12}, \tilde{M}_{12}]$ we obtain M_{22}. Then $[M_{23}, M_{22}]$ leads to $\tilde{M}_{23}$ and from $[M_{23}, \tilde{M}_{23}]$ we derive M_{33}, and so on. The generators N_{jk} are derived from $N_{p,p+1}$ by using commutators with $\tilde{M}_{p+1,p+2}$; $\tilde{M}_{p+2,p+3}, \ldots, \tilde{M}_{p+q-1,p+q}, \tilde{M}_{p-1,p}, \tilde{M}_{p-2,p-1}, \ldots, \tilde{M}_{1,2}$.

Thus we may distinguish three cases :

(i) M_{11} : The constant C in (7) trivially exists since

$$\| (M_{11})^n \mid m \rangle \| = (m_{11})^n.$$

(ii) $M_{k-1,k}$ ($k = 2, 3, \ldots, p$ and $k = p + 2, p + 3, \ldots, p + q$) : In this case the subspace of $\mathcal{H}$ spanned by vectors $\{ (M_{k-1,k})^n \mid m \rangle \}_{n=1}^{\infty}$, k and $\mid m \rangle$ fixed but arbitrary, are finite dimensional (generators $M_{k-1,k}$ change $k - 1$ row in $\mid m \rangle$ that for $k = 2, 3, \ldots, p$ and $k = p + 2, p + 3, \ldots, p + q$ contains m_{ik-1} ($i = 1, \ldots, k - 1$) which are bounded [*see* (1), (4), (5)] and thus C obviously exists).

(iii) $N_{p,p+1}$: In this case

$$N_{p,p+1} \mid m \rangle = i \sum_{j=1}^{p} [b_p^j(m) \mid m_p^j + 1 \rangle - a_p^j(m) \mid m_p^j - 1 \rangle].$$

Let us first consider the numbers $b_p^j(m)$. If $j \leq \alpha$:

$$\begin{aligned}(8) \quad b_p^j(m) = & \prod_{i=1}^{j-1} \left(\frac{m_{ip-1} - m_{jp} - i + j - 1}{m_{ip} - m_{jp} - i + j - 1} \right)^{1/2} \\ & \times \prod_{i=j}^{p-1} \left(\frac{m_{ip-1} - m_{jp} - i + j - 1}{m_{i+1p} - m_{jp} - (i + 1) + j} \right)^{1/2} \\ & \times \prod_{i=1}^{j-1} \left(\frac{m_{ip+1} - m_{jp} - i + j}{m_{ip} - m_{jp} - i + j} \right)^{1/2} \\ & \times \prod_{i=j}^{\alpha-1} \left(\frac{m_{ip+1} - m_{jp} - i + j}{m_{i+1p} - m_{jp} - (i + 1) + j - 1} \right)^{1/2} \\ & \times \prod_{i=\alpha+2}^{p+1} \left(\frac{m_{ip+1} - m_{jp} - i + j}{m_{i-1,p+1}\, m_{jp} - (i - 1) + j - 1} \right)^{1/2} \times \text{phase factor} \\ & \times (- (m_{\alpha,p+1} - m_{jp} - \alpha + j) \\ & \times (m_{\alpha+1,p+1} - m_{jp} - (\alpha + 1) + j))^{1/2}.\end{aligned}$$

Using the inequalities (4) one can easily show that the absolute values of all of the factors, except of the last one, are smaller or equal to 1. Therefore,

$$\begin{aligned}(9) \qquad | b_p^j(m) | \leq & | (m_{\alpha,p+1} - m_{jp} - \alpha + j) \\ & \times (m_{\alpha+1,p+1} - m_{jp} - (\alpha + 1) + j) |^{1/2} \\ \leq & (m_{1p} - m_{pp} + p) \qquad (j \leq \alpha).\end{aligned}$$

If $j > \alpha$ instead of (8) one writes

$$(8') \quad b_p^j(m) = \prod_{i=1}^{j-1} \left(\frac{m_{ip-1} - m_{jp} - i + j - 1}{m_{ip} - m_{jp} - i + j - 1} \right)^{1/2}$$
$$\times \prod_{i=j}^{p-1} \left(\frac{m_{ip-1} - m_{jp} - i + j - 1}{m_{i+1p} - m_{jp} - (i+1) + j - 1} \right)^{1/2}$$
$$\times \prod_{i=1}^{\alpha} \left(\frac{m_{ip+1} - m_{jp} - i + j}{m_{ip} - m_{jp} - i + j} \right)^{1/2}$$
$$\times \prod_{i=\alpha+3}^{j+1} \left(\frac{m_{ip+1} - m_{jp} - i + j}{m_{i-2p} - m_{jp} - (i-2) + j} \right)^{1/2}$$
$$\times \prod_{i=j+2}^{p-1} \left(\frac{m_{ip+1} - m_{jp} - i + j}{m_{i-1p} - m_{jp} - (i-1) + j} \right)^{1/2} \times \text{phase factor}$$
$$\times [- \ (m_{\alpha+1, p+1} - m_{jp} - (\alpha + 1) + j)$$
$$\times (m_{\alpha+2, p+1} - m_{jp} - (\alpha + 2) + j)]^{1/2}.$$

As before we get

$$(9') \qquad | b_p^j(m) | \geqq m_{1p} - m_{pp} + p \qquad (j > \alpha).$$

In a similar way we can show that

$$(10) \qquad a_p^j(m) \leqq m_{1\gamma} - m_{pp} + p.$$

Consequently

$$(11) \quad \| (N_{p,p+1})^n \mid m \rangle \| = \left\| \sum a_p^{j_1}(m^{(n-1)}) \, a_p^{j_2}(m^{(n-2)}) \, b_p^{j_3}(m^{(n-3)}) \dots \right.$$
$$\left. a_p^{j_{n-1}}(m^{(1)}) \, b_p^{j_n}(m^{(0)}) \mid m^{(n)} \rangle \right\|$$
$$\leqq \sum | a_p^{j_1}(m^{(n-1)}) \dots b_p^{j_n}(m^{(0)}) |$$
$$\leqq (2p)^n . \Delta (\Delta + 1) \dots (\Delta + n)$$
$$\leqq \Delta . n! \, (2p(\Delta + 1))^n$$

where $\Delta = m_{1p} - m_{pp} + p$ and the sum is over all possible combinations of three things the a_p and b_p factors and $m^{(k)}$ $(k = 1, 2, \dots, n-1)$, $m^{(0)} = m$.

Numbers $m^{(k)}$ are obtained from numbers $m^{(k-1)}$ by adding ± 1 to one of the numbers $m_{jp}^{(k-1)}$ $(j = 1, 2, \dots, p)$, i. e., $| m^k \rangle$ represents any vector in $\mathcal{H}$ which can be reached from $| m^{(k-1)} \rangle$ by acting once by operator $N_{p,p+1}$.

Thus we have proved that every basis vector $| m \rangle$ in $\mathcal{H}$ is analytic for the given set of generators of $u(p, q)$ and consequently, that every discrete skew symmetric representation of $u(p, q)$ is the differential (on D) of a unitary representation (on $\mathcal{H}$) of a connected and simply connected Lie group $\widetilde{U(p, q)}$. Since, in this unitary representation, all elements of the discrete center of $\widetilde{U(p, q)}$ are represented by the unit operator in $\mathcal{H}$ (m_{ij} are integers), the unitary representation of $\widetilde{U(p, q)}$ is a one-valued unitary representation of group $U(p, q)$.

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