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Section A :
Physique théorique.

Charged dust spheres in general relativity

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RÉSUMÉ. — Les équations d'Einstein-Maxwell sont étudiées afin d'en déduire une solution pour une distribution non-statique, en symétrie sphérique, de poussière chargée (à conductivité nulle), et la solution générale est présentée sous une formulation implicite. Les solutions sont raccordées avec la solution de Reissner-Nordström, et les équations gouvernant l'effondrement sont examinées. On utilise alors les résultats afin de déduire une solution intérieure générale pour une sphère statique de poussière chargée, en coordonnées de courbure.

1. INTRODUCTION

In 1968, A. Hamoui [1] presented two particular new solutions of the Einstein-Maxwell equations corresponding to a non-static spherically symmetric distribution of charged dust. In this paper the general solution is presented in an implicit form.

In section 2 Maxwell's equations are used to obtain an expression for the ratio of charge to mass densities. This ratio is found to be a function of the comoving coordinate r only. The ratio is then used in section 3 to express the metric coefficients in terms of the curvature distance R and three arbitrary functions of r. The equation of motion of matter is obtained and is shown to reduce to Tolman's equation [2] in the absence of charge. In section 4 the solutions are matched over a boundary to the Reissner-Nordström solution. The mass defect of a charged sphere is shown to be given by an arbitrary function of r as in the case of pure dust.

The analysis of the external gravitational field of a charged spherical body by J. Graves and D. Brill [3] showed that the Reissner-Nordström metric has an oscillatory character. The collapse of uniformly charged spheres has been investigated by V. de la Cruz and W. Israël [4] and by I. Novikov [5]. They found that after each shell of matter crossed its inner horizon it can avoid a central singularity by re-expanding into a region of space-time different from the one in which the collapse originated. In section 5, I examine the general solution to find the conditions on the parameters that will allow a gravitational bounce for a particular comoving layer.

Finally, in section 6, I use the results to obtain a general static pressure-free interior solution (*in* curvature coordinates) for a Reissner-Nordström particle. The solution includes arbitrary $\frac{e}{m}$ ratio although only solutions with $e^2 = m^2$ are free of singularities.

2. THE CHARGE TO MASS DENSITY RATIO

In comoving coordinates the most general form for a non-static line element representing a spherically symmetric distribution of matter is

$$ds^2 = e^{\gamma}\, dt^2 - e^{\alpha}\, dr^2 - R^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2.1}$$

where α, γ and R are functions only of t and of the comoving coordinate r. It will be assumed that $R(r, t) > 0$ for all $r \neq 0$.

The four velocity of matter is the unit timelike vector

$$u^a = e^{-\frac{\gamma}{2}}\, \delta^a_4. \tag{2.2}$$

The energy-momentum tensor for charged dust is

$$T^a{}_b = \rho\, u^a\, u_b + E^a{}_b, \tag{2.3}$$

where ρ is the energy density and the Maxwell tensor E_{ab} is defined in terms of the skew tensor F_{ab} by

$$4\pi E_{ab} = \frac{1}{4} g_{ab}\, F_{hk}\, F^{hk} + F_{ak}\, F^k{}_b, \tag{2.4}$$

while F_{ab} satisfies the equations (1) :

$$F_{ab,c} + F_{ca,b} + F_{bc,a} = 0 \tag{2.5}$$

and

$$F^{ka}{}_{;a} \equiv \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\, F^{ka}\right)_{,a} = 4\pi J^k, \tag{2.6}$$

(1) Commas and semi-colons denote partial and covariant differentiation, while dots and dashes denote partial differentiation with respect to t and r respectively.

where $J^k = \varepsilon u^k$ is the convection current, ε being the density of electric charge. In the case of spherical symmetry the only non-zero component of F_{ab} is $F_{14} = F_{14}(r, t)$.

$k = 1$ in equation (2.6) gives

$$F^{41} = E(r) R^{-2} e^{-\frac{\alpha+\gamma}{2}}, \tag{2.7}$$

where E is an arbitrary function of r, while $k = 4$ implies that

$$E' = 4\pi\varepsilon R^2 e^{\frac{\alpha}{2}}, \tag{2.8}$$

which expresses the conservation of charge inside a sphere comoving with the fluid.

The conservation of the energy-momentum tensor and the relationship $E^{ab}{}_{;a} = F^{bc} J_c$ give [6] :

$$(\rho u^a)_{;a} = 0 \tag{2.9}$$

and

$$\rho u^b{}_{;a} u^a = -\varepsilon F^{bc} u_c. \tag{2.10}$$

Equation (2.10) with $b = 1$ gives

$$\rho R^2 \gamma' = 2\varepsilon E(r) e^{\frac{\alpha}{2}} \tag{2.11}$$

and (2.9) gives, on integration,

$$4\pi\rho R^2 e^{\frac{\alpha}{2}} = M'. \tag{2.12}$$

M′ being an arbitrary function of r. M is then, by definition, the invariant mass contained within coordinate radius r. We now have by (2.8) and (2.12) :

$$\frac{\varepsilon}{\rho} = \frac{E'}{M'} = K(r), \tag{2.13}$$

so that the ratio of charge to mass densities is a function of r only.

3. THE FIELD EQUATIONS

The non-trivial field equations for the metric (2.1) are :

$$8\pi T_1^1 = \frac{E^2}{R^4} = \left\{\frac{2\ddot{R}}{R} - \frac{\dot{\gamma}\dot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2\right\} \times e^{-\gamma} - \left\{\frac{\gamma' R'}{R} + \left(\frac{R'}{R}\right)^2\right\} e^{-\alpha} + \frac{1}{R^2}, \tag{3.1}$$

$$(3.2)\quad 8\pi T^2_2 = 8\pi T^3_3 = -\frac{E^2}{R^4} = \left\{\frac{\ddot{\alpha}}{2} + \frac{\dot{\alpha}}{4}(\dot{\alpha} - \dot{\gamma}) + \frac{\ddot{R}}{R} + \frac{\dot{R}}{2R}(\dot{\alpha} - \dot{\gamma})\right\} e^{-\gamma}$$
$$- \left\{\frac{\gamma''}{2} + \frac{\gamma'}{4}(\gamma' - \alpha') + \frac{R''}{R} + \frac{R'}{2R}(\gamma' - \alpha')\right\} e^{-\gamma},$$

$$(3.3)\quad 8\pi T^4_4 = 8\pi\rho + \frac{E_2}{R^4}$$
$$= \left\{\frac{\dot{\alpha}\dot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2\right\} e^{-\gamma} - \left\{\frac{2R''}{R} - \frac{\alpha' R'}{R} + \left(\frac{R'}{R}\right) e^{-\alpha} + \frac{1}{R^2}\right.,$$

$$(3.4)\quad 8\pi T^1_4 = -8\pi T^4_1 = \left\{\frac{2\dot{R}'}{R} - \frac{\gamma'\dot{R} + \dot{\alpha}R'}{R}\right\} e^{-\alpha} = 0.$$

The last equation may be rewritten in the form

$$(3.5)\quad 2\dot{R}' = \gamma'\dot{R} + \dot{\alpha}R'.$$

Now, using (2.11) and (2.13), this field equation becomes

$$(3.6)\quad 2\dot{R}' = \frac{2KE}{R^2} e^{\frac{\alpha}{2}} \dot{R} + \dot{\alpha}R',$$

which may be integrated to give

$$(3.7)\quad e^{\frac{\alpha}{2}} = \frac{RR'}{\Gamma R - KE},$$

where $\Gamma(r)$ is an arbitrary function of integration. γ is then obtained from (3.7), (2.11) and (2.13) :

$$(3.8)\quad e^{\gamma} = \frac{e^{2\chi}}{R^2},$$

where

$$(3.9)\quad \chi = \int \frac{\Gamma R'}{\Gamma R - KE} dr.$$

Equation (3.1) can now be integrated to give

$$(3.10)\quad e^{-\gamma}\dot{R}^2 = \Gamma^2 - 1 + \frac{2F}{R} - \frac{(1 - K^2)E^2}{R^2},$$

where $F = F(r)$ is again a constant of integration. This equation reduces to Tolman's equation when no electric forces are present ($E = 0$) [2].

The density is given by (3.3) :

$$(3.11)\quad 4\pi\rho = \frac{(F + KE\Gamma)'}{R^2 R'} - \frac{EE'}{R^3 R'}.$$

An alternative expression for ρ may be obtained from (2.12) and (3.7) :

$$4\pi\rho = \frac{\Gamma M'}{R^2 R'} - \frac{EE'}{R^3 R'}. \tag{3.12}$$

Comparison of these equations shows that the five functions of r (F, K, E, Γ, M) are related by

$$(F + KE\,\Gamma)' = \Gamma M', \tag{3.13}$$

as well as (2.13). Thus the solutions depend on only three independent arbitrary functions of r.

The remaining field equation (3.2) is identically satisfied. In fact it can be shown in general that, given spherical symmetry and comoving coordinates, the T^2_2 equation will be automatically satisfied as a consequence of the Bianchi identities if the T^1_1, T^4_4 and T^1_4 equations are.

The cosmological constant Λ could have been included in the analysis. This would only have changed (3.10) — the term $+\dfrac{\Lambda R^2}{3}$ would have to be added to the right hand side of this equation.

4. THE BOUNDARY CONDITIONS

If we consider a charged dust sphere of radius $r = r_b =$ const. then the exterior field is given by the Reissner-Nordström metric

$$ds^2 = h\,d\bar{t}^2 - h^{-1}\,d\bar{r}^2 - \bar{r}^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{4.1}$$

with

$$h = 1 - \frac{2m}{\bar{r}} + \frac{e^2}{\bar{r}^2}, \tag{4.2}$$

where m and e are the gravitational mass and total charge of the sphere.

I use Darmois' conditions [7] at the boundary B which require that the first and second fundamental forms should be the same whether obtained from the interior or exterior metrics. Equivalence of the g_{22}'s on B imply that on B $\bar{r} = R\,(r_b, t)$ while the g_{44}'s give, on B,

$$h\dot{\bar{t}}^2 - h^{-1}\dot{\bar{r}}^2 = e^{\gamma}. \tag{4.3}$$

The equivalence of the second fundamental forms on B then gives the equation of motion of the boundary :

$$e^{-\gamma}\dot{R}^2 = -\left(1 - \frac{2m}{R} + \frac{e^2}{R^2}\right) + R'^2 e^{-\alpha}, \tag{4.4}$$

where all functions are calculated at $r = r_b$. With the help of (3.7), this can be rewritten as

$$e^{-\Upsilon}\dot{R}^2 = -\left(1 - \frac{2m}{R} + \frac{e^2}{R^2}\right) + \left(\Gamma - \frac{KE}{R}\right)^2. \tag{4.5}$$

Using (3.10) this implies that

$$e = E_{(r_b)} \tag{4.6}$$

and

$$m = (F + KE\,\Gamma)_{(r_b)}. \tag{4.7}$$

Since the gravitational mass reduces to $F_{(r_b)}$ in the cases where the dust is uncharged ($K \equiv 0$) and when the total electric charge is zero, it seems reasonable to assume that F is nonnegative as is done in section 5.

We are now in a position to interpret equation (3.13) — it gives the relationship between the active gravitational mass m and the invariant mass M :

$$m' = \Gamma M' \tag{4.8}$$

and only in the case where $\Gamma = 1$ can the two be equal. This expression also holds for uncharged dust where Γ determines not only the mass defect but also the total energy of the system and the geometry of 3-spaces $t = \text{const.}$ [8]. These interpretations are not possible here.

It can be shown that regularity [9] at the centre $r = R(0, t) = 0$ requires that $F(0) = E(0) = 0$, $\Gamma^2(0) = 1$ and $K(0) = \text{const.}$

5. GRAVITATIONAL COLLAPSE

In this section the solutions are examined to find the conditions necessary for a particular comoving particle (with $F \geqq 0$) to avoid a central singularity. The differential operator $D_t \equiv e^{-\frac{\Upsilon}{2}} \frac{\partial}{\partial t}$ is introduced so that $D_t R$ is then the proper velocity of the fluid [10]. The equation of motion for the interior (3.10) can now be rewritten as

$$R^2 (D_t R)^2 = f R^2 + 2FR - (1 - K^2) E^2, \tag{5.1}$$

where $f = \Gamma^2 - 1$. When the proper velocity in zero (5.1) will then give two positive finite roots if and only if

$$f < 0, \qquad K^2 < 1 \qquad \text{and} \qquad F^2 < f(K^2 - 1) E^2. \tag{5.2}$$

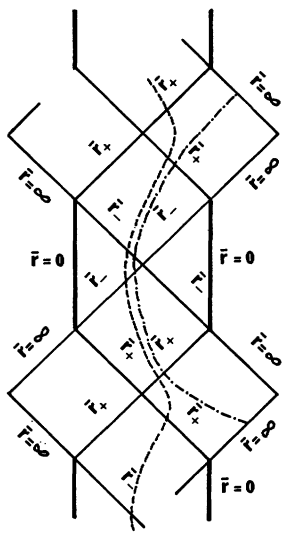
The comoving particle will then oscillate between these two values of the curvature distance. The only circumstances other than (5.2) under which the collapse of a comoving particle will not result in a singularity at $R = 0$ is when either

(5.3) $$f = 0 \quad \text{and} \quad K^2 < 1,$$

or

(5.4) $$F = 0, \quad f > 0 \quad \text{and} \quad K^2 < 1.$$

These particles, falling inward, come to rest at a certain minimum radius and then rebound out to infinity.



Cross-sections θ = const., φ = const. of the extension of the Reissner-Nordström metric in the case $0 < e^2 < m^2$. Heavy lines represent irremovable singularities. - - - - and -·-·- represent the history of oscillating and " bouncing " comoving particles respectively.

It is interesting to note that I. Novikov [5] has shown that the matter density of any uncharged layers ($K = 0$) will become infinite ($R' = 0$) during the collapse due to the crossing of dust particles.

It is well known, however, that if $e^2 \leqq m^2$ an external observer sees the surface of the sphere collapse asymptotically on to the gravitational radius $\bar{r}_+ = m + (m^2 - e^2)^{1/2}$, thus an external observer never sees the re-expanding sphere. The paradox is resolved by the extension of the Reissner-Nordström manifold given by J. Graves and D. Brill [3] for $e^2 < m^2$ and by Carter [11] for $e^2 = m^2$. After crossing its inner horizon, $\bar{r}_- = m - (m^2 - e^2)^{1/2}$, the sphere bounces and re-expands into another asymptotically flat region of space-time different from the one in which the collapse originated. This can be shown using (4.5) if $e^2 \leq m^2$. It then follows from this equation that the comoving particle (5.2) oscillates between a maximum greater than $\bar{r}_+$ and a minimum smaller than $\bar{r}_-$ while the minimum radius attained by the particles (5.2) (5.4) is less than $\bar{r}_-$ (see *fig.*). Normal oscillations are possible if $e^2 > m^2$ for then the metric (4.1) can be used throughout the whole of the exterior space time.

6. STATIC SOLUTIONS

If we consider the case in which R is a function of r only, the metric will then be static. $\ddot{R} = \dot{R} = 0$ implies that the arbitrary functions of r must further satisfy the equation $F^2 = f(K^2 - 1) E^2$. However, regularity now requires that $F \equiv 0$ and it then follows from $\ddot{R} = \dot{R} = 0$ that $K^2 = \Gamma^2 = 1$. We now have, from (4.6) and (4.7), that $\frac{e}{m} = \pm 1$ as it should be for a static solution free of singularities ([12], [13]).

A change of coordinates ($R = r$) can now be made so that the singularity-free interior solution becomes, from (3.7) and (3.9),

$$ds^2 = e^{\gamma} dt^2 - \left(1 - \frac{m(r)}{r}\right)^{-2} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2), \tag{6.1}$$

where

$$\gamma = 2 \int \frac{m(r)\, dr}{r^2 \left(1 - \frac{m}{r}\right)}, \tag{6.2}$$

$m(r)$ being an arbitrary function of r such that the Reissner-Nordström parameter is $m(r_b)$. The density of matter is given by

$$4\pi\rho = \frac{m'}{r^2}\left(1 - \frac{m}{r}\right). \tag{6.3}$$

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