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Section A :
Physique théorique

Topology of quantizable dynamical systems and the algebra of observables

by

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ABSTRACT. — Following H. Poincaré, G. Birkhoff and G. Reeb, a class of dynamical systems, called fibered dynamical systems, is studied. If the total space is a homotopy sphere, then the fibered dynamical system is homotopically equivalent to a quantizable dynamical system, namely the harmonic oscillator. And a manifold homotopically equivalent to the phase space of a harmonic oscillator gives rise to infinitely many differentiably distinct quantizable dynamical systems. Other results on the topology of fibered and quantizable dynamical systems are reviewed. Finally the only possible candidates for total spaces of quantizable dynamical systems, under the requirement that the total spaces are odd dimensional simply connected Finslerian B_x-manifolds, are shown to be homotopy spheres.

RÉSUMÉ. — D'après H. Poincaré, G. Birkhoff et G. Reeb, on étudie une classe de systèmes dynamiques appelée les systèmes dynamiques fibrés. Soit que l'espace total soit une sphère homotopique, le système dynamique fibré est équivalent homotopiquement, à un système dynamique qui est quantifiable; en particulier à l'oscillateur harmonique. La variété qui est équivalente homotopiquement à l'espace phase de l'oscillateur harmonique nous donne une infinité de systèmes dynamiques qui soit différentiablement distincts et quantifiables. On examine d'autres résultats sur la topologie de systèmes dynamiques fibrés qui sont quantifiables. Finalement, les seuls espaces totaux des systèmes dynamiques qui sont quantifiables simplement connexes finslériennes B-$_x$variétés de dimensions impaire sont des sphères de homotopie.

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INTRODUCTION

There is recurrent interest among physicists in the topology of dynamical systems. For example, Misner and Wheeler [35] noted that electric and magnetic charges could be interpreted via de Rham isomorphism as periods in multiply connected manifolds. This involves triangulation of the manifold. More recently Bohm, Hiley and Stuart [6] have studied an approach to aspects of quantum theory using combinatorial manifolds. (Confer also the papers by Penrose, Hiley, Atkin and Bastin in *Quantum Theory and Beyond*, ed. T. Bastin, Cambridge University Press, 1971, p. 147-226.)

We study below the problems which arise if a dynamical system is no longer differentiably equivalent but merely topological equivalent to the harmonic oscillator, which is a quantizable dynamical system. The *algebra of observables* or *dynamical variables*, A, for a classical dynamical system is generally taken to be the collection of C^∞ functions on the phase space M (*cf.* Mackey [32], Souriau [46]). The question then arises whether or not the topology of M affects the algebra of observables A. Recall briefly that a *topological manifold* is a Hausdorff space with a countable basis such that each point x in M has a neighborhood homeomorphic with an open subset of $\mathbf{R}^n$. And a differentiable manifold can be viewed as a ringed Hausdorff space (M, A) — i. e. M is a Hausdorff space and A, the *differentiable structure*, is the sheaf whose fiber A_x is the local commutative associative algebra of germs of continuous functions at x in M with unity 1_x in A_x. Certain axioms are placed on (M, A) — (*cf.* [16]). Two differentiable manifolds (M, A) and (N, B) are differentiably isomorphic or *diffeomorphic* if M and N are homeomorphic and A is isomorphic to B.

One question is : should it be excepted that there is associated to each dynamical system, whose phase space is a specific topological manifold M, at most one algebra of observables, up to a diffeomorphism? In topology this was an open question until 1956 when Milnor [33] showed that the standard seven dimensional sphere S^7 has several inequivalent differentiable structures; that is, several compact oriented differentiable manifolds Σ_i^7 homeomorphic to S^7 but each carrying a differentiable structure not equivalent to the standard differentiable structure of S^7, and the Σ_i^7 nondiffeomorphic to S^7. In 1960 Munkres [38] showed that any topological manifold M of dimension ≤ 3, and in 1962 Stallings [48] showed that any euclidean space $\mathbf{R}^n$ ($n \neq 4$), has a differentiable structure which is unique up to diffeomorphism.

A *triangulation* on a topological manifold M is a finite simplicial complex K and a homeomorphism h of the geometric realization $|K|$ of K onto M (*cf.* [47]). Under a certain axiom (*cf.* [39], [47]) this is called a combinatorial triangulation, and a maximal set of combinatorially equivalent combinatorial triangulations is called a *combinatorial structure.* A topological manifold with a specific combinatorial structure is called a *combinatorial manifold.* It is unknown whether every topological manifold admits a triangulation or whether a triangulated manifold admits one or more combinatorial structures; however by the Cairns-Whitehead theorem there is associated with every differentiable manifold a specific differentiable combinatorial structure (i. e. h restricted to any closed simplex of K is a diffeomorphism).

A second question is : should it be expected that there is associated with each dynamical system, with phase space a specific topological manifold M, at least one algebra of observables? Again in topology this was an open problem until 1960 when Kervaire [27] constructed an example of a combinatorial manifold which admits no differentiable structure. This has some interesting implications for the approaches cited in the first paragraph.

An abstract dynamical system in the sense of Birkhoff and Poincaré is a differentiable manifold E and a vector field Z on E. Under suitable conditions, e. g. E compact, Z generates the action of a topological group G on E. Thus the general theory of dynamical systems concerns the study of topological transformation groups on E. An especially interesting class of dynamical systems (E, G) are those for which $G = S^1$ acts differentiably and freely on E. This arises in the case of Hamiltonian dynamical systems and also in our study [22]-[26] of quantizable dynamical systems (QDS). Namely the harmonic oscillator with equal periods is QDS of the form $\xi : S^1 \rightarrow S^{2n+1} \rightarrow \mathbf{C}P(n)$. The aim of this paper is to study the topology of these special dynamical systems (E, G), where $G = S^1$ acts freely, and their relationship to QDSs. Indirectly results are obtained regarding the algebras of observables on manifolds, in particular for QDSs $S^1 \rightarrow E \rightarrow M$ whose orbit spaces M are topologically equivalent to the orbit space $\mathbf{C}P(n)$ of the QDS ξ. In paragraph 1 fibered dynamical systems (FDSs) are introduced. In paragraph 2 the topology of FDSs and QDSs is studied. Our main tool is the recent work in transformation groups on homotpoy spheres. In paragraph 3 B_x-manifolds are reviewed. These manifolds were first introduced into the study of dynamical systems by Reeb [41], [42]. Their relationship to QDSs is studied.

Notations. — We denote by S^n, the n dimensional sphere, $\mathbf{R}P(n)$, the n dimensional real projective space, $\mathbf{C}P(n)$, the $n = 2m$ dimensional complex projective space, $\mathbf{Q}P(n)$, the $n = 4m$ dimensional quaternion projective space, and $\mathbf{Ca}P(2)$, the 16 dimensional Cayley projective plane.

1. FIBERED DYNAMICAL SYSTEMS

Let (E, Z) be a *dynamical system*; that is, E is a differentiable manifold of dimension $n+1$ and Z is a C^∞ nonnull vector field on E. Assume the foliation associated to Z is proper with finite period — that is, it generates a global one-parameter group G of transformations on E with a finite period by

$$\left.\frac{d\varphi_t(x)}{dt}\right|_{t=t_0} = Z(\varphi_{t_0}(x)).$$

(For notations, *cf.* [25].) Then $G = S^1$ and $\varphi : G \times E \to E$ is a C^∞ map, denoting the action of Lie group G on E, and satisfies $\varphi_{ts}(x) = \varphi_t(\varphi_s(x))$ and $\varphi_{\mathrm{id}_G}(x) = x$ for x in E, t, s in G. Thus for each t, $\varphi_t : E \to E$ is a diffeomorphism. Conversely, if φ_t is a 1-parameter group of transformations, then

$$\left.\frac{d\varphi_t(x)}{dt}\right|_{t=0} = Z(x)$$

is a C^∞ vector field on E which generates φ_t. Thus let (E, φ, G) denote this transformation group or dynamical system. E is called the *total space* of the dynamical system.

Assume now that the action φ is free — that is, $\varphi_t(x) = x$ implies $t = \mathrm{id}_G$. Let $\frac{E}{G} = M$ be the orbit space, i. e. $M = \{ Gx \mid x \text{ in } E \}$. Then by Gleason's lemma [13] $G \to E \to M$ is a principal toral bundle and M is a differentiable manifold of dimension n. In this case (E, φ, G) is called a *fibered dynamical system* (FDS) (*cf.* Hurt [25]). Thus by [25] (Thm. 3.2) if $G \to E \to M$ is a FDS, then there is a connection form on E with respect to which every smooth path in M has horizontal lifts.

Given two fibered dynamical systems (E, φ, G) and (E′, φ', G), then map $f : E \to E'$ is called *equivariant* if $f\varphi_t(x) = \varphi'_t(f(x))$. Two fibered dynamical systems are said to be *topologically equivalent* (resp. *differentiably equivalent*) if there is an equivariant, homeomorphism (resp. diffeomorphism) $f : E \to E'$, i. e. homeomorphism (resp. diffeomorphism) f such that the following diagram is commutative :

$$\begin{array}{ccc} G \times E & \xrightarrow{\mathrm{id}\times f} & G \times E' \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} \\ E & \xrightarrow{f} & E' \end{array}$$

(*cf.* Smale [45]). Since f maps each orbit of E into an orbit of E′, it is direct to show :

PROPOSITION 1.1. — *Let* (E, φ, G) *and* (E′, φ', G) *be two fibered dynamical systems. An equivariant diffeomorphism* $f : E \to E'$ *induces a diffeomorphism* $\bar{f} : M = \frac{E}{G} \to M' = \frac{E'}{G'}$ *such that the following diagram is commutative :*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{\bar{f}} & M' \end{array} \tag{1.1}$$

And conversely, a diffeomorphism $\bar{f} : M \to M'$ *induces an equivariant diffeomorphism* f *such that* (1.1) *is commutative. Thus, equivalence classes of differentiably equivalent fibered dynamical systems are in one-one correspondence with diffeormorphism classes of manifolds* $M = \frac{E}{G}$.

A *homotopy sphere* Σ^n is a closed, simply connected oriented n-dimensional differentiable manifold with the same homotopy as the standard sphere S^n — i. e. $\pi_i(\Sigma^n) \simeq \pi_i(S^n)$ for $i \leq n$ (*cf.* [49]). Recall that $\pi_i(S^n) = 0$ for $i < n$ and $\pi_n(S^n) = \mathbf{Z}$. A *homotopy complex projective space* M is a closed differentiable simply connected manifold of real dimension $2n$ such that $\pi_i(M) \simeq \pi_i(\mathbf{C}P(n))$ for $i \leq 2n$. Clearly $\mathbf{C}P(n)$ is a homotopy complex projective space. An *integral cohomology complex projective space* is a space with the same integral cohomology as $\mathbf{C}P(n)$, i. e. $H^*(M; \mathbf{Z}) \simeq H^*(\mathbf{C}P(n); \mathbf{Z})$. Thus $H^*(M; \mathbf{Z}) = \frac{\mathbf{Z}[\alpha]}{\alpha^{n+1}}$ — the quotient of $\mathbf{Z}[\alpha]$, the polynomial ring over $\mathbf{Z}$ in one indeterminant α of degree 2 $[\alpha \in H^2(M; \mathbf{Z})]$, over the ideal generated by α^{n+1}; that is, a truncated polynomial ring over $\mathbf{Z}$ generated by α of degree two and height $n+1$ (*cf.* [47]). A homotopy complex projective space is a closed simply connected integral cohomology complex projective space; but an integral cohomology complex projective space is not necessarily a homotopy complex projective space as it may not be simply connected.

A principal G-bundle $\xi : G \to E \to M$ is called *n-universal* if E is arcwise connected and $\pi_i(E) = 0$ for $1 \leq i < n$; then $\pi_i(M) \simeq \pi_{i-1}(G)$ for $1 \leq i \leq n-1$. Let ξ be an n-universal bundle, let X be a CW-complex of dimension $< n$ (*cf.* [47]), let [X, M] be the set of homotopy classes of maps of X into M, and let $H^1(X, \mathcal{O}(G))$ be the equivalence classes of principal G-bundles over X. Then the map $\varphi : [X, M] \to H^1(X, \mathcal{O}(G))$ defined by $\varphi : [f] \to f^* \xi$, the induced bundle over X (*cf.* [49], § 10), is a bijection (*cf.* [49], p. 101].

2. TOPOLOGY OF QUANTIZABLE AND FIBERED DYNAMICAL SYSTEMS

A quantizable dynamical system (E, Z) is an odd dimensional, proper regular contact manifold E with a finite period, where Z is the associated vector field of the contact structure. (For notations, *cf.* [25].) Z generates a free action of $G = S^1$ on E; thus (E, Z, G) is a FDS. Furthermore, $\eta : G \to E \to M = \frac{E}{G}$ is a principal toral bundle over a symplectic manifold (M, Ω) and Ω determines an integral cocycle on M. That is, Ω is a 2-form on M with $d\Omega = 0$, $(\Omega)^n = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n} \neq 0$ and $\Omega \in H^2(M; \mathbf{Z})$ under de Rham isomorphism. Theorem 6.6 of [25] states:

PROPOSITION 2.1. — *If* (M, Ω) *is a symplectic manifold and the closed 2-form* Ω *represents an integral cohomology class on* M, *then there exists a QDS* $\eta : G \to E \to M$ *over* M.

If the contact structure on E is normal, i. e. E is a Sasakian manifold (*cf.* [25]), then by [25] (Prop. 6.7) the phase space M is a Hodge manifold; and conversely by [25] (Cor. 6.8) there is canonically associated to every Hodge manifold a normal QDS. The classical example of a normal simply connected compact QDS is $\xi : G \to S^{2n+1} \to \mathbf{C}P(n)$.

Abbreviate homotopy complex projective space by $H\mathbf{C}P(n)$. Then from the definition of a $H\mathbf{C}P(n)$ we have

PROPOSITION 2.2. — *If* M *is an* $H\mathbf{C}P(n)$, *then there is a QDS* $\eta : G \to E \to M$ *over* M.

Proof. — The generator $\alpha \in H^2(M, \mathbf{Z})$ under de Rham isomorphism corresponds to an integral closed 2-form and clearly $\alpha^n \neq 0$. The result follows then from Proposition 2.1.

Let $\eta : G \to E \to M$ be a simply connected QDS and let

$$\xi : \quad G \to S^{2n+1} \to \mathbf{C}P(n)$$

be the classical QDS. If M is homeomorphic to $\mathbf{C}P(n)$ or if M is an $H\mathbf{C}P(n)$, then by the homotopy sequences of the fibrations η and ξ we have $\pi_i(E) \simeq \pi_i(S^{2n+1})$ for $i > 1$. Since E is simply connected, E is thus homotopically equivalent to S^{2n+1}; so E is homeomorphic to S^{2n+1} by Smale [44] for $n \geq 2$. To summarize we state

PROPOSITION 2.3. — *If* $G \to E \to M$ *is a simply connected QDS where* M *is an* $H\mathbf{C}P(n)$, *then* E *is a homotopy sphere.*

If M^n is a compact Kähler manifold which is homeomorphic to $\mathbf{C}P(n)$, then by Kodaira [30] M is a Hodge manifold. Thus by [25] (Cor. 6.8) there is a normal QDS $G \to E \to M$ over M and by Proposition 2.3 above we have

PROPOSITION 2.4. — *If* M^n *is a compact Kähler manifold homeomorphic to* $\mathbf{C}P(n)$, *then there is a normal QDS* $G \to E \to M$ *over* M *and* $\pi_i(E) \simeq \pi_i(S^{2n+1})$ *for* $i > 1$.

Consider now the FDSs $(\Sigma^{2n+1}, \varphi, G)$ with total spaces the homotopy spheres Σ^{2n+1}. Since $\xi : G \to S^{2n+1} \to \mathbf{C}P(n)$ is $(2n+1)$-universal, then as noted in paragraph 1, $\pi_i(\mathbf{C}P(n)) \simeq \pi_{i-1}(G)$ for $i \leqq 2n$; and the principal G-bundle $\eta : G \to \Sigma^{2n+1} \to M = \dfrac{\Sigma^{2n+1}}{G}$ is classified by a map $f : M \to \mathbf{C}P(n)$. That is $f^*\xi = \eta$. By the homotopy exact sequence of the fiber bundle η and the spectral sequences of ξ, η, we have : $\pi_i(M) \simeq \pi_{i-1}(G)$ for $i \leqq 2n$, M is simply connected, and

$$H^*(M; \mathbf{Z}) \simeq H^*(CP(n); \mathbf{Z}).$$

Thus $H_*(M; \mathbf{Z}) \simeq H_*(\mathbf{C}P(n); \mathbf{Z})$ and by the Whitehead theorem ([2]), p. 307) M is homotopically equivalent to $\mathbf{C}P(n)$, since M and $\mathbf{C}P(n)$ are simply connected. Conversely if M is an H$\mathbf{C}P(n)$ and $f : M \to \mathbf{C}P(n)$ is the homotopy equivalence, then $f^*\xi$ is a homotopy sphere with a free differentiable action by G such that M is the orbit space. And by Proposition 2.1, $f^*\xi$ is a QDS. To summarize :

PROPOSITION 2.5. — *If* $(\Sigma^{2n+1}, \varphi, G)$ *is a FDS then* $\eta : G \to \Sigma^{2n+1} \to M$ *is homotopically equivalent to the classical QDS* $\xi : G \to S^{2n+1} \to \mathbf{C}P(n)$. *And conversely, if* M *is an* H$\mathbf{C}P(n)$, *then there is canonically associated a QDS* $G \to E \to M$ *over* M *where* E *is a homotopy sphere.*

By Proposition 1.1 we have

PROPOSITION 2.6. — *Differentiable equivalence classes of FDSs* $(\Sigma^{2n+1}, \varphi, G)$ *are in one-one correspondence with diffeomorphism classes of manifolds homotopically equivalent to* $\mathbf{C}P(n)$.

Cf. [9], Proposition 3.2.

W. C. Hsiang [19] has shown that there are infinitely many differentiable manifolds M^n of the same homotopy type as $\mathbf{C}P(n)$ distinguished by the first rational Pontrjagin class $p_1(M)$ for $n \geqq 4$. By Proposition 2.6 we have

PROPOSITION 2.7. — *There are infinitely many differentiably distinct QDSs with total spaces being homotopy spheres* Σ^{2n+1} *for* $n \geqq 4$. *And since* $p_1(M)$ *is a topological invariant, each diffeomorphism class is a homeomorphism class !*

Thus even if a dynamical system is homotopically equivalent to the harmonic oscillator with equal periods, there are infinitely many differentiably nonisomorphic phase spaces and algebras of observables (M, A).

Montgomery and Yang [36] showed

PROPOSITION 2.8. — *There are infinitely many differentiably inequivalent FDSs* (Σ^7, φ, G); *in fact, the diffeomorphism classes of* (Σ^7, φ, G) *form an infinite cyclic group. Again the orbit spaces have distinct* $p_1(M)$, *so each diffeomorphism class is a homeomorphism class.*

In addition by [37] Theorem 1, we have

PROPOSITION 2.9. — *There are exactly* 10 *oriented homotopy 7-spheres not diffeomorphic to one another such that for each of them* (Σ^7, φ, G) *is a FDS.*

In particular Montgomery and Yang [37] showed that if (Σ^7, φ, G) is a FDS then Σ^7 is diffeomorphic to $k\,\Sigma^7_M$ for some $k = 0, \pm 4, \pm 6, \pm 8, \pm 10$ or 14 (mod 28) where Σ^7_M is the Milnor 7-sphere (*cf.* [28]). And if (Σ^7, φ, G) is one of these FDS, then by Proposition 2.8 there are infinitely many topologically distinct actions which can be distinguished by $p_1(M)$.

Hsiang and Hsiang [20] showed

PROPOSITION 2.10. — *There are infinitely many differentiably distinct FDSs* (S^{11}, φ, G).

However not every homotopy sphere can be the total space of a FDS since Lee [3] showed

PROPOSITION 2.11. — *There exist homotopy spheres* Σ^{8k+1} *for* $k \geq 1$ *which do not admit free differentiable* S^1*-action.*

(Regarding the classification of combinatorial HCP (n) confer D. Sullivan's, *Geometric Seminar Notes*, Princeton, 1967.)

Much more is known regarding the topology of the total space E of a QDS if further restrictions are made on E. We review briefly the type of results now available. Recall from [25] that a QDS admits a positive definite Riemannian metric g. Then the sectional curvature of a plane P spanned by X, Y is

$$K(X, Y) = -\frac{g(R(X, Y)X, Y)}{g(X, Y)\,g(Y, Y) - g(X, Y)^2}$$

where R is the Riemannian curvature tensor.

LEMMA (Goldberg [14]) 2.12. — *If a normal QDS* (E, Z) *has positive sectional curvature, then the base space* M *has positive sectional curvature.*

From Lemma 2.12 the following two Propositions arise directly from work of Bishop and Goldberg [4].

PROPOSITION (Goldberg [14]) 2.13. — *If* (E, Z) *is a complete normal QDS of positive curvature, then* b_2 (E) = dim H^2 (E; **R**) = 0.

Cf. Tanno [50].

PROPOSITION (Goldberg [14]) 2.14. — *If* (E, Z) *is a complete* 5-*dimensional simply connected normal QDS of positive sectional curvature then* E *has the same homotopy as* S^5.

Cf. Tanno [50].

In addition

PROPOSITION (Goldberg [14]) 2.15. — *A complete simply connected normal QDS of positive sectional curvature and constant scalar curvature is isometric to a sphere.*

Cf. Tanno and Moskal in [50].

PROPOSITION (Goldberg [15]) 2.16. — *If* (E, Z) *is a compact normal QDS with nonnegative sectional curvature, then* b_2 (E) = 0.

Recall from [22] Theorem A that a contact manifold E, homogeneous with respect to a connected Lie group G, is a regular contact manifold, so a QDS. If in addition E is compact and simply connected, then by [7], [25], Theorem C, and [23], Prop. 6.7, E is a normal QDS. If (E, Z) is a homogeneous QDS and is Riemannian symmetric, then (E, Z) is called a *symmetric* QDS.

PROPOSITION (Goldberg [15]) 2.17. — *If* (E, Z) *is a simply connected (normal) symmetric QDS, then* E *is isometric with a sphere.*

PROPOSITION (Blair and Goldberg [5]) 2.18. — *The fundamental group* π_1 (E) *of a compact symmetric normal QDS is finite.*

PROPOSITION (Golberg [15]) 2.19. — *A compact torsion free* 5-*dimensional normal QDS with negative sectional curvature is a homotopy sphere.*

Cf. Tanno [50].

Using the notations of [25], let (ω, φ, Z, g) denote the contact metric structure on E. Then $d\omega$ (X, Y) = kg (φ X, Y) where k is a positive constant. Let $h = \inf \{ K (X, \Phi X) \mid X \perp Z$ and $(X, \Phi X)$ form orthonormal basis of plane P $\}$.

Then

PROPOSITION (Harada [17]) 2.20. — *If* (E, Z) *is a compact normal QDS with* $h > k^2$, *then* π_1 (E) *is cyclic.*

3. B_x-MANIFOLDS AND QUANTIZABLE DYNAMICAL SYSTEMS

Let $p : T(M) \to M$ be the tangent bundle over a manifold M and let $p_0 : \mathfrak{T}(M) \to M$ be the subbundle of nonnull vectors. A C^0 map $L : T(M) \to \mathbf{R}^+$ wich is C^∞ on $\mathfrak{T}(M)$ and which is positively homogeneous of degree one is called *Lagrangian*. The function $E : T(M) \to \mathbf{R}^+$ given by $E = L^2$ is called the *energy*. E is positively homogeneous of degree 2, C^1 on T (M), C^∞ on $\mathfrak{T}(M)$, and has canonically associated to it a homogeneous of degree zero symmetric covariant tensor of rank two, $g : \otimes^2 p_0^{-1} T(M) \to \mathbf{R}$ given in local coordinates by $g_{ij} = \frac{1}{2} \frac{\partial^2 E}{\partial X_i \partial X_j}$. If this g is positive definite, then (M, L) is called a *Finslerian manifold.* Clearly a Riemannian manifold is a Finslerian manifold.

Let $[a, b] \subset \mathbf{R}$. Then a geodesic $g : [a, b] \to M$ is a *geodesic loop* if $g(a) = g(b)$ (with self-intersections permitted). A *closed* or *periodic geodesic* is a non-constant geodesic loop with $g'(a) = g'(b)$. A geodesic is *simple* if g on $[a, b]$ is injective.

The QDS $\xi : S^1 \to S^{2n+1} \to \mathbf{CP}(n)$ has certain special properties. Namely there exists a point x in $E = S^{2n+1}$ such that :

(3.1) $\left\{\begin{array}{l}\text{all geodesics through } x \text{ are closed,}\\ \text{simple and of the same length.}\end{array}\right.$

In fact, for *every* point x in E and for every nonnull X in $T(E)_x$, the geodesics g with $g'(a) = X$ are geodesic loops, closed, simple and of the same length. Manifolds for which there exists a point x in E with property (3.1) are called B_x*-manifolds.*

LEMMA (Dazord [11]) 3.1. — *Finslerian* B_x*-manifolds are compact.*

The *(Morse) index* λ of a geodesic with initial point a is the number of conjugate points, counted with multiplicity, of point a on the geodesic arc $g(t)$ for $a < t < b$.

LEMMA (Dazord [11]) 3.2. — *If* E *is a Finslerian* B_x*-manifolds, then all geodesics from* x *have the same index* λ. *If* $\lambda > 0$, *then* $\pi_1(E) = 0$; *and if* $\lambda = 0$, *then* $\pi_1(E) = 0$ *or* $\mathbf{Z}_2$.

In Cartan's study of symmetric spaces he found

THEOREM (Cartan [10]) 3.3. — *The irreducible Riemannian symmetric spaces of rank one, namely* S^{n+1}, $\mathbf{C}P(n+1)$, $\mathbf{Q}P(n+1)$, $\mathbf{Ca}P(2)$, *are Riemannian* B_x*-manifolds; and these are the only Riemannian* B_x*-manifolds among the Riemannian symmetric manifolds.*

Bott [8] and Samelson [43] in the Riemannian case and Dazord [11] in the Finslerian case have studied the cohomological properties of B_x-manifolds and they have found :

THEOREM (Bott-Samelson-Dazord) 3.4. — *If* E *is a Finslerian* B_x*-manifold, and*

(1) *if* $\dim E = 2$, *then the universal covering of* E *is homeomorphic to* S^2 *and* $\pi_1(E) = 0$ *or* $\mathbf{Z}_2$;

(2) *if* $\dim E \geq 3$, *and*

- (*a*) *if* $\lambda = 0$, *then the universal covering of* E *is homological sphere*;
- (*b*) *if* $\lambda > 0$, *then* E *is simply connected and the integral cohomology ring* $H^*(E; \mathbf{Z})$ *is a truncated polynomial ring generated by a (homogeneous) element* X *of degree* $\lambda + 1$. [*That is,* $H^*(E^n; \mathbf{Z}) = \frac{\mathbf{Z}[\alpha]}{\alpha^{m+1}}$. *Or in other words* $H^{k(\lambda+1)}(E; \mathbf{Z}) = \mathbf{Z}$ *for* $k = 0, 1, 2, \ldots, m$, $H^p(E; \mathbf{Z}) = 0$ *otherwise, and* $n = m(\lambda + 1)$.]

Cf. also Nakagawa [40], and Allamigeon [3].

According to results of Adem [2], Milnor [34] and Adams [1], if a manifold has an integral cohomology ring

$$H^*(E^n; \mathbf{Z}) = \frac{\mathbf{Z}[\alpha]}{\alpha^{m+1}}, \tag{3.2}$$

then either $\alpha^2 = 0$ and $\deg(\alpha) = \dim E$, or $\alpha^2 \neq 0$ and $\lambda = \deg(\alpha) - 1$ is equal to 0, 1, 3, or 7. (Adem [2] showed that if $\deg \alpha = 2^n$, $\alpha^2 \neq 0$ and $n \geq 3$, then $\alpha^3 = 0$.) Thus if $\lambda = 7$, then $n = 16$. Summarizing we have

LEMMA 3.5. — *If* E *satisfies* (3.2), *then one of the following holds :*

(*a*) $\lambda = 0$, $n = m$, *and*

$$H^*(E; \mathbf{Z}) = H^*\left(\frac{SO(n+1)}{O(n)} = \mathbf{R}P(n); \mathbf{Z}\right);$$

(*b*) $\lambda = 1, n = 2m$, *and*

$$H^*(E; \mathbf{Z}) = H^*\left(\frac{SU(n+1)}{U(n)} = \mathbf{C}P(n); \mathbf{Z}\right);$$

(*c*) $\lambda = 3, n = 4m$, *and*

$$H^*(E; \mathbf{Z}) = H^*\left(\frac{Sp(n+1)}{Sp(n) \times Sp(1)} = \mathbf{Q}P(n); \mathbf{Z}\right);$$

(*d*) $\lambda = 7,\ m = 2,\ n = 16$, *and*

$$H^*(E; \mathbf{Z}) = H^*\left(\frac{F_4}{SO(9)} = \mathbf{Ca}P(2); \mathbf{Z}\right);$$

(*e*) $\lambda = n - 1,\ m = 1$, *and*

$$H^*(E; \mathbf{Z}) = H^*\left(\frac{SO(n+1)}{SO(n)} = S^n; \mathbf{Z}\right).$$

From Lemma 3.5 and Theorem 3.4 every simply connected Finslerian B_x-manifold has the same integral cohomology ring as Cartan's irreducible symmetric spaces of rank one in Theorem 3.3; and a non simply connected B_x-manifold has $H^*(E^n; \mathbf{Z}) \simeq H^*(\mathbf{R}P(n); \mathbf{Z})$.

Certain other results are known. Under restrictions on the conjugate locus, E in case (*b*) of Lemma 3.5 has the same homotopy type as $\mathbf{C}P(n)$ (*cf.* Klingenberg [29]). However, Eells and Kuiper [12] have constructed compact simply connected manifolds with the same integral cohomology as $\mathbf{Q}P(n)$ and $\mathbf{Ca}P(2)$, but which do not have the same homotopy type. Varga [51] has shown that even dimensional homogeneous Riemannian B_x-manifolds are homeomorphic to symmetric spaces of rank one.

By the Bott-Samelson-Dazord Theorem the only possible candidates for QDSs whose total space is an odd dimensional simply connected Finslerian B_x-manifold are the odd dimensional integral cohomology spheres [case (*e*) of Lemma 3.5]. By the Hurewicz and Whitehead theorems (*cf.* [21], [47]), E is a homotopy sphere and so homeomorphic to a sphere for $n \geq 5$ by Smale [44]. Which homotopy spheres are actually total spaces of QDSs or even FDSs remains, in general, an open question as was seen in paragraph 2. Harada [18] has shown that compact normal QDSs (E, Z) with $h > k^2$ (*cf.* § 2) have a structure very similar to Finslerian B_x-manifolds. Under the added condition that E has minimal diameter π, then E is a Finslerian B_x-manifolds but in fact isometric to S^{2n+1} with constant curvature 1.

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