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Galilei and Lorentz structures on space-time : Comparison of the corresponding geometry and physics (*)

by

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ABSTRACT. — A Galilei (Lorentz) structure on a manifold V is defined as a reduction of the bundle of linear frames to a subbundle of frames invariant under the homogeneous Galilei (Lorentz) group. Galileian or Newtonian and (general) relativistic theories are thus distinguished in a satisfactory way by nothing but the group operating at each space-time event on the admissible reference frames. This approach leads to a more unified formalism for theories in Galilei and Lorentz « invariant » versions. The existence of different types of Galilei connections is investigated and applied to a new characterization of the Newtonian gravitation theory. As another application an almost completely parallel treatment of canonical (Hamiltonian) formalisms for external forces on a particle un Galileian and Lorentzian space-time is obtained.

RÉSUMÉ. — Une structure de Galilée (Lorentz) sur une variété V est définie comme restriction du fibré principal des repères au groupe de Galilée (Lorentz) homogène. Les théories galiléennes ou newtoniennes et celles de relativité générale sont ainsi distinguées d'une façon satisfaisante par rien d'autre que le groupe qui opère en chaque point de l'espace-temps sur les référentiels admissibles. Ce point de vue mène à un formalisme plus unifié pour les versions galiléennes et lorentziennes des théories physiques. On étudie l'existence de différents types de

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connexions galiléennes, et on les utilise pour une caractérisation nouvelle de la théorie de la gravitation de Newton. Une autre application consiste en un traitement presque parallèle des formalismes canoniques (hamiltoniens) pour les forces extérieures sur une particule dans l'espace-temps galiléen ou lorentzien.

1. INTRODUCTION

Space-time formulations of Newtonian mechanics have been discussed by many authors, starting probably with Frank [8], Weyl [24] and Cartan ([2], [3]). Most of this earlier work is summarized in the review articles of Havas [10] and, in the case of continuum mechanics, of Truesdell and Toupin [23]. Toupin [20] pointed out the rôle of the different symmetry groups on the four dimensional formulation of (flat space) Galileian continuum theories. Trautman ([21], [22]) and Havas [10] stated Newton's gravitational field equations in termes of the curuature of a suitable connection in close analogy to Einstein's equations and emphasized the rôle of the Galilei and Lorentz group as asymptotical symmetry group. At about the same time Dombrowski and Horneffer ([6], [7]) introduced the concept of a general Galilei manifold and studied its differential geometry.

The purpose of this paper is to present a more uniform formalism for the two types of theories and to show from a new point of view that practically all the differences between Newtonian and Einsteinian physics — not just in the free particle case and for special relativity — can be readily inferred from differences between the structures of the homogeneous Galilei and Lorentz groups. In particular, there is no need to assume asymptotical flatness to recover these groups in a curved space-time. The Galilei and Lorentz group, respectively, are once and for all connected with the particular space-time structure by the fact that they define the admissible reference frames of an observer at any one space-time event. Several of the earlier given definitions seem less arbitrary in this approach. Einsteinian and Newtonian theories are put on a strictly equal footing with very little room for separate ad hoc assumptions in either theory.

The formalism appears to be useful for the comparison of any « non relativistic » and the corresponding « relativistic » theory, like the study of interacting particle systems, the foundations of continuum theories, coordinate invariant Newtonian limits of Einsteinian universes (1), etc. This paper treats in some detail the special conditions needed to derive

(1) Discussions of this limiting process, but with the use of special coordinate systems have been given by Friedrichs [9] and Dautcourt [5].

the classical Newtonian gravitation theory from a general Galilei space-time. These conditions can be made somewhat weaker than those imposed by Trautman [21], but one condition in addition to the analogue of Einstein's field équations is needed, due to the nonuniqueness of a symmetric Galilei connection. Only for these special symmetric Galilei connections — We call them Newtonian connections — the curvature tensor has the same number of independent components at any one point as for the Levi-Cività connection of a pseudo-Riemannian metric, before the field equations are imposed.

A second application of the formalism is made to the dynamics of a particle in an exterior force field in Galileian and Lorentzian space-times, since this may be useful as a first step to a more geometrical treatment of many particle systems. Most of these results (for the Galilei case) have already been obtained by Horneffer [11]. But though his formalism is geometrical it seems too much adapted to the Galilei structure, specifically, to admit direct comparison with the corresponding relativistic case. Here, the classical Hamiltonian formalisms for non relativistic systems and for the few known cases of Hamiltonian relativistic systems are almost completely unified. The remaining difference is that gyroscopic force terms (like the Lorentz force of a Maxwell field) can be considered as inertial forces in Galileian but not in Lorentzian space-times.

While some general terminology of the theory of G-structures (as presented in Sternberg ([19], chapter 7) is used in parts of sections 2 and 3 no familiarity with this theory is required for reading the rest of the paper, which uses tensor calculus only. Since the differential geometry of Lorentz manifolds is too well known to be repeated here most of the following discussion will be confined to Galilei structures.

2. HOMOGENEOUS GALILEI GROUP AND ALGEBRA

The *proper (inhomogeneous) Galilei group* $\tilde{\mathrm{G}}^0$ can be defined as the group of affine coordinate transformations of $\mathbf{R}^{n+1}$ of the form

$$(2.1) \qquad (x^0, x^{\mathrm{A}}) \to (\hat{x}^0, \hat{x}^{\mathrm{A}}) = (x^0 + \mathrm{b}^0, \mathrm{J}^{\mathrm{A}}_{\mathrm{B}}\, x^{\mathrm{B}} + \mathrm{a}^{\mathrm{A}}\, x^0 + \mathrm{b}^{\mathrm{A}})$$

where b^a, a^{A} and $(\mathrm{J}^{\mathrm{A}}_{\mathrm{B}}) \in \mathrm{SO}\,(n, \mathbf{R})$ are constants [2]. The *proper homogeneous Galilei group* G^0 is the subgroup of those transformations (1) that leave the origin of $\mathbf{R}^{n+1}$ invariant, that is, for which $\mathrm{b}^a = 0$. It can be considered as a subgroup of $\mathrm{GL}\,(n+1, \mathbf{R})$ regarded as group of

(2) Small Latin and small Greek indices run from 0 to n, capital Latin indices (barred or unbarred) from 1 to n.

$(n+1)\times(n+1)$-matrices operating on $\mathbf{R}^{n+1}$ on the left by

$$\mathrm{GL}\,(n+1,\mathbf{R})\times\mathbf{R}^{n+1}\to\mathbf{R}^{n+1} : ((\Lambda^k_l),(x^r))\to(\hat{x}^n=\Lambda^k_r\,x^r).$$

With respect to this action $\tilde{\mathrm{G}}^0$ is the semi-direct product $\mathbf{R}^{n+1}$ (s) G^0 (3). As a Lie group only (i. e. not as a Lie transformation group) G^0 is isomorphic to the group of n-dimensional Euclidean (rigid) motions $\mathrm{E}\,(n)=\mathbf{R}^n$ (s) $\mathrm{SO}\,(n)$.

For comparison purposes we will often refer to the *proper* (*homogeneous*) *Lorentz group* L^0 and the proper Poincaré group $\tilde{\mathrm{L}}^0=\mathbf{R}^{n+1}$ (s) L^0 where L^0 is also considered as a subtransformation group of $\mathrm{GL}\,(n+1)$ on $\mathbf{R}^{n+1}$.

If we represent the Lie algebra $\mathfrak{gl}\,(n+1)$ of $\mathrm{GL}\,(n+1)$ by the set of $(n+1)$-square matrices with the bracket operation

$$[\mathrm{A},\mathrm{B}]^k_l=\mathrm{A}^k_r\,\mathrm{B}^r_l-\mathrm{B}^k_r\,\mathrm{A}^r_l$$

then $\Omega\in\mathfrak{gl}\,(n+1)$ is in the (homogeneous) Galilei algebra $\mathfrak{g}^0$, or in the (homogeneous) Lorentz algebra $\mathfrak{l}^0$ iff

$$(2.2)\qquad \Omega^0_0=0,\qquad \Omega^0_l=\nu\,\Omega^{\mathrm{A}}_0\qquad\text{and}\qquad \Omega^{\mathrm{A}}_{\mathrm{B}}+\Omega^{\mathrm{B}}_{\mathrm{A}}=0$$

for $\nu=0$ or 1, respectively. The matrices E^l_k defined by $(\mathrm{E}^l_k)^r_s=\delta^r_k\,\delta^l_s$ form the standard basis of $\mathfrak{gl}\,(n+1)$. As a suitable basis for $\mathfrak{g}^0$ and $\mathfrak{l}^0$ we choose $\{\mathrm{E}_{\mathrm{AB}},\mathrm{F}_{\mathrm{A}}\}$ (for $1\leqq\mathrm{A}<\mathrm{B}\leqq n$, $\mathrm{E}_{\mathrm{BA}}=-\mathrm{E}_{\mathrm{AB}}$) defined by $\mathrm{E}_{\mathrm{AB}}=-\mathrm{E}^{\mathrm{B}}_{\mathrm{A}}+\mathrm{E}^{\mathrm{A}}_{\mathrm{B}}$ and $\mathrm{F}_{\mathrm{A}}=-\mathrm{E}^0_{\mathrm{A}}-\nu\,\mathrm{E}^{\mathrm{A}}_0$. The multiplication table then is

$$(2.3)\qquad \begin{cases}[\mathrm{E}_{\mathrm{AB}},\mathrm{E}_{\mathrm{CD}}]=\delta_{\mathrm{AC}}\,\mathrm{E}_{\mathrm{BD}}-\delta_{\mathrm{AD}}\,\mathrm{E}_{\mathrm{BC}}-\delta_{\mathrm{BC}}\,\mathrm{E}_{\mathrm{AD}}+\delta_{\mathrm{BD}}\,\mathrm{E}_{\mathrm{AC}},\\ [\mathrm{E}_{\mathrm{AB}},\mathrm{F}_{\mathrm{C}}]=\delta_{\mathrm{AC}}\,\mathrm{F}_{\mathrm{B}}-\delta_{\mathrm{BC}}\,\mathrm{F}_{\mathrm{A}},\\ [\mathrm{F}_{\mathrm{A}},\mathrm{F}_{\mathrm{B}}]=-\nu\,\mathrm{E}_{\mathrm{AB}}.\end{cases}$$

Most of the qualitative differences between Galileian and Lorentzian (or, more generally, pseudo-Riemannian) geometry are due to the fact that the first prolongation of $\mathfrak{so}\,(n,m)$ vanishes while it does not for the Galilei algebra $\mathfrak{g}^0$. In fact, the Galilei algebra is of infinite type. We follow here Sternberg ([19], chapter 7).

Let G be any subgroup of $\mathrm{GL}\,(n+1)$ considered as transformation group of $\mathbf{R}^{n+1}$. Then the Lie algebra $\mathfrak{g}$ is a subvector space of the set $\mathrm{Hom}\,(\mathbf{R}^{n+1},\mathbf{R}^{n+1})$ of all linear maps of $\mathbf{R}^{n+1}$ into itself. The *first prolongation* of $\mathfrak{g}$ is defined by

$$(2.4)\qquad \mathfrak{g}^{(1)}=\{\,\Omega\in\mathrm{Hom}\,(\mathbf{R}^{n+1},\mathfrak{g})/\partial\Omega=0\,\},$$

where $\partial\Omega$ is an antisymmetric bilinear map of $\mathbf{R}^{n+1}$ into $\mathbf{R}^{n+1}$ defined by $\partial\Omega\,(x\wedge y)=\Omega\,(x)\,y-\Omega\,(y)\,x,\ \ \forall\,x,y\in\mathbf{R}^{n+1}$.

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(3) *Cf.*, for example, Simms [16].

Let explicitly $\Omega(x) = \Omega^k_{lr}\, x^r\, E^l_k$, then $\Omega(x) \in \mathfrak{g}^{0(1)}$ or $\mathfrak{l}^{0(1)}$ iff

$$\Omega^k_{rs} = \Omega^k_{sr} \tag{2.5}$$

and

$$\Omega^0_{0r} = 0, \qquad \Omega^0_{Ar} = \nu\, \Omega^A_{0r}, \qquad \Omega^A_{Br} + \Omega^B_{Ar} = 0 \tag{2.6}$$

for $\nu = 0$ or 1, respectively. Now (5) and (6) imply $\Omega^k_{rs} = 0$ for $\nu = 1$ and $\Omega(x) = \varphi^A_r\, x^r\, F_A$ for $\nu = 0$ with $(\varphi^k_l) = (-\Omega^k_{l0}) \in \mathfrak{g}^0$, arbitrary. Thus the first prolongation $\mathfrak{l}^{0(1)}$ of the Lorentz algebra vanishes, while, for the Galilei algebra, $\mathfrak{g}^{0(1)}$ is isomorphic (as a vector space) to $\mathfrak{g}^0$.

The higher prolongation of a Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n+1)$ are defined by $\mathfrak{g}^{(k)} = (\mathfrak{g}^{(k-1)})^{(1)}$, explicitly,

$$\mathfrak{g}^{(k)} = \{\, \Omega \in \mathrm{Hom}(\mathbf{R}^{n+1}, \mathfrak{g}^{(k-1)}) / \Omega(x)\, y = \Omega(y)\, x,\ \forall x, y \in \mathbf{R}^{n+1} \,\}. \tag{2.7}$$

Thus if $\Omega(x) = (\Omega^r_{s_1 \ldots s_k m}\, x^m) \in \mathfrak{g}^{(k-1)}$ then $\Omega \in \mathfrak{g}^{(k)}$ iff $\Omega^r_{s_1 \ldots s_{k+1}}$ is totally symmetric in $s_1, \ldots, s_{k+1}$. A Lie group (or the corresponding Lie algebra) is called of *finite type* if there exists a k such that $\mathfrak{g}^{(k)} = 0$, otherwise it is of *infinite type*.

For $\mathfrak{g} = \mathfrak{g}^0$ we have $\Omega^0_{s_1 \ldots s_{k+1}} = 0$ ans $\Omega^A_{B s_2 \ldots s_{k+1}}$ antisymmetric in A and B. It follows that only $\Omega^A_{0 \ldots 0}$ and

$$\Omega^A_{B0 \ldots 0} = \Omega^A_{0B0 \ldots 0} = \ldots = \Omega^A_{0 \ldots 0B}$$

do not vanish. We have therefore proved

THEOREM 1. — *The homogeneous Galilei group* G^0 *is of infinite type. All prolongations* $\mathfrak{g}^{0(k)}$ *of the Lie algebra* $\mathfrak{g}^0$ *of* G^0 *are isomorphic* (*as vector spaces*) *to* $\mathfrak{g}^0$. □

In the rest of this section we review the geometrical characterization of the Galilei group as symmetry group of space-time and its operation on frames. Suppose $x = x^r e_r = \hat{x}^r \hat{e}_r \in V$ where $\{ e_r \}$ and $\{ \hat{e}_r \}$ are two bases of the $(n+1)$-dimensional vector space V. Then the linear coordinate transformation $x^k \mapsto \hat{x}^k = \Lambda^k_r x^r$ corresponds to a basis transformation

$$e_k = \hat{e}_r\, \Lambda^r_k \tag{2.8}$$

and if $\{ \theta^k \}$ is the basis of V^* dual to $\{ e_k \}$ then

$$\hat{\theta}^k = \Lambda^k_r\, \theta^r. \tag{2.9}$$

Now, for the Galilei group

$$\Lambda^0_0 = 1, \qquad \Lambda^0_B = 0, \qquad \Lambda^A_0 = a^A, \qquad (\Lambda^A_B) = (J^A_B) \in SO(n), \tag{2.10}$$

so that (8) and (9) become explicitly

$$\hat{e}_0 = e_0 - e_A (J^{-1})^A_B a^B, \tag{2.11}$$

$$\hat{e}_A = e_B (J^{-1})^B_A, \tag{2.12}$$

$$\hat{\theta}^0 = \theta^0, \tag{2.13}$$

$$\hat{\theta}^A = a^A \theta^0 + J^A_B \theta^B. \tag{2.14}$$

We see that equations (12) and (13) are particularly simple. They express that $\psi = \theta^0 \in V^*$ and $\underset{\sim}{\gamma} = \delta^{AB} e_A \otimes e_B \in V \otimes V$ (where $\delta^{AB} \equiv \delta^A_B \equiv \delta_{AB}$ is the Kronecker delta) are invariants under Galilei transformations. Conversely, these two quantities characterize (a certain action of) the proper homogeneous Galilei group (on V), namely

THEOREM 2. — *Proper homogeneous Galilei transformations of the* $(n + 1)$*-dimensional vector space* V *are characterized by leaving invariant a given linear* 1*-form* $\psi \in V^*$ *and a given positive semi-definite symmetric tensor* $\underset{\sim}{\gamma} \in V \otimes V$ *of rank n such that* ψ *is in the kernel of* $\underset{\sim}{\gamma}$, *i. e.* $\psi_r \gamma^{rs} = 0$. □

Remarks. — That this is a more convenient characterization than one using a degenerate metric (i. e. a covariant tensor) $\tilde{\gamma} \in V^* \otimes V^*$ has been recognized in many though not all previous discussions. It will greatly simplify the definition and formalism of a Galilei manifold in the next section as compared to the ones given by Dombrowski and Horneffer [7].

The full homogeneous Galilei group G (including space and time inversions) can be described similary if ψ is replaced by a symmetric 2-form Ψ of rank one satisfying $\gamma^{kr} \Psi_{rl} = 0$ (namely $\Psi = \psi \otimes \psi$). Then $\Lambda \in G$ iff $\Lambda^k_r \gamma^{rs} \Lambda^l_s = \gamma^{kl}$ and $\Lambda^r_k \Psi_{rs} \Lambda^s_l = \Psi_{kl}$ (*cf.* Havas [10]).

3. GALILEI STRUCTURES

This section treats the definition and some basic properties of Galilei structures that do not involve the use of a connection. It is a fairly straightforward application of the theory of G-structures — as in Sternberg ([19], chapter 7) where all the terminology is explained in detail — to the Galilei group. Since the latter is of infinite type theorems about local flatness do not follow from the general theory. But it turns out that they are very easily proved directly. In this sense Galilei structures are simpler (far more readily reduced to Riemannian geometry) than other related structures of infinite type, like e. g. the degenerate metric structures studied by Crampin [4].

Let V be an $(n + 1)$-dimensional C^∞-manifold, $(\mathcal{G}l(V), \pi, V)$ the principal bundle of linear frames over V. Then GL $(n + 1)$ acts freely on each fibre of $\mathcal{G}l(V)$ to the right by $\varphi : \mathrm{GL}(n+1)\times\pi^{-1}(x) \to \pi^{-1}(x)$: $((\Lambda^k_l), \{ e_r \}) \mapsto \{ \hat{e}_a = e_r \Lambda^r_a \}$ where $\{ e_r \}$ is any basis of $\mathrm{T}_x V$ and $e_r = e^a_r \partial_a$ in terms of a local coordinate system of V. A reduction $\mathcal{G}^0(V)$ of $\mathcal{G}l(V)$ to the (proper) homogeneous Galilei group G^0 is then called a *(proper) Galilei structure on V* and V together with this structure a *(proper) Galilei manifold* (4).

More explicity, $\mathcal{G}^0(V)$ is a subbundle of $\mathcal{G}l(V)$ over V such that if $p\in\mathcal{G}^0$ then $\varphi(\Lambda, p)\in\mathcal{G}^0$ iff $\Lambda\in\mathrm{G}^0$. Suppose $\mathcal{G}^0 \ni p = (x, \{ e_a \})$ and let $\{ \theta^a \}$ be the basis of $\mathrm{T}^*_x V$ dual to $\{ e_a \}$ then $\hat{p} = (x, \{ \hat{e}_a \})\in\mathcal{G}^0$ iff $\hat{e}_a = e_r \Lambda^r_a$ for $\Lambda\in\mathrm{G}^0$ or iff $\theta^a = \Lambda^a_r \hat{\theta}^r$. Thus, for a Galilei structure [by (2.10)], $\theta^0 = \hat{\theta}^0$ and $\hat{\underset{\sim}{\gamma}} = \delta^{AB}\hat{e}_A\times\hat{e}_B = \delta^{AB} e_A\times e_B = \underset{\sim}{\gamma}$. Theorem 2 leads immediately to

THEOREM 3. — *A proper Galilei structure $\mathcal{G}^0$ on V defines and is characterized by a pair $(\underset{\sim}{\gamma}, \psi)$ of a positive semi-definite contravariant symmetric tensor field $\underset{\sim}{\gamma} = \gamma^{\alpha\beta}(x)\,\partial_\alpha\times\partial_\beta$ of rank n and a never vanishing* 1-form $\psi = \psi_\alpha\, dx^\alpha$, *subject to*

$$\gamma^{\alpha\beta}\,\psi_\beta = 0. \quad \square \tag{3.1}$$

Globally a proper Galilei structure exists on a manifolds V iff there exists a time orientable and space orientable Lorentz structure. In particular, V must be orientable and non compact or have Euler characteristic zero. The proof is the same as for Lorentz structures (e. g. Steenrod [18]).

If a Galilei manifold $(V, \underset{\sim}{\gamma}, \psi)$ is given, a *Galilei frame* $\{ e_a \}$ at a point $x\in V$ is a basis of $\mathrm{T}_x V$ such that $e^\alpha_a\,\psi_\alpha = \delta^0_a$ and $\gamma^{\alpha\beta}\,\theta^a_\alpha\,\theta^b_\beta = \delta^a_A\,\delta^{AB}$. The 1-form ψ defines an n-dimensional subspace S_x of $\mathrm{T}_x V$ for each $x\in V$. A tangent vector $\mathrm{X}\in\mathrm{S}_x$, characterized by $\mathrm{X} \lrcorner\, \psi = 0$, is called *spacelike*. A vector $\mathrm{X}\in\mathrm{T}_x V$ is called *timelike* if $\mathrm{X} \lrcorner\, \psi \neq 0$, future (past) directed if $\mathrm{X} \lrcorner\, \psi \gtrless 0$, *a timelike unit vector* if $\mathrm{X} \lrcorner\, \psi = 1$.

The symmetric contravariant tensor $\gamma_x\in\mathrm{T}_x V\otimes\mathrm{T}_x V$ induces a positive definite scalar product on S_x by $(\mathrm{X}\,|\,\mathrm{X}) = \mathrm{X}^A\,\delta_{AB}\,\mathrm{X}^B$ where $\mathrm{X} = \mathrm{X}^A e_A\in\mathrm{S}_x$ and $\{ e_a \}$ is any Galilei frame at x. In particular, if the n-dimensional differential system S is integrable there is an induced Riemannian metric on each of its integral manifolds.

Every proper Galilei manifold carries a *canonical volume element* $\tau = \theta^0\wedge\theta^1\wedge\ldots\wedge\theta^n$ where $\{ e_a \}$ is any Galilei frame.

The *standard flat Galilei structure* on $V = \mathbf{R}^{n+1}$ is defined as follows in terms of the cartesian coordinates (x^0, x^A). Let $\{ e_a = e^\alpha_a\,\partial_\alpha \}$ be a

(4) Exclusion of time and space inversions means that a *proper* Galilei manifold is time and space oriented, hence oriented. The word « proper » will often be omitted in the following.

Galilei frame iff $e_a = E_r \Lambda^r_a$ where $\{E_r\}$ is the standard frame field on $\mathbf{R}^{n+1}$ [i. e. $E^\alpha_r(x) = \delta^\alpha_r$] and $(\Lambda^r_a) \in G^0$. Thus

(3.2) $\quad e^0_0 = 1, \qquad e^0_A = 0, \qquad e^A_0 = \text{const.}, \qquad (e^A_B) \in SO(n)$, const.

Then $\mathcal{G}^0(V)$ is the set of all $(x, \{e_a\})$ with $x \in \mathbf{R}^{n+1}$ and e_a satisfying (2). From (2) we find

(3.3) $$\theta^0_0 = 1, \qquad \theta^0_A = 0, \qquad \theta^A_B = e^B_A, \qquad \theta^A_0 = \theta^A_B\, e^B_0$$

whence

(3.4) $$\psi = \theta^0 = \theta^0_\alpha\, dx^\alpha = dx^0$$

and

(3.5) $$\underset{\sim}{\gamma} = \delta^{AB} e^\alpha_A e^\beta_B\, \partial_\alpha \otimes \partial_\beta = \delta^{AB}\, \partial_A \otimes \partial_B.$$

The differential system S is integrable and its maximal integral manifolds $\Sigma_t = (x^0)^{-1}(t) = \{x \in \mathbf{R}^{n+1} / x^0 = t\}$ carry the induced flat Riemannian metric

(3.6) $$\tilde{\gamma} = \delta_{AB}\, dx^A \otimes dx^B.$$

For any Lie group $G \subset GL(n+1)$ a diffeormophism $f : V \to W$ is an *isomorphism* of the G-structures $\mathcal{G}(V)$ and $\mathcal{G}(W)$ if the induced bundle map

(3.7) $$\tilde{f} : \mathcal{G}(V) \to \mathcal{G}(W) : (x, \{e_a\}) \mapsto (f(x), \{f_* e_a\})$$

(where f_* denotes the tangent map) maps G-frames into G-frames. For Galilei structures this is easily seen to be equivalent to the conditions

(3.8) $$f_* \underset{\sim}{\gamma}_V = \underset{\sim}{\gamma}_W \qquad \text{and} \qquad f^* \psi_W = \psi_V.$$

The map f is called a G-*automorphism* if $W = V$. A vector field X on V is an *infinitesimal* G-*automorphism* if it generates a local 1-parameter group of G-automorphisms. For Galilei structures this condition is equivalent to

(3.9) $$\mathcal{L}_X \underset{\sim}{\gamma} = 0 = \mathcal{L}_X \psi.$$

A G-structure is called *locally flat* if it is locally isomorphic to the standard flat G-structure.

Let H be any horizontal system on $\mathcal{G}(V)$, i. e. an $(n+1)$-dimensional differential system such that $H_p \subset T_p \mathcal{G}$ is mapped onto $T_{\pi(p)} V$ by π_* for all $p \in \mathcal{G}(V)$. Define $C_H : \mathcal{G} \to \operatorname{Hom}(\mathbf{R}^{n+1} \wedge \mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ by $C_H(x, \{e_a\})(\xi \wedge \eta) = \left\langle \underset{\sim}{\xi} \wedge \underset{\sim}{\eta} / d\tilde{\theta} \right\rangle$ where $\tilde{\theta} = (\theta^0, \theta^A)$ and $\underset{\sim}{\xi}$ is the unique vector field on $\mathcal{G}$ such that $\underset{\sim}{\xi}(p) \in H_p$ for all $p \in \mathcal{G}$ and $\underset{\sim}{\xi} \lrcorner\, \tilde{\theta} = \xi \in \mathbf{R}^{n+1}$. For any two horizontal systems H and H′ the antisymmetric linear map $C_H(p) - C_{H'}(p)$ satisfies

(3.10) $[C_H(p) - C_{H'}(p)](\xi \wedge \eta) = \partial T(\xi \wedge \eta) = T(\xi)\eta - T(\eta)\xi$

for some $T \in \mathrm{Hom}(\mathbf{R}^{n+1}, \mathfrak{g})$. Therefore the equivalence class $C(p)$ of $C_H(p) \in \mathrm{Hom}(\mathbf{R}^{n+1} \wedge \mathbf{R}^{n+1}, \mathbf{R}^{n+1})$ modulo $\partial\,\mathrm{Hom}(\mathbf{R}^{n+1}, \mathfrak{g})$ is independent of H and defines a function

$$C : \mathcal{G} \to \mathrm{Hom}(\mathbf{R}^{n+1} \wedge \mathbf{R}^{n+1}, \mathbf{R}^{n+1})/\partial\,\mathrm{Hom}(\mathbf{R}^{n+1}, \mathfrak{g}) \cong \mathfrak{g}^{(1)},$$

called the *first structure function* of $\mathcal{G}$.

The first structure function commutes with a G-automorphism f in the sense that $C_{H'} \circ \tilde{f} = C_{F}$. It always vanishes for the standard flat G-structure. Thus a necessary condition for local flatness is that $C = 0$. Such G-structures are called *first order flat* (*f. o. f*). They are particularly important because first order flatness is a necessary and sufficient condition for the existence of a torsion free connection on $\mathcal{G}(V)$. While all pseudo-Riemannian structures are f. o. f. (because $\mathfrak{g}^{(1)} = 0$), for Galilei structures this is a non trivial restriction. We have

THEOREM 4. — *A Galilei manifold is first order flat iff* $d\psi = 0$.

Proof :

(i) Let H be a horizontal system on $\mathcal{G}^0(V)$, $\xi = \xi^r E_r$ and $\eta = \eta^r E_r \in \mathbf{R}^{n+1}$. Then $\underset{\sim}{\xi} \lrcorner\, \tilde{\theta} = \xi$ implies $\underset{\sim}{\xi} = \xi^\alpha \dfrac{\partial}{\partial x^\alpha} + K^\alpha_a \dfrac{\partial}{\partial e^\alpha_a}$ with $\xi^\alpha = \xi^r e^\alpha_r$ and K^α_a depending on H. Now

$$C^a_{rs}\,\xi^r \eta^s E_a = C_H(\xi \wedge \eta) = \langle \underset{\sim}{\xi} \wedge \underset{\sim}{\eta} / d\theta^a \rangle E_a$$

implies $C^0_{kl} = 2\, e^\alpha_k\, e^\beta_l\, \partial_{[\alpha}\psi_{\beta]}$, thus $C^0_{kl} = 0$ iff $d\psi = 0$.

(ii) On the other hand $C = 0$ iff $C_H(p) = \partial T$ for some T for all choices of H. But componentwise $T = (T^k_{rs}) \in \mathrm{Hom}(\mathbf{R}^{n+1}, \mathfrak{g}^0)$ satisfies

(3.11) $$T^0_{rs} = 0 \quad \text{and} \quad T^A_{Bs} + T^B_{As} = 0$$

[*see* (2.2)]. Therefore the only nonzero components are T^A_{0s} and $T^A_{Bs} = T^B_{As}$. Now $C = 0$ iff there exists such $(T^k_{rs}) \in \mathrm{Hom}(\mathbf{R}^{n+1}, \mathfrak{g}^0)$ such that $C^k_{rs} = (\partial T)^k_{rs} = -T^k_{rs} + T^k_{sr}$. Thus $C = 0$ implies

$$C^0_{rs} = -T^0_{rs} + T^0_{sr} = 0$$

by (11).

(iii) Now suppose $C^0_{rs} = 0$, then the equations

$$C^A_{0B} = -T^A_{0B} + T^A_{B0} \quad \text{and} \quad C^A_{BC} = -T^A_{BC} + T^A_{CB}$$

have the solutions

$$T^{A}_{0B} = -\frac{1}{2}(C^{A}_{0B} + C^{B}_{0A}), \qquad T^{A}_{B0} = \frac{1}{2}(C^{A}_{0B} - C^{B}_{0A})$$

and

$$T^{A}_{BC} = \frac{1}{2}(- C^{A}_{BC} + C^{B}_{AC} + C^{C}_{AB})$$

which satisfy (11) for arbitrary C^{A}_{0B} and C^{A}_{BC}. Together with (i) this completes the proof. □

To find sufficient conditions for local flatness one would in general investigate higher order structure functions. For a group of infinite type, however, this does not necessarily lead to all needed conditions. But because the Galilei structures are so closely related to Riemannian structures satifsactory necessary and sufficient conditions can be found directly.

For any f. o. f. Galilei manifold the differential system S is integrable; we denote again its maximal connected integral manifolds by Σ_t. Then we have

THEOREM 5. — *A Galilei manifold is locally flat iff (a) it is first order flat, i. e. satisfies $d\psi = 0$, and (b) the induced Riemannian metrics on all Σ_t are locally flat.*

Proof :

(i) Local flatness implies (*a*) and (*b*) by (4) and (6), respectively.

(ii) If $d\psi = 0$ then there exists locally a function x^0 such that $\psi = dx^0$ and $\Sigma_t = \{ x^0 = t \}$ are regular hypersurfaces (since ψ never vanishes). Let $\tilde{\gamma}$ be the induced Riemannian metric. By (*b*) $(\Sigma_t \tilde{\gamma})$, is locally flat as a Riemannian manifold. Thus, for fixed t, a local coordinate system (x^A) can be found such that $\gamma_{AB} = \delta_{AB}$. Now with (x^0, x^A) as local coordinates of V (1) implies that $\gamma^{\alpha 0} = 0$ and the definition of $\tilde{\gamma}$ that (γ^{AB}) is the inverse of (γ_{AB}), i. e. $\gamma^{AB} = \delta^{AB}$, This coordinate system provides the local isomorphism with the standard flat structure (4) and (5). □

One major difference between Galilei and Lorentz structures is the maximal dimension of the group of automorphisms admitted. For pseudo-Riemannian structures, as for all G-structures of finite type this group is necessarily a finite dimensional Lie group (*cf.* Sternberg [19], p. 347). In particular, the group of automorphisms of the flat Lorentz structure is simply the inhomogeneous Lorentz group $\tilde{L}^0$, that is, the semi-direct product of $\mathbf{R}^{n+1}$ with the group L^0 defining the structure. For the Galilei structure the group of automorphisms of the flat structure is not finite dimensional, but is still easily computed and found to be what has been called the *kinematical group*, the group of *Euclidean transfor-*

mations (Toupin [20]) or the group of *non inertial Cartesian transformations* (Trautman [21]).

Let (x^α) be the standard coordinate system of $\mathbf{R}^{n+1}$ and

$$f: \mathbf{R}^{n+1} \to \mathbf{R}^{n+1} : (x^\alpha) \to (\hat{x}^\alpha)$$

a diffeomorphism. Then

$$\tilde{f}: \mathcal{G}^0 \to \mathcal{G}^0 : (x^\alpha, e^\alpha_a) \to \left(\hat{x}^\alpha, \frac{\partial \hat{x}^\alpha}{\partial x^\beta} e^\beta_a\right) \tag{3.12}$$

[see (7)] and f is a Galilei automorphism iff

$$\frac{\partial \hat{x}^\alpha}{\partial x^\beta} \mathrm{E}^\beta_a = \frac{\partial \hat{x}^\alpha}{\partial x^a} = \hat{e}^\alpha_a = (\mathrm{E}^\alpha_r \Lambda^r_a) = (\Lambda^\alpha_a) \in \mathrm{G}^0,$$

i. e. iff

$$\frac{\partial \hat{x}^0}{\partial x^0} = 1, \qquad \frac{\partial \hat{x}^0}{\partial x^{\mathrm{B}}} = 0, \qquad \left(\frac{\partial \hat{x}^{\mathrm{A}}}{\partial x^{\mathrm{B}}}\right) \in \mathrm{SO}\,(n), \tag{3.13}$$

while $\dfrac{\partial \hat{x}^{\mathrm{A}}}{\partial x^0}$ is arbitrary. There follows now easily

THEOREM 6. — *The automorphisms of the standard flat Galilei structure on* $\mathbf{R}^{n+1}$ *are given by*

$$\begin{cases} \hat{x}^0 = x^0 + e, \\ \hat{x}^{\mathrm{A}} = a^{\mathrm{A}}(x^0) + \mathrm{J}^{\mathrm{A}}_{\mathrm{B}}(x^0)\, x^{\mathrm{B}} \end{cases} \tag{3.14}$$

with $e = const. \in \mathbf{R}$ *and* $(\mathrm{J}^{\mathrm{A}}_{\mathrm{B}}) \in \mathrm{SO}\,(n)$. *The infinitesimal automorphisms are of the form*

$$\mathrm{X} = \varepsilon \frac{\partial}{\partial x^0} + [\alpha^{\mathrm{A}}(x^0) + \Omega^{\mathrm{A}}_{\mathrm{B}}(x^0)\, x^{\mathrm{B}}] \frac{\partial}{\partial x^{\mathrm{A}}} \tag{3.15}$$

with $\varepsilon = const.$ *and* $\Omega^{\mathrm{A}}_{\mathrm{B}} + \Omega^{\mathrm{B}}_{\mathrm{A}} = 0$. □

The fact that (14) is the group of automorphisms of the flat Galilei structure while the inhomogeneous Galilei group is obtained only as group of automorphisms of the flat Galilei structure together with a flat connection was pointed out by Toupin [20] (in the older terminology). For Lorentz structures these two groups of automorphisms coincide.

4. CONNECTIONS ON A GALILEI MANIFOLD

A *Galilei connection* ([5]) for a Galilei manifold V is a connection on $\mathcal{G}^0(V)$; it is characterized by its connection form $\tilde{\omega}$, a $\mathfrak{g}^0$-valued 1-form on $\mathcal{G}^0$ satisfying

$$\underset{\sim}{\mathrm{A}} \lrcorner\, \tilde{\omega} = \mathrm{A}, \qquad \forall\, \mathrm{A} \in \mathfrak{g}^0 \tag{4.1}$$

([5]) This name was used differently by Toupin [20].

and

$$\varphi_a^* \tilde{\omega} = \mathrm{ad}_{a^{-1}} \tilde{\omega}, \qquad \forall\, a \in \mathrm{G}^0 \tag{4.2}$$

where φ denotes again the action of G^0 on the principal bundle $\mathcal{G}^0$ and $\underset{\sim}{\mathrm{A}}$ is the vertical vector field on $\mathcal{G}^0$ induced by $\mathrm{A} \in \mathfrak{g}^0$ (*cf.*, for example, Kobayashi and Nomizu [12], chapter 2). It follows that an arbitrary linear connection Γ for V is a Galilei connection iff the covariant derivatives of γ and ψ vanish, i. e. if in local coordinates

$$\nabla_\alpha \gamma^{\beta\gamma} = \partial_\alpha \gamma^{\beta\gamma} + 2\,\Gamma^{(\beta}_{\alpha\rho}\,\gamma^{\gamma)\rho} = 0 \tag{4.3}$$

and

$$\nabla_\alpha \psi_\beta = \partial_\alpha \psi_\beta - \Gamma^{\rho}_{\alpha\beta}\,\psi_\rho = 0. \tag{4.4}$$

From (4) it follows that the torsion tensor $\mathrm{T}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}$ satisfies

$$\mathrm{T}^{\rho}_{\alpha\beta}\,\psi_\rho = 2\,\partial_{[\alpha}\,\psi_{\beta]}, \tag{4.5}$$

which provides a direct proof of the earlier stated fact that a Galilei manifold must satisfy $d\psi = 0$ in order to admit a torsion free connection. Unless this is the case there appears to be no particularly distinguished condition that can be imposed on the torsion. The torsion of a Galilei connection can be arbitrarily prescribed, subject to (5).

For the rest of the paper we restrict ourselves to f. o. f. Galilei manifolds with symmetric connections. Contrary to the situation for Lorentz structures equations (3) and (4) do not determine a unique symmetric connection, but an explicit particular solution can be written down in terms of an arbitrarily chosen (and fixed) timelike unit vector field $\underset{\sim}{u} = u^\alpha\,\partial_{\alpha'}$ namely [6] :

$$\Gamma^{\gamma}_{\alpha\beta} = \gamma^{\gamma\rho}\,\partial_\rho\,u^\sigma\,\gamma_{\sigma(\alpha}\,\psi_{\beta)} - \gamma_{\rho(\alpha}\,\partial_{\beta)}\,\gamma^{\rho\gamma} + \frac{1}{2}\,\gamma^{\gamma\rho}\,\partial_\rho\,\gamma^{\mu\nu}\,\gamma_{\mu\alpha}\,\gamma_{\nu\beta} - \psi_{(\alpha}\,\partial_{\beta)}\,u^\gamma \tag{4.6}$$

where $\gamma_{\alpha\beta}$ is defined by $\gamma_{\alpha\rho}\,u^\rho = 0$ and $\gamma_{\alpha\rho}\,\gamma^{\rho\beta} = \delta^\beta_\alpha - \psi_\alpha\,u^\beta = \gamma^\beta_\alpha$ (γ^β_α is the projection parallel to $\underset{\sim}{u}$ of $\mathrm{T}_x\,V$ onto S_x). Then, by the way, the integral curves of $\underset{\sim}{u}$ are geodesics with respect to Γ.

On the other hand, suppose $\mathrm{S} = \Gamma' - \Gamma$ is the difference of two symmetric Galilei connections and thus a tensor. Then the only restrictions on S are that it be symmetric in its covariant indices and that it satisfy

$$\mathrm{S}^{\rho}_{\alpha\beta}\,\psi_\rho = 0 \qquad \text{and} \qquad \mathrm{S}^{(\beta}_{\alpha\rho}\,\gamma^{\gamma)\rho} = 0. \tag{4.7}$$

The general solution of equations (7), however, can be written in the form

$$\mathrm{S}^{\gamma}_{\alpha\beta} = 2\,\psi_{(\alpha}\,\varkappa_{\beta)\rho}\,\gamma^{\rho\gamma} \tag{4.8}$$

---

[6] Γ in (6) agrees with the connection Ψ of Toupin ([20, p. 197) in the case of flat a Galilei structure.

for an arbitrary 2-form $\tilde{\varkappa} = \frac{1}{2}\varkappa_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$ on V. These remarks prove

THEOREM 7. — *The symmetric Galilei connections on a first order flat Galilei manifold V are in one-to-one correspondence with the set* $\Omega^2(V)$ *of all 2-forms on V. If* Γ *is any symmetric Galilei connection any other symmetric Galilei connection* $\bar{\Gamma}$ *is of the form*

$$(4.9) \qquad \bar{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + 2\,\psi_{(\alpha}\varkappa_{\beta)\rho}\gamma^{\rho\gamma}$$

for some $\tilde{\varkappa} \in \Omega^2(V)$. □

(*Cf.* Dombrowski and Horneffer [7]).

In any f. o. f. Galilei manifold local coordinates can be introduced such that $\psi_\alpha = \delta^0_\alpha$ and hence $\gamma^{\alpha 0} = 0$. We call this an adapted coordinate system. If, moreover, the timelike unit vector field $\underset{\sim}{u}$ is chosen to be given by $u^\alpha = \delta^\alpha_0$ (whence $\gamma_{\alpha 0} = 0$ and $\gamma_{AB}\gamma^{BC} = \delta^C_A$) then the most general connection has according to (6) and (9) the components

$$(4.10) \quad \begin{cases} \Gamma^0_{\alpha\beta} = 0, \qquad \Gamma^A_{00} = 2\,\gamma^{AC}\varkappa_{0C}, \qquad \Gamma^A_{0B} = \left(\varkappa_{BC} - \frac{1}{2}\partial_0\gamma_{BC}\right)\gamma^{CA}, \\ \Gamma^A_{BC} = \frac{1}{2}\gamma^{AD}\left(-\partial_D\gamma_{BC} + 2\,\partial_{(B}\gamma_{C)D}\right). \end{cases}$$

Noting that γ_{AB} are just the local components of the induced Riemannian metric on Σ_t in this adapted coordinate system, one derives easily (*cf.* [13]).

THEOREM 8. — *The submanifolds* Σ_t *are autoparallel (hence totally geodesic) and the connection induced on* Σ_t *by any symmectric Galilei connection on V coincides with the Levi-Cività connection of the induced Riemannian metric.* □

The curvature tensor of a symmetric Galilei connection satisfies, in addition to the Ricci and Bianchi identities,

$$(4.11) \qquad R^\alpha{}_{[\beta\gamma\delta]} = 0 = R^\alpha{}_{\beta[\gamma\delta/\lambda]},$$

the additional relations

$$(4.12) \qquad \psi_\rho R^\rho{}_{\alpha\beta\gamma} = 0$$

and

$$(4.13) \qquad R^{(\alpha\beta)}{}_{\gamma\delta} = \gamma^{\rho(\beta} R^{\alpha)}{}_{\rho\gamma\delta} = 0$$

which follow from (3) and (4) by covariant differentiation and antisymmetrization. The Ricci tensor $R_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta}$ is symmetric as a consequence of (12) and (13).

If these relations are written in an adapted coordinate system it is not hard to count the independent components of the curvature tensor at any one point. There are $(1/12)\, n^2\, (n + 1)\, (n + 5)$ of them, which is $(1/6)\, n\, (n^2 - 1)$ more than for the curvature tensor of an $(n + 1)$-dimensional pseudo-Riemannian manifold.

This indicates that a generalized Newtonian gravitation theory should probably not be formulated in terms of an arbitrary Galilei connection on space-time. Rather, if the field equations are to resemble those of General Relativity on Lorentz manifolds, then the connections on the Galilei manifold must be restricted *a priori*. A useful restriction (imposing the missing symmetries on the curvature tensor) seems to be the following.

Call a symmetric Galilei connection *Newtonian* (and the Galilei manifold together with such a connection a *Newtonian manifold*) if the curvature satisfies

$$\text{(4.14)} \qquad R^{[\alpha}{}_{(\beta}{}^{\gamma]}{}_{\delta)} = \gamma^{\rho[\alpha} R^{\gamma]}{}_{(\beta\delta)\rho} = 0.$$

Equations (14) together with the Ricci identities imply that

$$\text{(4.15)} \qquad R^{[\alpha\beta\gamma]}{}_{\delta} = 0$$

and if these conditions are applied to the components of the curvature tensor in an adapted coordinate system there remain just $(1/12)\,(n + 1)^2\,[(n + 1)^2 - 1]$ independent components at a point as in the pseudo-Riemannian case.

Conditions (14) in terms of Γ read

$$\gamma^{\rho[\alpha}\, \partial_{(\gamma}\, \Gamma^{\beta]}_{\delta)\rho} - \gamma^{\rho[\alpha}\, \partial_{\rho}\, \Gamma^{\beta]}_{\gamma\rho} + \gamma^{\rho[\alpha}\, \Gamma^{\beta]}_{\sigma(\gamma}\, \Gamma^{\sigma}_{\delta)\rho} - \gamma^{\rho[\alpha}\, \Gamma^{\beta]}_{\rho\sigma}\, \Gamma^{\sigma}_{\gamma\delta} = 0$$

which with (10) becomes in an adapted coordinate system

$$\text{(4.16)} \qquad \partial_0\, \varkappa_{AB} + 2\, \partial_{[A}\, \varkappa_{B]0} = 0 = \partial_{[A}\, \varkappa_{BC]},$$

that is

$$\text{(4.17)} \qquad \partial_{[\alpha}\, \varkappa_{\beta\gamma]} = 0 \qquad \text{or} \qquad d\tilde{\varkappa} = 0.$$

Thus

THEOREM 9. — *On any first order flat Galilei manifold V there exists a Newtonian connection. The set of Newtonian connections is in one-to-one correspondence with the set of closed 2-forms on V.*

Proof. — Choose a timelike unit vector field $\underset{\sim}{u}$ and the connection Γ defined by (6) in terms of $\underset{\sim}{u}$. (Clearly if $\underset{\sim}{u}$ is given on all of V then Γ is globally defined although (6) was formulated in terms of local coordinates.) Then, in an adapted coordinate system Γ has the form (10) with $\tilde{\varkappa} = 0$. In particular, (17) is satisfied. This proves existence.

All other Newtonian connections are obtained by adding to Γ a tensor S of the form (8) whit $\tilde{\alpha}$ satisfying (17). □

In particular, the special Galilei connection associated to a timelike unit vector field by (6) is thus Newtonian. This result in a somewhat different formulation is also due to Dombrowski and Horneffer [7].

The rest of this section is devoted to a comparison of Einstein's and Newton's gravitation theories. If the field equations of a Galilei covariant gravitation theory are formulated on a general four dimensional Newtonian manifold as closely analogous to Einstein's equations as possible then the classical Newtonian gravitational field results, at least if two additional, physical assumptions are made. The first is, that in the Newtonian theory only the *matter density* ρ, not, however, the energy density and stresses act a source of the gravitational field (i. e. the connection). Secondly, an asymptotic or global condition seems necessary.

One adopts the field equations

$$R_{\alpha\beta} = 4\,\pi\rho\psi_\alpha\,\psi_\beta \tag{4.18}$$

$\left[\text{which are equivalent to } G_{\alpha\beta} = 4\,\pi\rho\psi_\alpha\,\psi_\beta \text{ if } G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}(\gamma^{\rho\sigma}\,R_{\rho\sigma})\,\psi_\alpha\,\psi_\beta\right]$
and has

THEOREM 10. — *A four dimensional Newtonian manifold V satisfying equations* (18) *for a given (scalar) matter density* ρ *is a locally flat Galilei manifold. The connection* Γ *interpreted as gravitational field, is equivalent to a classical Newtonian gravitational field with a potential* U *satisfying the Poisson equation*

$$\Delta U = 4\,\pi\rho, \tag{4.19}$$

provided that Γ *falls off asymptotically on the space-sections* Σ_t *or that* $H^2(\Sigma_t, \mathbf{R}) = 0$.

Proof. — We consider the connection as a gravitational force field in the standard way by interpreting $-\Gamma^A_{BC}\dfrac{dx^B}{dt}\dfrac{dx^C}{dt}$ as the A-component of the force on the particle of unit mass in a particular Cartesian coordinate system. (This will be elaborated somewhat in the next section.) The first part of the theorem follows from the

LEMMA. — *For a four dimensional Galilei manifold with a symmetric Galilei connection* Γ *the following are equivalent* :

(*a*) $R^{\alpha\beta} = \gamma^{\alpha\rho}\,\gamma^{\beta\sigma}\,R_{\rho\sigma} = 0$,

(*b*) $R_{\alpha\beta} = 2\,\lambda_{(\alpha}\,\psi_{\beta)}$ *for some* λ_α,

(*c*) *the Galilei structure is locally flat.*

Proof. — In an adapted coordinate system (*a*) and (*b*) are equivalent to $R_{AB} = 0$ which implies that the Ricci curvature of the space sections is zero. Since they are three dimensional they are locally flat. The equivalence with (*c*) now follows from Theorem 5.

We can now find a flat (7) local coordinate system such that $\psi_\alpha = \delta^0_\alpha$, $\gamma^{\alpha\beta} = \delta^\alpha_A \, \delta^{AB} \, \delta^\beta_B$ and the induced metric on Σ_t is $\gamma_{AB} = \delta_{AB}$. According to (10) we then have for the most general Galilei connection

$$(4.20) \quad \Gamma^0_{\alpha\beta} = 0, \qquad \Gamma^A_{00} = 2\,\delta^{AB}\,\varkappa_{0B}, \qquad \Gamma^A_{0B} = \varkappa_{BC}\,\delta^{CA} \qquad \text{and} \qquad \Gamma^A_{BC} = 0,$$

which gives for the Ricci tensor components

$$\begin{aligned} R_{00} &= -\,2\,\delta^{KL}\,\partial_K\,\varkappa_{0L} + \varkappa_{KL}\,\delta^{KM}\,\delta^{LN}\,\varkappa_{MN}, \\ R_{0A} &= \delta^{KL}\,\partial_K\,\varkappa_{LA} \qquad \text{and} \qquad R_{AB} = 0. \end{aligned}$$

Equations (18) applied to the components R_{0A} imply $\delta^{KL}\,\partial_K\,\varkappa_{LA} = 0$, equations (15) together with (20) give $\partial_{[A}\,\varkappa_{BC]} = 0$. The 2-form $\frac{1}{2}\varkappa_{AB}\,dx^A \wedge dx^B$ on Σ_t is thus harmonic and a mild form of asymptotical flatness (for the connection, hence for $\varkappa_{AB}$) or a global condition like $H^2(\Sigma_t, \mathbf{R}) = 0$ assures that it vanishes. We now interpret $E^A = \delta^{AB}\,\varkappa_{0B}$ as a vector field on the locally Euclidean manifold Σ_t; then (16) gives $\partial_{[A}\,E_{E]} = 0$ implying that locally $E_A = \partial_A\,U$ for a function U. The field equations (18) applied to R_{00} then reduce to (19). This proves the theorem. □

Remark. — The condition

$$(4.21) \qquad R^{[\alpha}{}_\beta{}^{\gamma]}{}_\delta = 0$$

of Trautman [21, 22], imposed on a symmetric Galilei connection, already implies a locally flat Galilei structure and the existence of a classical Newtonian gravitational potential field. In particular, it implies (18) up to the interpretation of ρ. No global condition is needed in this case. Trautman's second condition, $\psi_{[\alpha}\,R^\beta{}_{\gamma]\delta\varepsilon} = 0$, seems redundant.

Replacing (21) by (14) and (18) we have achieved a better analogy of the Newtonian and Einsteinian gravitation theories. It seems likely that if a sequence of Lorentz structures $\mathcal{L}_n$ on a given manifold *V* has a Galilei structure $\mathcal{G}^0$ as a limit then the corresponding Levi-Cività connections tend to a Newtonian connection compatible with $\mathcal{G}^0$. Note also that for Lorentz manifolds (14) is always satisfied while (21) implies flat space.

(7) Flat with respect to the Galilei structure, not the connection.

5. CANONICAL DYNAMICS OF A POINT PARTICLE IN GALILEI AND LORENTZ SPACE-TIMES

This last section contains a simple application of the developed space-time formalism to the Hamiltonian description of a point particle subject to external forces.

In the Galilei case the following formalism is not restricted to point particles but will describe any system with a Lagrangean of the form $L = T - V$ where V is a function defined on the $(n + 1)$-dimensional manifold $\mathbf{R} \times Q$ (with Q the classical configuration manifold) and T, the kinetic energy, a function quadratic in the velocities and defined on $\mathbf{R} \times TQ$.

Apart from the new formalism this material is, of course, very classical. It has, in particular, been treated by Havas [10] (for flat space-times) and by Horneffer [11] for general Galilei manifolds. (*Cf.* also Śniatycki and Tulczyjew [17].)

We consider a system of n degrees of freedom and assume it satisfies a second order equation for the position variables. Classically (*cf.*, for example, Abraham and Marsden [1]) it is described by an n-dimensioal configuration manifold Q (coordinates x^A) and a vector field X on $\mathbf{R} \times TQ$ (coordinates $t, x^A, \bar{v}^A$) of the form $X = \frac{\partial}{\partial t} + \bar{v}^A \frac{\partial}{\partial x^A} + F^A(t, x^B, \bar{v}^B) \frac{\partial}{\partial \bar{v}^B}$ where t is interpreted as the time. The motions are then the (parametrized) integral curves $\bar{\gamma} : \mathbf{R} \rightarrow \mathbf{R} \times T\,Q$ of X and are described in Q alone by the solutions of

$$(5.1) \qquad \frac{d^2 x^A}{dt^2} = F^A\left(t, x^B, \frac{dx^B}{dt}\right).$$

Since the use of any fixed time is not convenient for our purpose we describe the same system in the homogeneous formalism. That is, a motion is given by an unparametrized curve γ in configuration space-time V (diffeomorphic to $\mathbf{R} \times Q$) that would satisfy (1) if a time coordinate t was distinguished and chosen as curve paramenter. This means that instead of a vector field X on $\mathbf{R} \times T\,Q$ there is given an integrable two dimensional differential system E on T V of the form

$$(5.2) \qquad E_p = \{ X \in T_p(T\,V)/X = a\left(v^\alpha \partial_\alpha + \xi^\alpha \partial_{\dot{\alpha}}\right) + b\, v^\alpha \partial_{\dot{\alpha}}, \quad \text{for any } a, b \in \mathrm{R} \}$$

where ξ^α is a given function of x^α and v^α (a fibre coordinate system of T V), $\partial_\alpha = \dfrac{\partial}{\partial x^\alpha}$ and $\partial_{\dot{\alpha}} = \dfrac{\partial}{\partial v^\alpha}$. Such a differential system E on T V shall be called a *second order system over* V.

The condition that E be integrable, i. e. by Frobenius's theorem that for two vector fields X and Y parallel to E also [XY] be parallel to E implies that ξ^α *must be homogeneous of second order in* v^α in the sense that

$$\xi^\alpha(x, \lambda v) = \lambda^2 \xi^\alpha(x, v) + \mu v^\alpha, \tag{5.3}$$

or, equivalently,

$$v^\rho \partial_{\dot{\rho}} \xi^\alpha(x, v) = 2\xi^\alpha(x, v) + \nu v^\alpha \tag{5.4}$$

for some functions μ and ν on T V [since, according to (2), ξ^α is only determined up to a term parallel to v^α]. The leaves of the foliation defined by E on T V (i. e. the maximal connected integral manifolds) project then onto the unparametrized curves γ in V that satisfy the second order equation, and, on the other hand, these leaves are generated by the lifts of γ's into T V with respect to all possible parametrizations.

If a fixed time $t = x^0$ is given and used as curve parameter the quantities ξ^α are obtained from a given force law F^A by the relation

$$\xi^A(x^\alpha, v^\alpha) = (v^0)^2 F^A(x^0, x^B; (v^0)^{-1} v^B) + (v^0)^{-1} v^A \xi^0(x^\alpha, v^\alpha) \tag{5.5}$$

with ξ^0 arbitrary. Similarly, a given E determines a unique set of F^A's. Clearly, for purposes of explicit integration the homogeneous formulation is not convenient. But it does facilitate the study of the evolution of mechanical systems in space-time independently from the specific (Galileian or Lorentzian) structure of the latter.

It is well known that the quantities ξ^α as well as the F^A do not transform like vectors under configuration space coordinate transformations. Since this is inconvenient in view of the general use of tensor calculus we will resort to the following device. Let a *velocity dependent tensor field* K *over* V be an expression

$$K = K^{\beta_1 \dots \beta_p}_{\gamma_1 \dots \gamma_q}(x, v)\, \partial_{\beta_1} \otimes \dots \otimes \partial_{\beta_p} \otimes dx^{\gamma_1} \otimes \dots \otimes dx^{\gamma_q}$$

whose components may depend on v^α, but transform like ordinary tensor components under coordinate transformation in V [8]. If the covariant derivative $\tilde{\nabla}$K with respect to a given connection Γ on V is defined by

$$\tilde{\nabla}_\alpha K^{\beta_1 \dots \beta_p}_{\gamma_1 \dots \gamma_q}(x, v) = \nabla_\alpha K^{\beta_1 \dots \beta_p}_{\gamma_1 \dots \gamma_q} - \Gamma^\rho_{\alpha\sigma} v^\sigma \partial_{\dot{\rho}} K^{\beta_1 \dots \beta_p}_{\gamma_1 \dots \gamma_q} \tag{5.6}$$

---

[8] More intrinsically, K can be defined as a map $K : \mathrm{T}\, V \to \mathrm{T}^p_q(V)$ such that $\tau^p_{q,V} \circ K = \tau_V$ where $\tau^p_{q,V}$ and τ_V are the projections of the bundles T V and $\mathrm{T}^p_q(V)$, respectively.

where the first term on the right is formally the usual covariant derivative of K (with the v^α-dependence ignored) then one observes that $\tilde{\nabla}$ K is a again velocity dependent tensor field over V, as well as

$$(5.7)\quad \dot{\partial}\mathrm{K} = \partial_{\dot{\alpha}}\ \mathrm{K}^{\beta_1\dots\beta_p}_{\gamma_1\dots\gamma_q}\,\partial_{\beta_1}\oplus\dots\otimes\partial_{\beta_p}\otimes dx^\alpha\otimes dx^{\gamma_1}\otimes\dots\otimes dx^{\gamma_q}.$$

In particular, $\underset{\sim}{v} = v^\alpha\,\partial_\alpha$ is velocity dependent vector field and satisfies

$$(5.8)\qquad \tilde{\nabla}\,\underset{\sim}{v} = 0.$$

Now the mechanical system can be described by covariant equations of motion.

THEROEM 11. — *Let E be a given second order system over configuration space-time V. For any chosen connection Γ on V there exists a velocity dependent vector field $\underset{\sim}{f} = f^\alpha\,\partial_\alpha$ over V, determined up to a term parallel to $\underset{\sim}{v} = v^\alpha\,\partial_\alpha$ such that*

$$(5.9)\qquad \xi^\alpha = f^\alpha - \Gamma^\alpha_{\rho\sigma}\,v^\rho\,v^\sigma + \lambda\,(x, v)\,v^\alpha$$

for some λ. The motions of E are then the (unparametrized) solution curves of

$$(5.10)\qquad \ddot{x}^\alpha = f^\alpha\,(x, \dot{x}) + \mu\,\dot{x}^\alpha$$

where $\dot{x}^\alpha = \dfrac{dx^\alpha}{d\tau}$, ˙denotes the covariant derivative with respect to $\dot{x}$ and μ is an arbitrary function of τ.

Proof :

(i) Suppose $\xi^\alpha\,(x, v)$ is given subject to (3). Let $x^\alpha \to \hat{x}^\alpha\,(x^\rho)$ be a V-coordinate transformation. Then

$$\hat{v}^\alpha = \frac{\partial\hat{x}^\alpha}{\partial x^\rho}\,v^\rho \qquad \text{and} \qquad v^\alpha\,\partial_\alpha + \xi^\alpha\,\partial_{\dot{\alpha}}$$
$$= \hat{v}^\alpha\,\hat{\partial}_\alpha + \left[\xi^\rho\left(\frac{\partial\hat{x}^\alpha}{\partial x^\rho}\right) + \frac{\partial^2\,\hat{x}^\alpha}{\partial x^\rho\,\partial x^\sigma}\,v^\rho\,v^\sigma\right]\hat{\partial}_{\dot{\alpha}}.$$

whence

$$\hat{\xi}^\alpha = \frac{\hat{\partial}^\alpha}{\partial x^\rho}\,\xi^\rho + \frac{\partial^2\,\hat{x}^\alpha}{\partial x^\rho\,\partial x^\sigma}\,v^\rho\,v^\sigma.$$

Combining this with the transformation law for $\Gamma^\alpha_{\rho\sigma}$ and (9), considered as definition of $\underset{\sim}{f}$ shows that $\underset{\sim}{f}$ is a velocity dependent vector field.

(ii) An integral curve $\gamma : \tau \to (x\,(\tau),\ v\,(\tau))$ of a general vector field X parallel to E satisfies

$$\frac{dx^\alpha}{d\tau} = a\,v^\alpha \qquad \text{and} \qquad \frac{dv^\alpha}{d\tau} = a\,f^\alpha - a\,\Gamma^\alpha_{\rho\sigma}\,v^\rho\,v^\sigma + b\,v^\alpha,$$

whence

$$\frac{d^2 x^\alpha}{d\tau^2} = \frac{da}{d\tau} a^{-1} \frac{dx^\alpha}{d\tau} + a^2 f^\alpha (x, v) - a^2 \Gamma^\sigma_{\rho\alpha} v^\rho v^\sigma + ab\, v^\alpha,$$

that is

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = f^\alpha \left(x, \frac{dx}{d\tau} \right) + \mu \frac{dx^\alpha}{d\tau},$$

since f^α is second order homogeneous in v^α in the same sense as ξ^α. □

Whenever the configuration space-time has a well defined concept of a timelike unit vector one can restrict the velocities of a motion to unit vectors, or, in other words use *proper* time as the curve parameter. In particular, for V equipped with a Galilei or Lorentz structure, let

(5.11) $$\dot{x}^\alpha \psi_\alpha = 1 \qquad \text{or} \qquad \dot{x}^\alpha g_{\alpha\beta} \dot{x}^\beta = -1,$$

respectively. Then, if Γ is chosen to be a symmetric Galilei (Lorentz) connection

(5.12) $$\ddot{x}^\alpha \psi_\alpha = 0 \qquad (\ddot{x}^\alpha g_{\alpha\beta} \dot{x}^\beta = 0),$$

and if the arbitrary μ in equations (10) is put equal to zero and the arbitrariness in f^α eliminated by

(5.13) $$f^\alpha \psi_\alpha = 0 \qquad (f^\alpha g_{\alpha\beta} \dot{x}^\beta = 0)$$

the equations of motion become

(5.14) $$\ddot{x}^\alpha = f^\alpha (x, \dot{x})$$

and their integral curves are automatically parametrized by world (proper) time.

Up to this point it seemed natural to regard forces as contravariant time dependent vector fields. The possibility of regarding them as 1-forms arises only when there is a given space-time structure on V (or there is a given Hamiltonian). In the Lorentz case one naturally defines $f_\alpha = g_{\alpha\beta} f^\beta$, which in view of the condition (13) can also be written

(5.15) $$f_\alpha = \gamma_{\alpha\beta} f^\beta$$

where

(5.16) $$\gamma_{\alpha\beta} = g_{\alpha\beta} - (v_\lambda v^\lambda)^{-1} v_\alpha v_\beta,$$

the metric of the rest space orthogonal to v^α, is a velocity dependent tensor over V. (All indices in the Lorentz case are raised and lowered with respect to $g_{\alpha\beta}$.) It is characterized by

(5.17 *a*) $$\gamma_{\alpha\rho} v^\rho = 0 \qquad \text{and} \qquad \gamma_{\alpha\rho} \gamma^{\rho\beta} = \delta^\beta_\alpha - \frac{1}{v_\lambda v^\lambda} v_\alpha v^\beta = \gamma_\cdot{}^\alpha_\beta$$

An analogous velocity dependent metric of the rest space can be introduced in Galilei space-times, simply by replacing $(-v_\alpha)$ in (17 *a*) by ψ_α :

(5.17 *b*) $\gamma_{\alpha\rho}\, v^\rho = 0$ and $\gamma_{\alpha\rho}\, \gamma^{\rho\beta} = \delta^\beta_\alpha - \frac{1}{\psi_\lambda\, v^\lambda}\, \psi_\alpha\, v^\beta = \gamma^\beta_\alpha.$

An explicit solution as (16) in the Lorentz case cannot be obtained covariantly, but if $\{ e_a \}$ is any Galilei frame and $v^a = v^\alpha\, \theta^a_\alpha$ then

(5.18) $\gamma_{\alpha\beta} = (v^0)^{-2}\, \delta_{KL}\, v^K\, v^L\, \psi_\alpha\, \psi_\beta - 2\,(v^0)^{-1}\, \delta_{KL}\, v^K\, \theta^L_{(\alpha}\, \psi_{\beta)} + \theta^K_\alpha\, \delta_{KL}\, \theta^L_\beta.$

For later use we calculate

(5.19) $$\partial_{\dot\lambda}\gamma_{\alpha\beta} = -\,2\,(\psi_\rho\, v^\rho)^{-1}\, \gamma_{\lambda(\alpha}\psi_{\beta)}$$

and

(5.20) $\partial_\lambda\, \gamma_{\alpha\beta} = -\,\gamma_{\alpha\rho}\, \partial_\lambda\, \gamma^{\rho\sigma}\, \gamma_{\sigma\beta} = 2\,\Gamma^\rho_{\lambda(\alpha}\, \gamma_{\beta)\rho} - 2\,(\psi_\rho\, v^\rho)^{-1}\, \Gamma^\sigma_{\lambda\mu}\, v^\mu\, \gamma_{\sigma(\alpha}\, \psi_{\beta)}$

where Γ is any (not necessarily symmetric) Galilei connection. Equations (19) and (20) also hold in the Lorentz case, provided ψ_α is replaced by $-v_\alpha$. In both cases $\gamma_{\alpha\beta}$ is homogeneous of order zero in v^α and satisfies

(5.21) $$\tilde{\tilde\nabla}_\alpha\, \gamma_{\beta\gamma} = 0$$

with respect to any Galilei (Lorentz) connection.

Finally we discuss the *canonical formalism for second order systems* over both, Galileian and Lorentzian, configuration space-times. The general programme which consists of equipping the set of all motions with a symplectic structure via introducing a suitable presymplectic form ω on the evolution manifold of the system — in this case T V — has been described previously ([14], [15]). The problem is to find a closed 2-form ω on T V such that

(5.22) $$E_p = \ker_p \omega = \{ X \in T_p\,(T\, V)/X \,\lrcorner\, \omega = 0 \}.$$

In general, ω is not unique for a given E. A presymplectic ω always exists at least locally. But in the physics literature almost exclusively a special type of ω is considered, namely one induced by a Lagrangean.

If $L : T\,V \to \mathbf{R}$ is a Lagrangean (homogeneous of first order in v^α) then ω is defined by

(5.23) $$\omega = -\,d\left(\partial_{\dot\alpha}\, L\, dx^\alpha\right)$$

or

(5.24) $$\omega = \frac{1}{2}\,\omega_{\alpha\beta}\, dx^\alpha \wedge dx^\beta + \sigma_{\alpha\beta}\, dx^\alpha \wedge dv^\beta$$

with

(5.25) $$\omega_{\alpha\beta} = \partial_{\dot\alpha\beta}\, L - \partial_{\alpha\dot\beta}\, L \qquad \text{and} \qquad \sigma_{\alpha\beta} = \partial_{\dot\alpha\dot\beta}\, L.$$

The Lagrangean is non degenerate iff rank $(\sigma_{\alpha\beta}) = n$. Then $X \lrcorner\, \omega_L = 0$ implies that X is of the general form (2).

Second order systems E that are defined by (22) for a presymplectic form of the type (24) (i. e. a form ω whose restriction to the fibres of T V vanishes) will be called *Lagrangean second order systems.* Actually this is no generalization, since if ω has the form (24) there always exists (locally) a Lagrangean such that (25) holds. In particular, $\sigma_{\alpha\beta}$ is always symmetric [9]. This is so, because being closed, ω is locally of the form $\omega = -d\,\theta = -d\,(\varkappa_\alpha\, dx^\alpha + \lambda_\alpha\, dv^\alpha)$, whence

$$(5.26) \quad \omega_{\alpha\beta} = -\partial_\alpha \varkappa_\beta + \partial_\beta \varkappa_\alpha, \sigma_{\alpha\beta} = -\partial_\alpha \lambda_\beta + \partial_{\dot\beta} \varkappa_\alpha, \partial_{[\dot\alpha}\, \lambda_{\beta]} = 0.$$

The last of these equations implies that there exists a function K (x, v) such that $\lambda_\alpha = \partial_{\dot\alpha}$ K. Therefore also $\sigma_{\alpha\beta} = -\partial_{\alpha\dot\beta}\,\mathrm{K} + \partial_{\dot\beta}\,\varkappa_\alpha$. But in order for ω to define a second order system we must have

$$(5.27) \qquad \sigma_{\alpha\beta}\, v^\beta = 0 = v^\beta\, \sigma_{\beta\alpha}.$$

It follows that $\varkappa_\alpha = \partial_{\dot\alpha}\,(\varkappa_\beta\, v^\beta - v^\beta\, \partial_\beta\, \mathrm{K}) + \partial_\alpha\, \mathrm{K}$. Let $\mathrm{L} = \varkappa_\beta\, v^\beta - v^\beta\, \partial_\beta\, \mathrm{K}$ then (25) follows immediately.

For the rest of the section we only deal with Lagrangean second order systems. First note that the ξ^α (up to a term parallel to v^α) are obtained from

$$(5.28) \qquad \sigma_{\alpha\beta}\, \xi^\beta = v^\beta\, \omega_{\beta\alpha}.$$

It turns out that $\sigma_{\alpha\beta}$ is a velocity dependent tensor over V, but $\omega_{\alpha\beta}$ is not. However, one checks easily that

$$(5.29) \qquad w_{\alpha\beta} = \omega_{\alpha\beta} - 2\, \sigma_{\rho[\alpha}\, \Gamma^\rho_{\beta]\mu}\, v^\mu$$

is a velocity dependent 2-form over V, for any chosen connection Γ. Then (28), together with (9) gives

$$(5.30) \qquad \sigma_{\alpha\beta}\, f^\beta = v^\beta\, w_{\beta\alpha}.$$

The condition that $d\omega = 0$ is expressed in terms of $w_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ by

$$(5.31) \qquad \partial_{\dot\alpha}\, \sigma_{\beta\gamma} - \partial_{\dot\beta}\, \sigma_{\alpha\gamma} = 0,$$

$$(5.32) \qquad \partial_{\dot\gamma}\, w_{\alpha\beta} + 2\, \tilde\nabla_{[\alpha}\, \sigma_{\beta]\gamma} + \mathrm{T}^\rho_{\alpha\beta}\, \sigma_{\rho\gamma} = 0$$

and

$$(5.33) \qquad \tilde\nabla_{[\alpha}\, w_{\beta\gamma]} - \mathrm{R}^\rho_{\mu[\alpha\beta}\, \sigma_{\gamma]\rho}\, v^\mu = 0.$$

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[9] The point is, that there may exist a global or covariant form for ω, but not for L.

Contracting (31) with v^α and (32) with v^γ and using (27) and (8) shows that $v^\rho \partial_{\dot\rho} \sigma_{\alpha\beta} = - \sigma_{\alpha\beta}$ and $v^\rho \partial_{\dot\rho} w_{\alpha\beta} = 0$, i. e. $\sigma_{\alpha\beta}$ is homogeneous of order -1 and $w_{\alpha\beta}$ of order 0 in v^γ.

So far no assumption was made about the specific space-time structure of V. The question arises which ω's (or also which forces f^α) are compatible with a Galilei or Lorentz structure on V, respectively. Since we cannot require invariance of ω under the group of space-time automorphisms without reducing all forces to zero, and there seems to be no distinguished action of G^0 or L^0 on the fibres of T V only, this problem appears to be non trivial. We make in the following a simple assumption that looks admittedly somewhat ad hoc from the present point of view, but actually covers all the classical Hamiltonian forces normally considered for this type of systems.

Note that the velocity dependent tensor $\gamma_{\alpha\beta}$ is of rank n and satisfies $\gamma_{\alpha\beta} v^\beta = 0$. It would be a good candidate for $\sigma_{\alpha\beta}$ except for being homogeneous of order 0 in v^γ, instead of order-1. Thus, the simplest choice for $\sigma_{\alpha\beta}$, compatible with the appropriate space-time structure seems to be

$$(5.34) \qquad \sigma_{\alpha\beta} = (v^\rho \psi_\rho)^{-1} \gamma_{\alpha\beta} \qquad \text{and} \qquad \sigma_{\alpha\beta} = (- v_\rho v^\rho)^{-1/2} \gamma_{\alpha\beta}$$

for the Galilei and Lorentz case, respectively [10]. (Assume now that ω need be defined only on $T_0 V = \{ (x, v) \in T V / v^\rho \psi_\rho > 0 \}$ with ψ_ρ replaced by $- v_\rho$ in the Lorentz case.) We call a Lagrangean E satisfying (34) a *classical Galileian (Lorentzian) Lagrangean second order system.*

Observe first that, if moreover $w_{\alpha\beta} = 0$, the choice (34) for $\sigma_{\alpha\beta}$ corresponds to the well known free particle Lagrangean $L = - \sqrt{- g_{\rho\sigma} v^\rho v^\sigma}$ in the Lorentz case. In the Galilei case a space-time covariant expression for L cannot be given, but it will turn out soon that the expression $L = \frac{1}{2} m \, \delta_{AB} v^A v^B$ can be obtained for a special coordinate choice.

Assume that a Galilei structure on V is f. o. f. Then we can choose Γ to be a symmetric Galilei (Lorentz) connection. Equations (31) are already satisfied for a $\sigma_{\alpha\beta}$ of the form (34), as can be seen with the help of (19). Equations (32) reduce to $\partial_{\dot\gamma} w_{\alpha\beta} = 0$ showing that $\frac{1}{2} w_{\alpha\beta} \, dx^\alpha \wedge dx^\beta = \tilde{w}$ is actually on ordinary 2-form on V. If, in the Galilei case, we let Γ be a Newtonian connection it follows from (4.15) that the second term in (33) vanishes, as it does for a symmetric Lorentz connection. Thus (33) reduces to $d\tilde{w} = 0$.

[10] For the flat Galilei (Lorentz) structure this expression follows frome the invariance of ω under the inhomogeneous Galilei (Lorentz) group.

Equations (30) can now be solved for f^α with the help of (17) and yield

$$(5.35) \qquad f^\alpha = -(v^\rho \psi_\rho)\, \gamma^{\alpha\lambda}\, w_{\lambda\beta}\, v^\mu$$

and

$$(5.36) \qquad f^\alpha = -\sqrt{-v^\rho v_\rho}\, \gamma^{\alpha\lambda}\, w_{\lambda\mu}\, v^\mu = -\sqrt{-v^\rho v_\rho}\, g^{\alpha\lambda}\, w_{\lambda\mu}\, v^\mu$$

for the Galilei and Lorentz case, respectively [where f^α is already normalized according to (13)]. For a Lorentz space-time the interpretation is immediate.

THEOREM 12. — *A classical Lagrangean second order system over a Lorentz speca-time V describes a particle subject to the Lorentz force of a Maxwell field* $\tilde{w}$ *according to* $\ddot{x}_\alpha = -g^{\alpha\lambda}\, w_{\lambda\mu}\, \dot{x}^\mu$. $\square$.

For Galilei configuration space-times the situation is the same, except that, since the (Newtonian) connection is still arbitrary, any force of type (35) can be regarded as an inertial force. To see this, define a new connection $\hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \gamma^{\alpha\rho}\, w_{\rho(\beta}\, \psi_{\gamma)}$. Since Γ is Newtonian and $d\tilde{w} = 0$ so is $\hat{\Gamma}$ according to theorems 7 and 9. But now

$$\xi^\alpha = f^\alpha - \Gamma^\alpha_{\rho\sigma}\, v^\rho\, v^\sigma = f^\alpha - \hat{\Gamma}^\alpha_{\rho\sigma}\, v^\rho\, v^\sigma + \gamma^{\alpha\rho}\, w_{\rho\beta}\, v^\beta\, \psi_\gamma\, v^\gamma = -\hat{\Gamma}^\alpha_{\rho\sigma}\, v^\rho\, v^\sigma$$

by (35). Thus

THEOREM 13. — *The classical Lagrangean second order systems E over a f. o. f. Galilei manifold V are in one-to-one correspondence with the Newtonian connections* Γ *on V. The set of motions of E coincides with the set of timelike geodesics (considered as point sets) with respect to a Newtonian connection.*

To recover finally the old non homogeneous formalism we introduce an adapted coordinate system such that $\psi_\alpha = \delta^0_\alpha$. Then

$$(\gamma_{AB}) = \text{inverse of } (\gamma^{AB}), \qquad \gamma_{0A} = -(v^0)^{-1}\, \gamma_{AK}\, V^K,$$
$$\gamma_{00} = (v^0)^{-2}\, \gamma_{KL}\, V^K\, V^L.$$

Choosing the connection such that $\tilde{w} = 0$ and using (4.10) yields

$$\begin{aligned}\omega = &-[\gamma_{AK}\,(\Gamma^K_{00} + (v^0)^{-1}\, \Gamma^K_{0L}\, v^L) - (v^0)^{-1}\, \Gamma^K_{A0}\, \gamma_{KL}\, v^L \\ &\quad - (v^0)^{-2}\, \Gamma^K_{AL}\, v^L\, \gamma_{KM}\, v^M]\, dx^0 \wedge dx^A \\ &+ [\gamma_{AK}\, \Gamma^K_{0B} + (v^0)^{-1}\, [A, BK]\, v^K]\, dx^A \wedge dx^B + (v^0)^{-3}\, \gamma_{KL}\, v^K\, v^L\, dx^0 \wedge dv^0 \\ &- (v^0)^{-2}\, \gamma_{AK}\, v^K\, (dx^0 \wedge dv^A + dx^A \wedge dv^0) + (v^0)^{-1}\, \gamma_{AB}\, dx^A \wedge dx^B,\end{aligned}$$

which can be derived according to (25) from the Lagrangean

$$L = -v^0\, \varphi + A_K\, v^K + \frac{1}{2}\,(v^0)^{-1}\, \gamma_{KL}\, v^K\, v^L$$

if

$$\Gamma^{\mathrm{A}}_{00} = \gamma^{\mathrm{AK}} (\partial_{\mathrm{K}} \varphi + \partial_0 \mathrm{A}_{\mathrm{K}}) \qquad \text{and} \qquad \Gamma^{\mathrm{A}}_{0\mathrm{B}} = \gamma^{\mathrm{AK}} \left(\frac{1}{2} \partial_0 \gamma_{\mathrm{KB}} + \partial_{[\mathrm{K}} \mathrm{A}_{\mathrm{B}]} \right),$$

since the connection is Newtonian. The equations of motion in this coordinate system become

$$(5.37) \quad \frac{d^2 x^{\mathrm{A}}}{(dx^0)^2} + \Gamma^{\mathrm{A}}_{\mathrm{KL}} \frac{dx^{\mathrm{K}}}{dx^0} \frac{dx^{\mathrm{L}}}{dx^0} = - \gamma^{\mathrm{AK}} \left\{ \partial_{\mathrm{K}} \varphi + \partial_0 \mathrm{A}_{\mathrm{K}} + (\partial_0 \gamma_{\mathrm{KL}} + 2\, \partial_{[\mathrm{K}} \mathrm{A}_{\mathrm{L}]}) \frac{dx^{\mathrm{L}}}{dx^0} \right\}.$$

They are seen to include most « non relativistic » Lagrangean force laws. It should be noted that in the above coordinate choice the unit vector field $\underset{\sim}{u}$ [*cf.* (4.6)] was tacitly left arbitrary. In many cases a choice is possible that eliminates some of the terms in (37).

REFERENCES

[1] R. ABRAHAM and J. E. MARSDEN, *Foundations of mechanics*, Benjamin, New York, 1967.

[2] E. CARTAN, *Ann. scient. Éc. Norm. Sup.*, t. 40, 1923, p. 325.

[3] E. CARTAN, *Ann. scient. Éc. Norm. Sup.*, t. 41, 1924, p. 1.

[4] M. CRAMPIN, *Proc. Camb. Phil. Soc.*, t. 64, 1968, p. 307.

[5] G. DAUTCOURT, *Acta Phys. Polon.*, t. 65, 1964, p. 637.

[6] H. D. DOMBROWSKI and K. HORNEFFER, *Nachr. Akad. Wiss. Göttingen, Math.-phys. Kl.*, 1964, p. 233.

[7] H. D. DOMBROWSKI and K. HORNEFFER, *Math. Z.*, t. 86, 1964, p. 291.

[8] P. FRANK, *Sitzungsber. Kgl. Akad. Wiss. Wien, Math.-naturw. Kl.*, t. 118, Abt. II *a*, 1909, p. 373.

[9] K. FRIEDRICHS, *Math. Annalen*, t. 98, 1927, p. 566.

[10] P. HAVAS, *Rev. Mod. Phys.*, t. 36, 1964, p. 938.

[11] K. HORNEFFER, *Arch. Rat. Mech. Anal.*, t. 23, 1966, p. 239.

[12] S. KOBAYASHI and K. NOMIZU, *Foundations of differential geometry*, I, Wiley New York, 1963.

[13] S. KOBAYASHI and K. NOMIZU, *Foundations of differential geometry*, II, Wiley, New York, 1969.

[14] H. P. KÜNZLE, *Ann. Inst. Henri Poincaré*, A, t. 11. 1969, p. 393.

[15] H. P. KÜNZLE, *J. Math. Phys.*, t. 13, 1972, p. 739.

[16] D. J. SIMMS, *Lie groups and quantum mechanics* (*Lect. Notes in Math.*, N° 52, Springer, Berlin, 1968.

[17] J. ŚNIATYCKI and W. M. TULCZYJEW, *Ann. Inst. Henri Poincaré*, A. t. 16, 1972, p. 23.

[18] N. Steenrod, *The topology of fibre bundles,* Princeton University, Press, Princeton, 1951.

[19] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, 1964.

[20] R. A. Toupin, *Arch. Rat. Mech. Anal.*, t. 1, 1958, p. 181.

[21] A. Trautman, *C. R. Acad. Sc.*, Paris, t. 257, 1963, p. 617.

[22] A. Trautman, in B. Hoffmann (ed.), *Perspectives in geometry and relativity,* Indiana University, Press, Bloomington, 1966.

[23] C. Truesdell and R. A. Toupin, *The classical field theories, in* Flügge (ed.), *Handbuch der Physik,* Bd. III/I, Springer, Berlin, 1960.

[24] H. Weyl, *Raum-Zeit-Materie,* 5, Aufl., Springer, Berlin, 1923.

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