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GÉRARD A. MAUGIN

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Section A :
Physique théorique.

An Action Principle in General Relativistic Magnetohydrodynamics

by

Gérard A. MAUGIN *

SUMMARY. — An action principle is presented in the Eulerian description for general relativistic magnetohydrodynamics. A generalized Clebsch's representation is thus obtained for the fluid, current and the specific enthalpy from which a Crocco-Varsonyi's theorem, the Euler equations of motion, a Bernoullian theorem and another remarkable from of the action principle where the matter Lagrangian is nothing but the thermodynamical pressure, follow. The case of general MHD (with nonlinear *electromagnetic* constitutive equations) and the case of perfect MHD (linear isotropic *magnetic* constitutive equation) are examined. Jump relations are obtained on an equal foot with the field equations. The differences and the points in common with an action principle given before in the Lagrangian description are discussed.

RÉSUMÉ. — On présente un principe variationnel en description eulérienne pour la magnétohydrodynamique en relativité générale. On obtient ainsi une représentation de Clebsch généralisée du courant fluide et de l'enthalpie spécifique. On en déduit un théorème de Crocco-Varsonyi, les équations d'Euler du mouvement, un théorème du genre « Bernoulli » et une autre forme remarquable du principe d'action où le lagrangien de la matière n'est autre que la pression thermodynamique. Le cas de la MHD où la loi de comportement *électromagnétique* est générale et non linéaire et le cas de la MHD parfaite (loi de comportement *magnétique*

* Formerly Research Associate, Princeton University, U. S. A. Presently with the CEDOCAR (*Centre de Documentation de l'Armement*), Ministry of National Defense, Paris, France.
Private address : 138, rue de la Madeleine, 49-Angers, France.

linéaire et isotrope) sont examinés. Les conditions de discontinuité sont obtenues en même temps que les équations du champ. Les différences et les points communs du présent article avec un principe variationnel donné précédemment en description lagrangienne sont discutés.

CONTENTS

1. INTRODUCTION

A current problem of mathematical physics (and pure mathematics by the same token) is the finding of a variational principle for a given system of differential equations. This is not always an amusement of mathematician, variational principles may be useful in view of applications e. g., the recent works of Whitham on wave propagation for conservative systems (Whitham, 1967) : by analogy, they may help to find out field equations yet unknown; they are used for setting down the differential equations of a given problem. The latter is particularly true of point mechanics where everyone is used to Lagrange's equations and the formulation of Hamilton's principle of action.

When one goes over to continuum mechanics, one is confronted with two possibilities : to use a Lagrangian (i. e., material or " reference ", or " undeformed state ") description or an Eulerian (i. e., spatial or " actual " or " deformed state ") description. In the former case, the extension of Hamilton's principle of point mechanics is straightforward. The similarity with a system of discrete particles is complete. This is mainly why we have established variational principles in the Lagrangian description : (*a*) in classical continuum mechanics, following the Cosserats' model (1909) (*nonlinear micromorphic media, cf.* Maugin, 1970; *magnetically saturated media,* Maugin and Eringen, 1972 *a*); (*b*) following Taub's type of action principle (Taub, 1949-1954-1957), in special relativistic continuum mechanics (*polarized nonlinear elastic solids endowed with a continuous repartition of electronic spins, cf.* Maugin and Eringen, 1972 *b*; Maugin, 1972 *b*; *relativistic continua with directors, cf.* Maugin and Eringen, 1972 *c*) and in general relativistic continuum mechanics (*nonlinear magnetized elastic materials, cf.* Maugin, 1971 *g*). In a later version of his variational principle (1969) applied to general relativistic magnetohydrodynamics, Taub uses co-moving coordinates as an artifice to replace an explicit reliance upon the generalized Lagrangian coordinates of relativistic continuum mechanics (*cf.* hereafter).

If one uses the so-called Eulerian description which is generally preferred in fluid dynamics and we guess, even more in magnetohydrodynamics, one loses the close similarity with a system of discrete particles. It is more difficult to construct variational principles and one gets more involved in straight mathematical manipulations. Mathematical problems are raised and it seems that " the Eulerian description is introduced primarily as a mathematical device " (Seliger and Whitham, 1968). As these authors say, " variational principles in Eulerian description were found at first by very special methods, merely by trials and errors ". Now, Clebsch (1859 *a, b*), Bateman (1929, 1944), Lin (1963), Herivel (1955), Eckart (1963), Serrin (1959) and Seliger and Whitham (1968) have contributed to a general background and it seems possible to outline a fairly general procedure. A significant analogy with the Pfaff's problem for differential forms as been pointed out by Seliger and Whitham (1968).

It is along these lines that we develop an action principle for general relativistic magnetohydrodynamics (compressible fluid) in the Eulerian description. In contrast with the variational formulation given before (*cf.* Maugin, 1971 *g*), we vary here independently as many variables as possible : the metric, the matter density, the four-velocity, the physical fields such as electromagnetic fields and the Lagrange multipliers. We do not arrive automatically at all field equations, i. e., we do not obtain directly from the variation, the Euler equations of motion. We need manipulate some of the equations resulting from the variation to obtain the latter. Remark that in Maugin (1971 *g*), we arrived at the conservation of energy-momentum on an equal foot with the Einstein equations by considering a combined variation of the metric and the particle path (the variation of the latter implying partly the variation of the former). However the present variational scheme allows to get a generalized Clebsch's representation for the fluid current (or for the *modified* fluid current in MHD) and for the enthalpy density. Direct consequences of those are a Crocco-Varsonyi's type of equation (called streamline equation by Lichnerowicz) and a Bernoullian theorem, the conditions of establishment of which do not imply irrotationality or stationarity but only conservation of entropy along a particle path, conservation (or continuity) of matter and conservation of the *identity* of particles. Moreover peculiar forms of the action principle can be deduced, quite remarkable and closely related to the form given by Bateman (1929, 1944) in classical hydrodynamics : the thermodynamical pressure is the matter Lagrangian. Finally we must note that by taking account of more constraints than it seems at first necessary, we have enlarged the class of possible flows (*cf.* the quite general representation of the fluid current).

The contents of the article extend to general relativity the results given by Maugin (1971 *e*, 1972 *a*) in special relativity. Of course the same method can be used for magnetized elastic solids however, while Eulerian descriptions are favored in fluid dynamics, one often uses indifferently Lagrangian or Eulerian (or even mixed) descriptions in solid mechanics. The emphasis must therefore be placed upon hydrodynamics. The theory of relativistic magnetohydrodynamics presented here is entirely consistent and self-contained, albeit the fact that the present article constitutes in many respects both a complement and a continuation to our precedent work (Maugin, 1971 *g*).

Notations and a precise definition of variations are given in the remainder of this section. Although no new material, we have recalled in section 2 some features of the conservation of gravitational energy-momentum. Section 3 deals with electromagnetic fields in matter and how one can introduce these concepts in the action principle. For the sake of generality, nonlinear constitutive equations for the electromagnetic fields are considered to start with. The matter Lagrangian for a compressible charged magnetized fluid and the constraints imposed upon the behavior of this fluid are examined in sections 4 and 5. The action principle and the resulting equations are given and commented upon in section 6. In section 7, the Euler equations of motion are deduced from the equations obtained in the preceding section. We look at the case of *perfect* magnetohydrodynamics in section 8, a linear constitutive equation being assumed for the magnetic field. A generalized Bernoullian theorem is given in section 9. Another possible form of the action principle is arrived at in section 10. The list of references given includes works which have not been referred to in the body of the text. These works may prove useful to the reader interested in further researches in the field.

Notations. — The notations used here are closely related to those of precedent notes or articles (*cf.* Maugin, 1971 *e*, 1971 *g*). V^4 is a Riemannian four-dimensional space-time manifold whose symmetric

metric $g_{\alpha\beta}$ (signature : $+, +, +, -$) is normal hyperbolic and assumed to be of class C^1, piecewise C^3. All greek indices assume the values 1, 2, 3, 4 while latin indices, unless mentioned, take the values 1, 2 and 3. x^α ($\alpha = 1, 2, 3, 4$; x^4 timelike) are the coordinates in V^4. The summation convention on diagonally repeated indices is used throughout the article. Brackets around a set of indices denote alternation. $\varepsilon^{\alpha\beta\gamma\delta}$ is the permutation symbol (not a tensor). Commas or symbols ∂ are used to denote partial differentiation with respect to x^α. Symbols ∇ are used for the covariant differentiation based on the metric $g_{\alpha\beta}$. g is the determinant of the $g_{\alpha\beta}$'s. The *direct* motion of a material particle is entirely described by the mapping of class C^2 $\{ \mathbf{x} : E^3 \times R \mapsto V^4 \}$

$$(1) \qquad x^\alpha = x^\alpha (X^\Delta), \qquad X^\Delta = (X^K, ic\,\tau), \qquad i = \sqrt{-1},$$

where X^K ($K = 1, 2, 3$) are a set of Lagrangian coordinates in E^3, the three-dimensional Euclidean space of reference. τ is chosen to be the propertime of the particle while c denotes the velocity of light in vacuum (X^Δ are generalized Lagrangian coordinates). X^K and, τ are independent variables such that the *inverse* motion of a particle

$$(2) \qquad X^K = X^K (x^\alpha), \qquad \text{with} \quad \frac{\partial X^K}{\partial \tau} = 0$$

is well-defined and assumed to be of class C^2. The world line in V^4 of a particle labeled (X^K) is denoted by $(\mathcal{C}_{X^K})$. u^α is the 4-velocity such that

$$(3) \qquad g_{\alpha\beta}\, u^\alpha u^\beta = -c^2, \qquad \text{thus} \quad u^\alpha \nabla_\beta u_\alpha = 0.$$

Moreover, we have

$$(4) \qquad \frac{\partial}{\partial \tau} \equiv u^\alpha \nabla_\alpha.$$

We recall that the operator of projection $P_{\alpha\beta}$ is defined according to (*cf.* Maugin, 1971 *b*) :

$$(5) \qquad \begin{cases} P_{\alpha\beta} = g_{\alpha\beta} + \dfrac{1}{c^2} u_\alpha u_\beta, & P^{\alpha}_{.\alpha} = 3, \\ P_{\alpha\beta} P^{\beta\gamma} = P_{\alpha}^{.\gamma}, & P_{\alpha\beta} u^\beta = 0. \end{cases}$$

We have symbolically the generalized Green-Gauss theorem (*cf.* Maugin, 1971 *g*), valid for an arbitrary tensor fiield $\mathbf{V}$:

$$(6) \qquad \int_{(\mathcal{B} - \Sigma)} \operatorname{div} \mathbf{V}\, d^4\mathbf{x} + \int_{(\Sigma)} [\mathbf{V}].\mathbf{N}_\Sigma\, d^3\sigma = \int_{(\partial\mathcal{B} - \Sigma)} \mathbf{V}.\mathbf{n}\, d^3 s,$$

where $(\mathcal{B})$, $(\partial\mathcal{B})$ and (Σ) indicate respectively a closed four-dimensional region of V^4, its regular boundary whose unit exterior oriented normal is $\mathbf{n}$, and a three-dimensional discontinuity hypersurface whithin $(\mathcal{B})$, whose unit oriented normal is $\mathbf{N}_\Sigma$. The familiar symbolism $[\ldots]$ denotes the jump across (Σ).

Definition of variations. — To avoid any misunderstanding in the subsequent developments, we need a non-ambiguous definition of the variations. Let $\mathcal{F}$ be a tensor-valued functional on V^4 of the indexed series of tensorial arguments $\mathbf{A}_{(i)}$, $(i) = 1, 2, \ldots, N$ (e. g., invariant scalars, 4-vectors, 2-forms, general tensors of n-th order). Each $\mathbf{A}_{(i)}$ is supposed to belong to a normed linear space $\mathcal{H}_{(i)}$ in which a norm $\|\ldots\|_{\mathcal{H}_{(i)}}$ is well-defined (we need not go further along these topological considerations). We suppose that $\mathcal{F}$ is continuously Fréchet differentiable throughout its domain of definition (with possible exceptions on curves, two- and three-dimensional hypersurfaces within this domain). This smoothness condition guarantees the existence of the following *Fréchet derivative* of $\mathcal{F}$ with respect to $\mathbf{A}_{(k)}$ (*cf.* Tapia, 1971; Maugin, 1972 *d*), for every tensor $\mathbf{L}_{(k)}$ belonging to $\mathcal{H}_{(k)}$, $1 \leqq k \leqq N$ and any scalar parameter λ :

$$\mathrm{D}_{\mathbf{A}_{(k)}} \mathcal{F} \{ \mathbf{A}_{(i)} \} = \frac{\partial}{\partial\lambda} \mathcal{F} \{ \mathbf{A}_{(i)\neq(k)}, \mathbf{A}_{(k)} + \lambda \mathbf{L}_{(k)} \} \Big|_{\lambda=0}. \tag{7}$$

This expression is linear in $\mathbf{L}_{(k)}$ and jointly continuous in $\mathbf{A}_{(k)}$ and $\mathbf{L}_{(k)}$. The variation of $\mathcal{F}$ implied by a variation of the tensorial argument $\mathbf{A}_{(k)}$ is then defined as

$$\left\{ \begin{aligned} &\delta_{\mathbf{A}_{(k)}} \mathcal{F} \{ \mathbf{A}_{(i)} \} = \mathrm{D}_{\mathbf{A}_{(k)}} \mathcal{F} \{ \mathbf{A}_{(i)} \} . \delta\mathbf{A}_{(k)}, \\ &\text{with} \quad \delta\mathbf{A}_{(k)} = \frac{\partial}{\partial\varepsilon} \mathbf{A}_{(k)} (x^\alpha, \varepsilon) \Big|_{\varepsilon=0}, \end{aligned} \right. \tag{8}$$

where ε is an arbitrary parameter equal to zero in absence of perturbation. In most cases. the functional dependence of $\mathcal{F}$ reduces to a dependence on the tensorial arguments at x^α and on their first spatial derivative (in V^4) i. e., $\mathrm{D}_{\mathbf{A}_{(k)}} \mathcal{F}$ degenerates into the classical Euler-Lagrange derivative

$$\mathrm{D}_{\mathbf{A}_{(k)}} \mathcal{F} \{ \mathbf{A}_{(i)} \} = \left[\frac{\partial\mathcal{F}}{\partial\mathbf{A}_{(k)}} - \partial_\alpha \frac{\partial\mathcal{F}}{\partial(\partial_\alpha \mathbf{A}_{(k)})} \right]_{\substack{\mathbf{A}_{(i)\neq(k)} \\ \text{kept fixed}}}. \tag{9}$$

In the sequel, the definitions (9) and $(8)_2$ are assumed with no further reference.

2. EINSTEIN EQUATIONS AND CONSERVATION OF ENERGY-MOMENTUM

We recall that the field equations that govern : (*a*) the geometry of the Einstein Riemannian space-time manifold; (*b*) the dynamical behavior of a self-gravitating material (whose nature and constitutive equations remain to be specified) in interaction with other fields than gravity, e. g., electromagnetic fields; (*c*) these other fields as well, can be derived from a general variational principle whose form is

$$\delta \mathrm{I}_{\text{total}} = 0, \qquad \mathrm{I}_{\text{total}} = \int_{(\mathcal{B})} (\mathrm{L}_{\mathrm{G}} + \mathrm{L}_{\mathrm{F}})\, \hat{\mathbf{1}}, \tag{10}$$

where $(\mathcal{B})$ is a closed four-dimensional region of V^4 [1] whose boundary $(\partial\mathcal{B})$ is regular enough to allow, if necessary, the use of integral vectorial analysis. The meaning of the variation symbol (i. e., what is varied) remains to be specified. This will be done later on. $\hat{\mathbf{1}}$ is the dual of unity or element of volume in V^4 defined as

$$\begin{cases} \hat{\mathbf{1}} = \sqrt{-g}\, d^4\,\mathbf{x}, \\ d^4\,\mathbf{x} = \bigwedge_{\alpha} \mathbf{d}x^{\alpha} \equiv \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 \wedge \mathbf{d}x^4, \end{cases} \tag{11}$$

the $\mathbf{d}x^{\alpha}$ being the elementary one-forms and the symbol $\wedge$ denoting the exterior product.

In equation (10), L_{G} is the Lagrangian density of the gravitational field, L_{F} being the total Lagrangian density of other fields. A more conventional form of equation (10) is

$$\delta \int_{(\mathcal{B})} (\mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{F}})\, d^4\,\mathbf{x} = 0, \tag{12}$$

where the usual form of $\mathcal{L}_{\mathrm{G}}$ is given by the *scalar density*

$$\mathcal{L}_{\mathrm{G}} = (2\,\varkappa)^{-1}\, \mathfrak{R} = (2\,\varkappa)^{-1} \sqrt{-g}\,\mathbf{R} \tag{13}$$

in which $\mathbf{R}$ is the Ricci curvature defined as

$$\mathbf{R} = \mathrm{R}_{\beta\delta}\, g^{\beta\delta}, \qquad \mathrm{R}_{\beta\delta} = \mathrm{R}^{\alpha}_{\cdot\,\beta\gamma\delta}\, g_{\alpha}^{\ \beta} \tag{14}$$

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[1] We are examining the *interior* problem.

with

$$R^{\alpha}_{.\beta\gamma\delta} = 2\,\Gamma^{\alpha}_{\beta.[\delta,\gamma]} + 2\,\Gamma^{\mu}_{\beta.[\delta}\,\Gamma^{\alpha}_{\gamma].\mu}; \tag{15}$$

$$\Gamma^{\alpha}_{\beta.\delta} = \frac{1}{2}\, g^{\alpha\delta}\,(\partial_{\gamma}\, g_{\beta\delta} + \partial_{\beta}\, g_{\delta\gamma} - \partial_{\delta}\, g_{\beta\gamma}) \tag{16}$$

$\varkappa$ is a constant proportional to Newton's gravitational constant k $\left(\varkappa = \frac{8\,\pi\,k}{c^4}\right)$. A useful form of equation (13) is (*cf.* Mitskevich, 1969, p. 60; or Weber, 1961, p. 72)

$$\mathcal{L}_{G} = \Lambda_{G} + \partial_{\alpha}\,\lambda^{\alpha}_{G} \tag{17}$$

with

$$\Lambda_{G} \equiv (2\,\varkappa)^{-1}\sqrt{-g}\; g^{\mu\nu}\,(\Gamma^{\alpha}_{\mu.\nu}\,\Gamma^{\beta}_{\alpha.\beta} - \Gamma^{\alpha}_{\beta.\mu}\,\Gamma^{\beta}_{\alpha.\nu}); \tag{18}$$

$$\lambda^{\alpha}_{G} \equiv (2\,\varkappa)^{-1}\sqrt{-g}\,(\Gamma^{\nu}_{\beta.\nu}\, g^{\alpha\beta} - \Gamma^{\alpha}_{\beta.\nu}\, g^{\beta\nu}). \tag{19}$$

The first expression Λ_G is only a function of $g_{\alpha\beta}$ and its first derivatives $\partial_{\gamma} g_{\alpha\beta}$. Other variations apart, let us consider a variation of the metric $g_{\alpha\beta}$. Assuming that in $\mathcal{L}_{F} = \sqrt{-g}\,L_{F}$ only $g_{\alpha\beta}$ and its first derivatives appear and, for the time being, the variations of $g_{\alpha\beta}$ vanishing on $(\partial\mathcal{B})$ [equivalently the material and the fields fill up the whole space-time manifold and we do not consider contributions at infinity [2]] then, the Euler-Lagrange equations corresponding to $\delta g_{\alpha\beta}$ are none other than the Einstein field equations

$$\delta g_{\alpha\beta}: \quad \frac{\delta\mathcal{L}}{\delta g_{\alpha\beta}} = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} - \partial_{\gamma}\frac{\partial\mathcal{L}}{\partial\,(\partial_{\gamma}\, g_{\alpha\beta})} = 0, \tag{20}$$

where

$$\mathcal{L} = \mathcal{L}_{G} + \mathcal{L}_{F}$$

or, according to (17), (18) and usual computations (*cf.* Landau and Lifshitz, 1962)

$$\delta g_{\alpha\beta}: \quad (A^{\alpha\beta} - \varkappa\, T^{\alpha\beta})\sqrt{-g} = 0, \tag{21}$$

where the Einstein-Cartan tensor $A_{\alpha\beta}$ and the stress-energy-momentum tensor $T_{\alpha\beta}$ have been defined by

$$\left\{\begin{aligned} &\frac{\delta\mathcal{L}_{G}}{\delta g_{\alpha\beta}} = (2\,\varkappa)^{-1}\, A^{\alpha\beta}\sqrt{-g}, \qquad A^{\alpha\beta} \stackrel{\text{def}}{=} R^{\alpha\beta} - \frac{1}{2}\, g^{\alpha\beta}\,\mathbf{R},\\ &\frac{\delta\mathcal{L}_{F}}{\delta g_{\alpha\beta}} = -\frac{1}{2}\, T^{\alpha\beta}\sqrt{-g}. \end{aligned}\right. \tag{22}$$

(2) This must be dealt with carefully in cosmology, depending on the type of universe considered.

According to Bianchi's identities,

$$\nabla_\beta A^{\alpha\beta} = 0,$$

the equations

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{23}$$

follow from equation (21). Equation (23) is the usual form of the conservation of energy and momentum (for instance, in special relativity with curvilinear coordinates in the space-time manifold of Minkowski M^4). It does not however include clearly the conservation of energy and momentum of the gravitational field, the definition of the two latter quantities being somewhat loose. Following a classical method of analytical mechanics (*cf.* Weber, 1962, p. 45), it is possible to construct a conservation law which includes the gravitational effects on an equal foot with those of other fields. Define a canonical stress-energy-momentum *pseudotensor* of the gravitational field by

$$t_\mu^{\ \nu}\sqrt{-g} = \left[\delta_\mu^\nu \mathcal{L}_G - (\partial_\mu g_{\gamma\beta}) \frac{\partial \mathcal{L}_G}{\partial(\partial_\nu g_{\gamma\beta})}\right], \tag{24}$$

then, it is easily shown from (20) that

$$\partial_\nu\left[(t_\mu^{\ \nu} + T_\mu^{\ \nu})\sqrt{-g}\right] = 0 \tag{25}$$

which is the conservation law looked for.

Before going to our specific subject, let us remark that :

(*a*) In most cases encountered, $\mathcal{L}_F$ depends only on the $g_{\alpha\beta}$'s and not on their derivatives. We may thus replace the Euler-Lagrange derivative in equation $(22)_2$ by the partial derivative and write

$$T^{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_F}{\partial g_{\alpha\beta}}. \tag{26}$$

(*b*) It is most convenient in order to calculate $T^{\alpha\beta}$ from a known form of $\mathcal{L}_F$ to note that, of course, physically, the $g_{\alpha\beta}$'s represent the gravitational potentials but also, from a mathematical viewpoint, that the metric $g_{\alpha\beta}$ of V^4 associates a one-form with a tangent vector or, quoting J. A. Wheeler (1962, 1968), is a " prescription to get a squared length from a tangent vector " and " a machine into which to drop two-vectors if taking a scalar product " i. e., $\mathbf{v}$ and $\boldsymbol{\omega}$ being respectively a tangent vector and a one-form ($\mathbf{e}_\alpha$ denoting the basis tangent vectors and $\mathbf{d}x^\alpha$ the basis one-forms), one has

$$\begin{cases} \mathbf{v} = v^\alpha \mathbf{e}_\alpha, \qquad \boldsymbol{\omega} = \omega_\mu \mathbf{d}x^\mu, \\ \omega_\mu = g_{\mu\nu} v^\nu, \qquad \langle \mathbf{v} . \mathbf{v} \rangle = g_{\alpha\beta} v^\alpha v^\beta, \\ \|\mathbf{v}\|^2 = \langle \mathbf{v} . \mathbf{v} \rangle \equiv \langle \mathbf{v}, \boldsymbol{\omega} \rangle. \end{cases} \tag{27}$$

The symbolism $\langle \ldots\ldots\ldots \rangle$ which represents the inner product, makes use of the metric while the symbolism $\langle \ldots, \ldots \rangle$ makes no use of the metric. Hence **u** being a tangent vector and **F** and **G** being two two-forms, an independent variation of $g_{\alpha\beta}$ yields

$$(28)\qquad \begin{cases} \delta_G \langle \mathbf{u}.\mathbf{u} \rangle = u^\alpha\, u^\beta\, \delta g_{\alpha\beta}, \\ \delta_G \langle \mathbf{F}.\mathbf{G} \rangle = \delta_G \left(\frac{1}{2} F_{\alpha\beta}\, G^{\beta\alpha} \right) = F_{\alpha\beta}\, G^{\beta}_{.\delta}\, \delta g^{\alpha\delta}. \end{cases}$$

(*c*) Finally, the forthcoming variations are useful in the derivation of the expression (26) :

$$(29)\qquad \begin{cases} \delta_G\, g^{\alpha\beta} = -\, g^{\alpha\beta}\, g^{\beta\delta}\, \delta g_{\gamma\delta}, \\ \delta_G \sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\, \delta g_{\alpha\beta}, \\ \delta_G\, \hat{\mathbf{1}} = -\frac{1}{2}\, g^{\alpha\beta}\, \delta g_{\alpha\beta}\, \hat{\mathbf{1}}. \end{cases}$$

In presence of a three-dimensional discontinuity hypersurface (Σ) within $(\mathcal{B})$, using the Generalized Green-Gauss theorem (6) and discarding the contribution on the boundary $(\partial\mathcal{B})$ [since variations shall be taken identically zero on $(\partial\mathcal{B})$], we will write after (18) and (19) :

$$(30)\qquad I_G = \int_{(\mathcal{B})} \Lambda_G\, d^4\, \mathbf{x} - \int_{(\Sigma)} [N_\alpha\, \lambda_G^\alpha]\, d^3\, \sigma.$$

Independent variations of $g_{\alpha\beta}$ in the surface term of this expression would yield, by setting the coefficients of $\delta g_{\alpha\beta}$ and $\delta\,(\partial_\gamma\, g_{\alpha\beta})$ separately equal to zero (*cf.* Maugin, 1971 *g*; Taub, 1957)

$$(31)\qquad \begin{cases} [\sqrt{-g}\,(g^{\alpha\beta}\, g^{\gamma\delta} - g^{\beta\delta}\, g^{\alpha\gamma})] = 0, \\ [\sqrt{-g}\,(g^{\alpha\beta}\, g^{\gamma\delta} - g^{\beta\delta}\, g^{\alpha\gamma})\,(\delta^\sigma_\alpha\, \Gamma^\lambda_{\beta.\gamma} + \delta^\sigma_\beta\, \Gamma^\lambda_{\alpha.\lambda})]\, N_\delta = 0 \end{cases}$$

across (Σ). These two jump relations are identically satisfied if one assumes the Lichnerowicz's conditions of continuity for $g_{\alpha\beta}$ (C^1, piecewise C^3).

3. ELECTROMAGNETIC FIELD

Let **F** be the *magnetic flux* two-form, **G** be the *electric displacement-magnetic intensity* two-form and **J** the 4-*electric current*. In a curved Riemannian space-time manifold, the Maxwell equations read :

$$(32)\qquad \nabla_\beta\, G^{\alpha\beta} = \frac{1}{c}\, J^\alpha \;\Rightarrow\; \nabla_\alpha\, J^\alpha = 0, \quad \text{in} \quad (\mathcal{B});$$

$$(33)\qquad \nabla_\delta\, [(-\,g)^{-\frac{1}{2}}\, \varepsilon^{\alpha\beta\gamma\delta}\, F_{\beta\gamma}] = 0, \quad \text{in } V^4.$$

Equations (33) are valid everywhere in V^4 while (32) are valid only in matter [in free space, (32) takes the form : $\nabla_\beta F^{\alpha\beta} = 0$]. Since $F_{\alpha\beta}$ and $G_{\alpha\beta}$ are skew-symmetric, equations (32) and (33) can be written as

$$\frac{1}{\sqrt{-g}}\,\partial_\beta\left(\sqrt{-g}\,G^{\alpha\beta}\right) = \frac{1}{c}\,J^\alpha, \quad \text{in } (\mathcal{B}); \tag{34}$$

$$g^{-1}\,\varepsilon^{\alpha\beta\gamma\delta}\,\partial_\delta\left(\sqrt{-g}\,F_{\beta\gamma}\right) = 0, \quad \text{in } V^4. \tag{35}$$

Equation (35) implies the existence of a one-form $\mathbf{A}$ called the *electromagnetic potential* from which $\mathbf{F}$ is defined by

$$F_{\alpha\beta} = 2\,\nabla_{[\alpha}\,A_{\beta]} \equiv 2\,\partial_{[\alpha}\,A_{\beta]}. \tag{36}$$

Corresponding to equations (34) and (35), we have the jump relations across the discontinuity hypersurface (Σ) :

$$\begin{cases} [N_\beta\sqrt{-g}\,G^{\alpha\beta}] = 0, \qquad [\sqrt{-g}\,J^\alpha\,N_\alpha] = 0, \\ \varepsilon^{\alpha\beta\gamma\delta}\,[N_\delta\,\sqrt{-g}\,F_{\beta\gamma}] = 0 \end{cases} \tag{37}$$

if there is no surface current on (Σ).

Following the *exposé* of our note (*cf.* Maugin, 1972 *a*), we consider the invariant of the electromagnetic field

$$\varphi = \frac{1}{2}\langle \mathbf{F},\ \mathbf{F}\rangle, \qquad \langle \mathbf{F},\ \mathbf{F}\rangle = \frac{1}{2}\,F_{\alpha\beta}\,F^{\beta\alpha} \tag{38}$$

and, using equation (34) as a constraint for which a Lagrange multiplier A_α is introduced, consider the integral invariant

$$I_{(em)} = \int_{(\mathcal{B})} \varphi\,\hat{\mathbf{1}} + \int_{(\mathcal{B})} A_\alpha\left[\frac{1}{\sqrt{-g}}\,\partial_\beta\left(\sqrt{-g}\,G^{\alpha\beta}\right) - \frac{1}{c}\,J^\alpha\right]\hat{\mathbf{1}}. \tag{39}$$

Independant variations of A_α in equation (39) yield, of course, equation (34) while, identifying A_α with the 4-electromagnetic potential, equation (35) is identically satisfied. Upon using equations (11), (36), noting ([3])

$$\Phi \stackrel{\text{def}}{=} \langle \mathbf{F}.\mathbf{G}\rangle \equiv \frac{1}{2}\,F_{\alpha\beta}\,G^{\beta\alpha} \tag{40}$$

and using the definitions of inner products given in (27), we write equation (39) as

$$I_{(em)} = \int_{(\mathcal{B})} \left(\Lambda_{(em)} + \partial_\alpha\,\lambda^\alpha_{(em)}\right) d^4\,\mathbf{x}, \tag{41}$$

([3]) Φ and φ are different from those defined in Maugin, 1971 g.

where we have defined

$$(42) \qquad \begin{cases} \Lambda_{(em)} \equiv \sqrt{-g}\,(\varphi - \Phi - c^{-1} \langle \mathbf{A}, \mathbf{J} \rangle), \\ \lambda^{\alpha}_{(em)} \equiv \sqrt{-g}\, A_\beta\, G^{\beta\alpha}. \end{cases}$$

That is, equation (39) can be written in the form already given to I_G. What is the effect of a variation of the metric upon the integral $I_{(em)}$? Discarding for the time being the case for which a discontinuity hypersurface exists within ($\mathcal{B}$), the variations of $g_{\alpha\beta}$ vanishing on ($\partial\mathcal{B}$) and using the results (28)-(29), we get after some calculations

$$(43) \quad \delta_G\, I_{(em)} = -\int_{(\mathcal{B})} \frac{1}{2}\left(-F^{\alpha}_{.\mu}\, G^{\mu\beta} + \varphi\, g^{\alpha\beta} - \frac{1}{c} J^\mu A_\mu\, g^{\alpha\beta}\right) \delta g_{\alpha\beta} \sqrt{-g}\, d^4\mathbf{x},$$

i. e., setting

$$(44) \qquad T^{\alpha\beta}_{(em)} = -F^{\alpha}_{.\mu}\, G^{\mu\beta} + \varphi\, g^{\alpha\beta} - \frac{1}{c} J^\mu A_\mu\, g^{\alpha\beta},$$

we have indeed

$$(45) \qquad T^{\alpha\beta}_{(em)} \equiv -\frac{2}{\sqrt{-g}} \frac{\partial \Lambda_{(em)}}{\partial g_{\alpha\beta}}$$

which does agree with the general formula (26). $T^{\alpha\beta}_{(em)}$ is the stress-energy-momentum tensor that results from the electromagnetic field in presence of matter. A form almost similar has been used by Grot and Eringen (1966 *a, b*), Grot (1970), Maugin and Eringen (1972 *b*) and Maugin (1971 *f, g*; 1972 *a, b, c*). Its form differs from that of the Maxwell tensor used by different authors. Although the Pirandellian assumption " to each one his truth " is never more salient than in dealing with the choice of the form of $T^{\alpha\beta}_{(em)}$, it can be shown by applying well-chosen Legendre transformations that some of these forms and the one given here are equivalent. This question is delayed until we have introduced the notion of internal energy of the material medium in presence of electromagnetic fields. It will be examined in another paper. Presently, the appearance of the invariant $\langle \mathbf{A}, \mathbf{J} \rangle$ in $T^{\alpha\beta}_{(em)}$ is rather peculiar although we note that currents and potentials appear in electromagnetic energy-momentum tensors of other theories e. g., Mie's theory of electrodynamics (*cf.* Sen, 1968, p. 41). In any case, as will be shown in section 7, the present treatment is entirely self-contained and needs no reference to other treatments except for the sake of comparison.

It is of interest for the sequel to consider the case of the discontinuity hypersurface (Σ). Assuming as before that variations vanish on ($\partial\mathcal{B}$), we can drop the surface contribution on ($\partial\mathcal{B}$) but, applying the theorem (6), $I_{(em)}$ will be written :

$$(46) \qquad I_{(em)} = \int_{(\mathcal{B})} \Lambda_{(em)}\, d^4\,\mathbf{x} - \int_{(\Sigma)} [2 \langle \mathbf{N} \otimes \mathbf{A},\ \mathbf{G} \rangle \sqrt{-g}]\, d^3\,\sigma$$

in which the symbol $\otimes$ indicates the tensorial product. Of course,

$$(47) \qquad \sqrt{-g} \langle \mathbf{N} \otimes \mathbf{A}, \mathbf{G} \rangle \equiv \frac{1}{2} \sqrt{-g}\, N_\alpha\, A_\beta\, G^{\beta\alpha} = \frac{1}{2} N_\alpha\, \lambda^\alpha_{(em)}.$$

An independent variation of $\mathbf{A}$ in the second term of equation (46), would yield the jump relation $(37)_1$.

Another variation shall be considered in the sequel. Let $\delta F_{\alpha\beta}$ be an *independent* variation of the magnetic flux two-form. The resulting variation in equation (41) is

$$(48) \qquad \delta_{\mathbf{F}}\, I_{(em)} = - \int_{(\mathcal{B})} \frac{1}{2} (F^{\alpha\beta} - G^{\alpha\beta})\, \delta F_{\alpha\beta}\, \hat{\mathbf{1}}.$$

4. THE MATTER LAGRANGIAN

Let ρ, $\in$ and η be respectively the proper density of matter, the relativistic internal energy density per unit of proper mass ($\in$ reduces to the internal energy density of classical continuum mechanics in a local rest frame) and the proper density of entropy. $\in$ takes account of the presence of matter, in the present case, of a *perfect compressible fluid* hence the classical dependence of $\in$ upon ρ and η. We also consider that $\in$ takes partly account of the interactions between this fluid and the electromagnetic fields from which follows the notion of polarization-magnetization, a synthesized entity in four-dimensional formalism. Note however that $\in$ does not account for the rest energy. We thus write :

$$(49) \qquad \begin{cases} I_m = \int_{(\mathcal{B})} \Lambda_m\, d^4\,\mathbf{x}, \\ \Lambda_m = - \rho\, [c^2 + \in (\rho, \eta\ F^{\alpha\beta})] \sqrt{-g}. \end{cases}$$

The first term $\rho\, c^2 \sqrt{-g}$ is the invariant which takes care of the rest energy.

We now examine the effects upon I_m of independent variations of $g_{\alpha\beta}$, η, ρ and $F_{\alpha\beta}$. First, we have

$$(50) \qquad \delta_{\mathbf{F}}\, I_m = - \int_{(\mathcal{B})} \rho \frac{\partial \in}{\partial F_{\alpha\beta}}\, \delta F_{\alpha\beta} \sqrt{-g}\, d^4\,\mathbf{x},$$

second,

$$(51) \qquad \delta_\eta\, I_m = - \int_{(\mathcal{B})} \rho \frac{\partial \in}{\partial \eta}\, \delta\eta \sqrt{-g}\, d^4\,\mathbf{x},$$

third,

$$\delta_\rho \, \mathrm{I}_m = -\int_{(\mathcal{B})} \left[c^2 + \epsilon + \rho \frac{\partial \epsilon}{\partial \rho} \right] \delta \rho \sqrt{-g} \, d^4 \mathbf{x}. \tag{52}$$

The effect of a variation of the metric is more involved since the quantity $\rho \sqrt{-g}$ is, as pointed out above, an invariant. Thus we can write

$$\frac{\delta \rho}{\rho} + \frac{\delta \sqrt{-g}}{\sqrt{-g}} = 0, \qquad \text{or} \quad \delta \rho = -\rho \frac{\delta \sqrt{-g}}{\sqrt{-g}}. \tag{53}$$

That is, a variation of $g_{\alpha\beta}$ induces a variation of the matter density ρ. From equations (53) and (29), we then have

$$\delta_{\mathrm{G}} \rho + \rho \, \delta_{\mathrm{G}} \sqrt{-g} = 0; \tag{54}$$

$$\delta_{\mathrm{G}} \rho = -\frac{1}{2} \rho \, g^{\alpha\beta} \, \delta g_{\alpha\beta}. \tag{55}$$

It follows that

$$\delta_{\mathrm{G}} \, \mathrm{I}_m = -\int_{(\mathcal{B})} \frac{1}{2} \rho^2 \frac{\partial \epsilon}{\partial \rho} g^{\alpha\beta} \, \delta g_{\alpha\beta} \sqrt{-g} \, d^4 \mathbf{x}. \tag{56}$$

Now use the thermodynamical differential equation for a polarized-magnetized perfect fluid (*cf.* Maugin, 1972 *a*; Fokker, 1939)

$$d\epsilon = \theta \, d\eta - p \, d\frac{1}{\rho} + \frac{1}{2} \tilde{\pi}_{\alpha\beta} \, d\mathrm{F}^{\beta\alpha}, \qquad \tilde{\pi}_{\alpha\beta} \equiv \frac{\pi_{\alpha\beta}}{\rho} \tag{57}$$

in which θ is the proper *thermodynamical temperature*, p is the *thermodynamical pressure*, and π is the *polarization-magnetization* two-form per unit volume and $\tilde{\pi}$ the magnetization two-form per unit of proper mass,

$$\frac{\partial \epsilon}{\partial \eta} = \theta, \qquad \frac{\partial \epsilon}{\partial \rho} = \frac{p}{\rho^2}, \qquad \frac{\partial \epsilon}{\partial \mathrm{F}_{\alpha\beta}} = \frac{1}{2} \tilde{\pi}^{\beta\alpha} \tag{58}$$

Equivalently, introducing the proper density of *magneto-enthalpy* $\mathbf{i}$ by the relation (*cf.* Fokker, 1939) [4] :

$$\mathbf{i} = \epsilon + \frac{p}{\rho} + \frac{1}{2} \tilde{\pi}_{\alpha\beta} \, \mathrm{F}^{\alpha\beta}, \tag{59}$$

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[4] This is nothing but a Legendre transformation with $\mathbf{i}$ as generating function $\left(\text{canonical transformation with variables } \frac{1}{\rho} \text{ and } \mathrm{F}^{\alpha\beta}\right)$. Another possibility consists in carrying out a Legendre transformation with another generating function (canonical transformation with variables η and $\mathrm{F}^{\alpha\beta}$). In the latter case, we would introduce in lieu of $\mathbf{i}$, a proper density of *magneto*-Helmholtz free energy function Ψ by

$$\Psi = \epsilon - \eta\theta + \frac{1}{2} \tilde{\pi}_{\alpha\beta} \, \mathrm{F}^{\alpha\beta},$$

the present treatment would then become closer to that given in Maugin (1971 *g*).

equation (57) can be written

(60) $$d\mathbf{i} = \theta\, d\eta + \frac{1}{\rho}\, dp + \frac{1}{2} F^{\alpha\beta}\, d\tilde{\pi}_{\alpha\beta},$$

hence, considering $\mathbf{i} = \mathbf{i}\,(\eta,\ p,\ \tilde{\pi}^{\alpha\beta})$, we have

(61) $$\frac{\partial \mathbf{i}}{\partial \eta} = \theta, \qquad \frac{\partial \mathbf{i}}{\partial p} = \frac{1}{\rho}, \qquad \frac{\partial \mathbf{i}}{\partial \tilde{\pi}_{\alpha\beta}} = \frac{1}{2} F^{\alpha\beta}.$$

Upon use of equations (58), the variations (50)-(52) and (56) read :

(62) $$\partial_{\mathbf{F}}\, I_m = -\int_{\mathcal{B}} \frac{1}{2}\pi^{\beta\alpha}\, \delta F_{\alpha\beta} \sqrt{-g}\, d^4\, \mathbf{x};$$

(63) $$\delta_\eta\, I_m = -\int_{(\mathcal{B})} \rho\theta\, \delta\eta \sqrt{-g}\, d^4\, \mathbf{x};$$

(64) $$\delta_\rho\, I_m = -\int_{(\mathcal{B})} \left(c^2 + \mathbf{i} + \frac{1}{2}\tilde{\pi}_{\alpha\beta}\, F^{\beta\alpha}\right) \delta\rho \sqrt{-g}\, d^4\, \mathbf{x};$$

(65) $$\delta_G\, I_m = -\int_{(\mathcal{B})} \frac{1}{2} p g^{\alpha\beta}\, \delta g_{\alpha\beta} \sqrt{-g}\, d^4\, \mathbf{x}.$$

Note that I_m does not depend explicity on u^α, thus

(66) $$\delta_{\mathbf{u}}\, I_m = 0.$$

5. CONSTRAINTS

We now examine the constraints imposed upon the behavior of the material under consideration and upon the flow of this material.

(*a*) The 4-velocity has a constant length. That is,

(67) $$g_{\alpha\beta}\, u^\alpha\, u^\beta + c^2 = 0, \qquad \text{or} \quad \langle \mathbf{u} . \mathbf{u} \rangle + c^2 = 0.$$

(*b*) As time goes on, the flow respects the continuity equation usually written as

(68) $$\nabla_\beta\, (\rho\, u^\beta) = 0.$$

(*c*) The flow is isentropic i. e., along a world line ($\mathcal{C}_{X^K}$), we have

(69) $$\frac{\partial \eta}{\partial \tau} = 0, \qquad \text{i. e.,} \quad \rho\, u^\beta\, \nabla_\beta\, \eta = 0,$$

or, using equation (68),

(70) $$\nabla_\beta\, (\rho\, u^\beta\, \eta) = 0.$$

In order to respect this condition, the Joule term of electric dissipation must vanish. This is satisfied if the four-electric current J^α introduced in equations (32) is due only to convection or, in other words, is proportional to the 4-velocity (5)

$$(71) \qquad J^\alpha = q u^\alpha, \qquad \text{hence} \qquad J^\alpha P_{\alpha\beta} = 0.$$

This implies that an independent variation of the 4-velocity yields a variation of $I_{(em)}$ equal to [discarding contributions on the boundary $(\partial\mathcal{B})$ and the case of the discontinuity hypersurface (Σ)]

$$(72) \qquad \delta_{\mathbf{u}} I_{(em)} = -\int_{(\mathcal{B})} \frac{1}{c} q A^\alpha \delta u_\alpha \sqrt{-g}\, d^4 \mathbf{x}.$$

Remark that the constraint (69) does not imply in general that the flow is irrotational [it is well-known that, in classical gaz dynamics, beyond a curved shock surface, we have an equation similar to (69) along streamlines while, η differing from a streamline to another, the flow is rotational, *cf.* Crocco-Varsonyi's theorem].

(*d*) The identity of a particle labeled (X^K) is preserved as time goes on (i. e., in the language of classical continuum mechanics, the variation is carried out in the Eulerian description). We have [*cf.* eq. $(2)_2$]

$$(73) \qquad \frac{\partial X^K}{\partial \tau} = 0, \qquad \text{i. e.,} \quad \rho u^\beta \nabla_\beta X^K = 0$$

or, using equation (68),

$$(74) \qquad \nabla_\beta (\rho u^\beta X^K) = 0.$$

This is known as a *Lin's constraint* (*cf.* Lin, 1963).

To take account of the constraints (67), (68), (70) and (74), we introduce six Lagrange multipliers f, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_K$ (K = 1, 2, 3) and construct the integral invariant

$$(75) \qquad I^* = \int_{(\mathcal{B})} \Big\{ \frac{1}{2} \rho f(\langle \mathbf{u} . \mathbf{u} \rangle + c^2) + \mathcal{A} \nabla_\beta (\rho u^\beta) + \mathcal{B} \nabla_\beta (\rho u^\beta \eta) + \mathcal{C}_K \nabla_\beta (\rho u^\beta X^K) \Big\} \sqrt{-g}\, d^4 \mathbf{x}.$$

This expression is convenient for varying independently f, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_K$ since we have at once : $\delta_f I^* \mapsto (67)$, $\delta_{\mathcal{A}} I^* \mapsto (68)$, $\delta_{\mathcal{B}} I^* \mapsto (70)$, $\delta_{\mathcal{C}_K} I^* \mapsto (74)$. In order to vary independently ρ, u^β and η, it is more convenient to sum by parts in equation (75) and, using the definition

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(5) q may take positive or negative values, e. g., for electrons, e being the electric charge $(e > 0)$ per electron and n_0 the number of electrons (baryions of rest mass m_0) per unit of proper volume, we have : $q = -e n_0 = -e \frac{\rho}{m_0}$.

of the covariant divergence of a four-vector $\mathbf{M}$,

$$\nabla_\beta \mathrm{M}^\beta = \frac{1}{\sqrt{-g}} \partial_\beta (\sqrt{-g}\, \mathrm{M}^\beta) \tag{76}$$

to write equation (75) in the following form :

$$\mathrm{I}^* = \int_{(\mathcal{B})} \left(\overset{*}{\Lambda} + \partial_\alpha \overset{*}{\lambda}{}^\alpha\right) d^4\, \mathbf{x} \tag{77}$$

with

$$\begin{aligned}\overset{*}{\Lambda} \equiv \sqrt{-g} \Big\{ & \frac{1}{2} \rho f(\langle \mathbf{u}.\mathbf{u} \rangle + c^2) \\ & - \frac{1}{\sqrt{-g}} \rho\, u^\beta \left[\partial_\beta (\mathcal{A} \sqrt{-g}) + \eta\, \partial_\beta (\mathcal{B} \sqrt{-g}) \right. \\ & \left. + \mathrm{X}^{\mathrm{K}}\, \partial_\beta \left(\mathcal{C}_{\mathrm{K}} \sqrt{-g}\right)\right] \Big\} ;\end{aligned} \tag{78}$$

$$\overset{*}{\lambda}{}^\alpha \equiv \sqrt{-g}\, \rho\, u^\alpha (\mathcal{A} + \mathcal{B}\, \eta + \mathcal{C}_{\mathrm{K}}\, \mathrm{X}^{\mathrm{K}}). \tag{79}$$

Neglecting the contributions on the boundary $(\partial\mathcal{B})$ but considering a discontinuity hypersurface (Σ), we can write equation (77) in the form :

$$\mathrm{I}^* = \int_{(\mathcal{B})} \overset{*}{\Lambda}\, d^4\, \mathbf{x} - \int_{(\Sigma)} \left[\mathrm{N}_\alpha \overset{*}{\lambda}{}^\alpha\right] d^3\, \sigma. \tag{80}$$

Let us examine what is implied by independent variations in the surface term of equation (80),

$$\left[\mathrm{N}_\alpha \overset{*}{\lambda}{}^\alpha\right] = [\mathrm{N}_\alpha\, \rho\, u^\alpha \sqrt{-g} (\mathcal{A} + \mathcal{B}\, \eta + \mathcal{C}_{\mathrm{K}}\, \mathrm{X}^{\mathrm{K}})] = 0. \tag{81}$$

First, note that the continuity equation (68) implies across (Σ) (*cf.* Maugin, 1971 *g*) :

$$[\mathrm{N}_\alpha\, \rho\, u^\alpha \sqrt{-g}] = 0. \tag{82}$$

Hence, equation (81) can be written

$$\rho\, u_{(\mathbf{N})} \sqrt{-g} [\mathcal{A} + \mathcal{B}\, \eta + \mathcal{C}_{\mathrm{K}}\, \mathrm{X}^{\mathrm{K}}] = 0, \qquad u_{(\mathbf{N})} \equiv u^\alpha\, \mathrm{N}_\alpha. \tag{83}$$

Then, independent variations of $\mathcal{B}$ and $\mathcal{C}_{\mathrm{K}}$, with $\rho\, u_{(\mathbf{N})} \sqrt{-g} \neq 0$ ([6]), yield

$$[\eta] = 0, \qquad [\mathrm{X}^{\mathrm{K}}] = 0, \quad \text{across } (\Sigma) \text{ along } \left(\mathcal{C}_{\mathrm{X}^{\mathrm{K}}}\right). \tag{84}$$

([6]) Obviously $\rho\, u_{(\mathbf{N})} \sqrt{-g} \neq 0$ since, from equation (81) [a superposed bar indicating the mean value on (Σ)] :

$$\rho\, u_{(\mathbf{N})} \sqrt{-g} = \overline{\rho\, u_{(\mathbf{N})} \sqrt{-g}} \quad \text{on } (\Sigma),$$

(Σ) being within $(\mathcal{B})$, we assume that there is some flow across (Σ), i. e., $\rho \neq 0$, $u_{(\mathbf{N})} \neq 0$.

The first of these is obviously verified for isentropic flows. The second one is identically satisfied since, along the world line $(\mathcal{C}_{X^K})$ which goes through (Σ), the *inverse* motion has been assumed to be of class C^2. Independent variations of ρ and u^α in equation (81) yield, on account of equation (82)

$$\text{(85)} \qquad \begin{cases} \delta\rho \ : \ \rho\, u_{(\mathbf{N})} \sqrt{-g}\,[\rho^{-1}(\mathcal{A} + \mathcal{B}\,\eta + \mathcal{C}_K\, X^K)] = 0, \\ \delta u^\alpha : [\rho\, N_\alpha \sqrt{-g}\,(\mathcal{A} + \mathcal{B}\,\eta + \mathcal{C}_K\, X^K)] \qquad = 0, \end{cases}$$

across (Σ). The field equations valid in $(\mathcal{B})$ that correspond to equations (85) will be given below. Independent variations of $g_{\alpha\beta}$ need not be considered here because of the presence of $\rho\sqrt{-g}$ in the original equation (81) [*cf.* eq. (54)]. Since $\mathbf{N}$ is unity, taking the inner product of $(85)_2$ with N^α, we get the unique equation

$$\text{(86)} \qquad \delta u^\alpha : \ [\rho\sqrt{-g}\,(\mathcal{A} + \mathcal{B}\,\eta + \mathcal{C}_K\, X^K)] = 0.$$

Finally we examine the implications of independent variations of the different variables as far as the 4-volume term of equation (77) is concerned. On account of the fact that most of the terms have been introduced as constraints and with (28), we get immediately :

$$\text{(87)} \qquad \delta_G\, I^* = \int_{(\mathcal{B})} \frac{1}{2}\rho\, f\, u^\alpha\, u^\beta\, \delta g_{\alpha\beta} \sqrt{-g}\, d^4\mathbf{x};$$

$$\text{(88)} \qquad \begin{cases} \delta_\rho\, I^* = \displaystyle\int_{(\mathcal{B})} \delta_\rho \overset{*}{\Lambda}\, d^4\mathbf{x}, \qquad \delta_{\mathbf{u}}\, I^* = \int_{(\mathcal{B})} \delta_{\mathbf{u}} \overset{*}{\Lambda}\, d^4\mathbf{x}, \\ \qquad\qquad \delta_\eta\, I^* = \displaystyle\int_{(\mathcal{B})} \delta_\eta \overset{*}{\Lambda}\, d^4\mathbf{x} \end{cases}$$

with

$$\text{(89)} \quad \delta_\rho \overset{*}{\Lambda} = \frac{1}{2} f\sqrt{-g}\,(\langle \mathbf{u}.\mathbf{u}\rangle + c^2) \\ \qquad - u^\beta\left[\partial_\beta\left(\mathcal{A}\sqrt{-g}\right) + \eta\, \partial_\beta\left(\mathcal{B}\sqrt{-g}\right) + X^K\, \partial_\beta\left(\mathcal{C}_K\sqrt{-g}\right)\right];$$

$$\text{(90)} \quad \delta_{u^\beta} \overset{*}{\Lambda} = \rho\, f\sqrt{-g}\, u_\beta \\ \qquad - \rho\left[\partial_\beta\left(\mathcal{A}\sqrt{-g}\right) + \eta\, \partial_\beta\left(\mathcal{B}\sqrt{-g}\right) + X^K\, \partial_\beta\left(\mathcal{C}_K\sqrt{-g}\right)\right];$$

$$\text{(91)} \qquad \delta_\eta \overset{*}{\Lambda} = -\rho\, u^\beta\, \partial_\beta\left(\mathcal{B}\sqrt{-g}\right) \equiv -\rho\sqrt{-g}\,\frac{\partial\mathcal{B}}{\partial\tau}.$$

Of course,

$$\text{(92)} \qquad \delta_{\mathbf{F}} \overset{*}{\Lambda} = 0.$$

6. THE ACTION PRINCIPLE

We can now put together the results of the foregoing sections. The total action considered is

$$(93)\qquad \mathrm{I}_{\text{total}} = \mathrm{I}_{\mathrm{G}} + \mathrm{I}_{(\mathrm{em})} + \mathrm{I}_m + \mathrm{I}^*,$$

where the different expressions of the right-hand-side are respectively given by equations (30), (46), (49) and (75) or (80). The independent variations considered are those of $g_{\alpha\beta}$, $\mathrm{F}_{\alpha\beta}$, ρ, u^β, η, f, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_{\mathrm{K}}$. Collecting the results $(22)_1$, (43), (48), (62)-(66) and (87)-(92), for arbitrary variations, we obtain the field equations in ($\mathcal{B}$) (*interior* problem)

$$(94)\qquad \delta g_{\alpha\beta}:\quad \mathrm{A}_{\alpha\beta} - \varkappa\,\mathrm{T}_{\alpha\beta} = 0;$$

$$(95)\qquad \delta \mathrm{F}_{\alpha\beta}:\quad \pi^{\alpha\beta} \equiv \mathrm{F}^{\alpha\beta} - \mathrm{G}^{\alpha\beta} = -\,2\,\rho\,\frac{\partial \epsilon}{\partial \mathrm{F}_{\alpha\beta}};$$

$$(96)\qquad \delta\rho:\quad \mathbf{i}\,(\rho, \eta, \mathrm{F}^{\alpha\beta}) + \frac{1}{2}\tilde{\pi}_{\alpha\beta}\,\mathrm{F}^{\beta\alpha} = -\,u^\beta\,(\nabla_\beta\,\mathcal{A} + \eta\,\nabla_\beta\,\mathcal{B} + \mathrm{X}^{\mathrm{K}}\,\nabla_\beta\,\mathcal{C}_{\mathrm{K}}) + \frac{1}{2} f\,(\langle \mathbf{u}.\mathbf{u}\rangle + c^2) - c^2;$$

$$(97)\qquad \delta u^\beta:\quad f\,u_\beta - \frac{q}{\rho\,c}\mathrm{A}_\beta = \nabla_\beta\,\mathcal{A} + \eta\,\nabla_\beta\,\mathcal{B} + \mathrm{X}^{\mathrm{K}}\,\nabla_\beta\,\mathcal{C}_{\mathrm{K}};$$

$$(98)\qquad \delta\eta:\quad \theta = -\,\frac{\partial\mathcal{B}}{\partial\tau};$$

$$(99)\qquad \delta f:\quad g_{\alpha\beta}\,u^\alpha\,u^\beta = -\,c^2;$$

$$(100)\qquad \delta\mathcal{A}:\quad \nabla_\beta\,(\rho\,u^\beta) = 0;$$

$$(101)\qquad \delta\mathcal{B}:\quad \nabla_\beta\,(\rho\,u^\beta\,\eta) = 0;$$

$$(102)\qquad \delta\mathcal{C}_{\mathrm{K}}:\quad 0 = \frac{\partial \mathrm{X}^{\mathrm{K}}}{\partial\tau}.$$

In equation (94), we have used the definition (26) which, on account of the results (44)-(45), (65) and (87), gives

$$(103)\qquad \mathrm{T}^{\alpha\beta} = -\,\frac{2}{\sqrt{-\,g}}\,\frac{\partial}{\partial g_{\alpha\beta}}\left(\overset{*}{\Lambda} + \Lambda_{(\mathrm{em})} + \Lambda_m\right) = \rho\,f\,u^\alpha\,u^\beta + p g^{\alpha\beta} + \mathrm{T}^{\alpha\beta}_{(\mathrm{em})}$$

in which $\mathrm{T}^{\alpha\beta}_{(\mathrm{em})}$ is given by equation (44). The form of f will be given here after.

COMMENTS :

(*a*) The ten equations (94) are Einstein's field equations. Equation (95) is the electromagnetic constitutive equation given here in a general nonlinear form. The polarization-magnetization two-form is derived from a potential, the relativistic internal energy. Dropping the second term of the right-hand-side of equation (96) which is zero after equation (99), the equation (96) gives a definition of the specific magneto-enthalpy as a linear combination of the proper time derivatives of the Lagrange multipliers introduced to take account of the continuity equation, the isentropy condition along a streamline and the Lin's constraint, this up to a constant (the rest energy per unit of proper mass, c^2) and a quantity known as the energy of a magnetic doublet.

(*b*) The significance of equation (97) is made clear if one drops the electromagnetic effects. In these conditions of a *perfect fluid* scheme, equation (97) reads :

$$u_\beta = f^{-1} (\nabla_\beta \mathcal{A} + \eta \nabla_\beta \mathcal{B} + X^K \nabla_\beta \mathcal{C}_K) \tag{104}$$

which, by analogy with the treatment of classical hydrodynamics (*cf.* Clebsch, 1859 *a*, *b*; Seliger and Whitham, 1958), may be called a *Clebsch's representation* of the 4-velocity. This representation is quite general since, although the flow is isentropic (along streamlines), no irrotationality is implied. Indeed, define the *fluid current* C_β by

$$C_\beta = f u_\beta. \tag{105}$$

The vorticity tensor and the space-like vorticity 4-vector are then defined as (Lichnerowicz, 1955)

$$\Omega_{\alpha\beta} = \partial_\alpha C_\beta - \partial_\beta C_\alpha \equiv 2 \nabla_{[\alpha} C_{\beta]}; \tag{106}$$

$$\omega^\alpha \equiv - \frac{1}{2\sqrt{-g}} \varepsilon^{\alpha\beta\gamma\delta} \Omega_{\beta\gamma} u_\delta, \qquad \text{i. e.,} \quad \omega^\alpha u_\alpha = 0. \tag{107}$$

The latters are obviously different from zero after equation (104) (*cf.* section 7). Note the analogy of $\mathcal{A}$ with a 4-current potential (in classical hydrodynamics, velocity potential) since, if the flow were isentropic throughout the whole region ($\mathcal{B}$) (and not only along streamlines), we would have

$$C_\alpha = \nabla_\alpha \mathcal{A} \tag{108}$$

as a consequence of full isentropy and continuity. Thus, we may say that, by introducing more constraints (Lin's constraints) in the formulation, we succeeded in enlarging the class of possible flows. With electromagnetic fields taken into account, equation (104) is replaced by

$$(109)\qquad \begin{cases} \tilde{C}_\beta \equiv \tilde{f} u_\beta - \dfrac{k}{c} A_\beta = \nabla_\beta \mathcal{A} + \eta \nabla_\beta + X^K \nabla_\beta \mathcal{C}_K, \\ k \equiv \dfrac{q}{\rho}, \end{cases}$$

where $\tilde{C}_\beta$ is what we have called the *modified* current of the fluid. In this definition, we have used the notation $\tilde{f}$ (*modified* index) in lieu of f, reserving the symbol f to the case for which there are no electromagnetic fields, i. e., according to Maugin (1971 *e*), when manipulations of equations (96) and (97) yield

$$(110)\qquad f = 1 + \frac{\mathbf{i}(\rho, \eta)}{c^2}.$$

The quantity k defined by $(109)_2$ is characteristic of the behavior of the fluid in *convection.*

Equation $(109)_1$ is then called the Clebsch's representation of the modified current. It generalizes to general relativity the *énoncé* given by Maugin (1971 *e*) in special relativity. The *modified* index $\tilde{f}$ is then determined as follows. Contracting equation (97) with u^β and taking account of equations (99), (96), $(32)_1$, (36) and $(71)_1$, we find

$$(111)\qquad \tilde{f} = 1 + \frac{1}{c^2}\mathbf{i}(\rho, \eta, F^{\alpha\beta}) + \frac{1}{2c^2}\tilde{\pi}_{\alpha\beta} F^{\beta\alpha} - \frac{1}{\rho c^3} J^\alpha A_\alpha$$

in which equation, $\mathbf{i}$ is given by equation (59). Thus,

$$(112)\qquad T^{\alpha\beta} = \rho\left[1 + c^{-2}\left\{\epsilon(\rho, \eta, F^{\alpha\beta}) + \frac{p}{\rho} - \frac{1}{\rho c} J^\mu A_\mu\right\}\right] u^\alpha u^\beta + p\, g^{\alpha\beta} + T^{\alpha\beta}_{(\mathrm{em})}.$$

Noting

$$(113)\qquad \begin{cases} J^2 \stackrel{\mathrm{def}}{=} -\, g_{\alpha\beta} \dfrac{J^\alpha A^\beta}{\rho c}, \\ \|\tilde{\mathbf{C}}\|^2 = -\, g_{\alpha\beta} \tilde{C}^\alpha \tilde{C}^\beta, \qquad \|\mathbf{A}\|^2 = -\, g_{\alpha\beta} A^\alpha A^\beta, \end{cases}$$

where, in $(113)_1$, we assumed the validity of equation $(71)_1$, we get by inversion of the definition of $\tilde{C}_\beta$

$$(114)\qquad c^2 \tilde{f} = J^2 + (J^4 + \|\tilde{\mathbf{C}}\|^2 + k^2 \|\mathbf{A}\|^2)^{\frac{1}{2}}.$$

In absence of electromagnetic fields, this equation reduces to the classical result (*cf.* Lichnerowicz, 1955)

$$(115)\qquad f = \frac{1}{c}(-\, g_{\alpha\beta} C^\alpha C^\beta)^{\frac{1}{2}}$$

from which it is shown that the streamline equation is solution of the extremal of the homothetical metric

$$\overline{ds}^2 = f^2\, ds^2, \qquad (ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta).$$

The same result holds true for the case $(109)_1$ with

$$(116) \qquad \left\{ \begin{aligned} \tilde{f}^* &= \frac{1}{c}\left(-\, g_{\alpha\beta}\, \tilde{\mathrm{C}}^\alpha\, \tilde{\mathrm{C}}^\beta\right)^{\frac{1}{2}}, & \tilde{\mathrm{C}}^\alpha &= \tilde{f}^*\, u^\alpha, \\ \tilde{f}^* &= \tilde{f} + \frac{1}{\rho\, c^3}\, \mathrm{J}^\alpha\, \mathrm{A}_\alpha, & \overline{ds}^2 &= \tilde{f}^{*2}\, ds^2. \end{aligned} \right.$$

As a consequence of the continuity equation (68) and the conservation of charge equation $(32)_2$, equation $(71)_1$ being valid, it is well-known that the quantity k defined by equation $(109)_2$ is constant along a streamline $(\mathcal{C}_{\mathrm{X}^\mathrm{K}})$ (*cf.* Lichnerowicz, 1955, p. 55), i. e.,

$$(117) \qquad \frac{\partial k}{\partial \tau} = 0.$$

Some simple identities which may prove useful in further studies can be deduced from the definitions (109). For instance,

$$(118) \qquad u^\alpha \frac{\partial \tilde{\mathrm{C}}_\alpha}{\partial \tau} \equiv -\, c^2 \frac{\partial \tilde{f}}{\partial \tau} - \frac{k}{c}\, u^\alpha \frac{\partial \mathrm{A}_\alpha}{\partial \tau}$$

which is obtained by taking the proper time derivative of equation $(109)_1$ and then the inner product of the result with u^α [while taking account of equations $(3)_2$ and (117)]. Another interesting result which reflects the significance of the definition $(109)_1$ is obtained by taking the divergence of equation $(109)_1$ and assuming the Lorentz gauge for the electromagnetic potential A_α *i.* e.,

$$(119) \qquad \nabla^\alpha\, \mathrm{A}_\alpha = 0,$$

we get

$$(120) \qquad \nabla^\alpha\, \tilde{\mathrm{C}}_\alpha = \rho \frac{\partial}{\partial \tau} \frac{\tilde{f}}{\rho} - \mathrm{A}^\alpha\, \nabla_\alpha \frac{k}{c}$$

in which we took account of equation (68) and used the definition (4). Integrating equation (120) over a four-dimensional region $(\mathcal{D})$ of V^4 and using the Green-Gauss theorem (6) in absence of discontinuity hypersurface, we get

$$(121) \qquad \int_{(\partial\mathcal{D})} n^\alpha\, \tilde{\mathrm{C}}_\alpha \sqrt{-\,g}\; d^3\, s = \int_{(\mathcal{D})} \left[\rho \frac{\partial}{\partial \tau} \frac{\tilde{f}}{\rho} - \mathrm{A}^\alpha\, \nabla_\alpha \frac{k}{c}\right] \sqrt{-\,g}\; d^4\, \mathbf{x}.$$

In absence of electromagnetic fields, this reads :

$$\int_{(\partial\mathcal{D})} n^\alpha \, C_\alpha \sqrt{-g} \, d^3 s = \int_{(\mathcal{D})} \rho \frac{\partial}{\partial\tau} \frac{f}{\rho} \sqrt{-g} \, d^4 \mathbf{x}, \tag{122}$$

i. e., *the flux of fluid current across* $(\partial\mathcal{D})$ *is equal to the volume integral over* $(\mathcal{D})$ *of the time variation of the index along all streamlines passing through* $(\mathcal{D})$. Equation (121) generalizes this statement. Moreover if the fluid scheme is said to be *homogeneous* from the point of view of convection, i. e., $\nabla_\alpha k = 0$, then equation (121) takes a form identical to that of equation (122), $\tilde{C}_\alpha$ and $\tilde{f}$ replacing C_α and f respectively. Nevertheless one must not read in an equation of the type (122) more than is contained in it. Equation $(109)_1$ is a mere definition and the derivation of equation (122) requires the equation of continuity (100) to be valid, i. e., equation (122) is equivalent to equation (100) or rather to the integral form given precedently [Maugin, 1971 *e*, equation (2.27)].

Finally, using the definition $(109)_1$, we get an equation equivalent to equation (111) which gives the form of the last term of the second volume integral of equation (39) :

$$-\frac{1}{c} J^\alpha A_\alpha \equiv \rho \left(u^\alpha \tilde{C}_\alpha + c^2 \tilde{f} \right). \tag{123}$$

(*c*) Equation (98) gives a representation of the thermodynamical temperature as the proper time derivative of an invariant scalar. In fact, we have incidentally found the " physical " meaning of the strange thermodynamical variable Θ that, following von Laüe (1921) and Taub (1957), we had introduced in another type of variational formulation (Maugin, 1971 *g*; Maugin and Eringen, 1972 *b*). Indeed according to the definition $(8)_2$, we could write

$$\delta\theta = \left. \frac{\partial\theta\,(x^\alpha, \varepsilon)}{\partial\varepsilon} \right|_{\varepsilon=0} = \left. \frac{\partial\theta\,(X^K, \tau, \varepsilon)}{\partial\varepsilon} \right|_{\varepsilon=0} \tag{124}$$

The second expression follows from equation (1). Or, with the result (98),

$$\delta\theta = \frac{\partial}{\partial\varepsilon} \left(- \frac{\partial\mathcal{B}}{\partial\tau} \right) \bigg|_{\varepsilon=0} = \frac{\partial}{\partial\tau} \left(- \frac{\partial\mathcal{B}}{\partial\varepsilon} \right) \bigg|_{\varepsilon=0} = \frac{\partial}{\partial\tau} (- \delta\mathcal{B}). \tag{125}$$

But equation (2.19) of Maugin (1971 *g*), that made use of the same type of variational definition, was

$$\delta\theta = \frac{\partial}{\partial\tau} (\delta\Theta). \tag{126}$$

Thus, the variable Θ is, up to a constant, nothing but minus the Lagrange multiplier which takes care of the isentropy condition (70) in the present treatment.

There is no need to comment upon equations (99) through (102).

(*d*) We must remark that, in contrast with the treatments given by Maugin (1971 *g*) and Maugin and Eringen (1972 *b*), we did not arrive at the Maxwell equations during the variational process [obviously, these were accounted for at the starting point when we chose the form (39) for the action representing the electromagnetic participation] nor did we arrive at the *streamline equations* (or Euler equations of the motion) while in the other approaches mentioned above, they were obtained on an equal foot with, for instance, equation (94), by considering combined variations of the metric and of the particle path in V^4. Of course, according to Bianchi's identities, equation (23) follows from equation (94) and, using the result (112) and by projection with the help of $P_{\gamma\alpha}$ onto a three-dimensional submanifold $V^3_{\perp}$ (*cf.* Maugin, 1971 *b*), we would get the desired equations. It is more learning to get these equations by manipulating equations (96)-(97) which result from the present formulation. This will be done in a subsequent section.

(*e*) Note that the jump relations across (Σ), that correspond to the field equations (94), (96), (97), (100), (101) and (102) have been given before; they are equations (31), $(85)_1$, (86), (82), $(84)_1$ and $(84)_2$ respectively.

(*f*) Finally, let us recall the " physical " meaning which has been granted to the different Lagrange multipliers introduced along the treatment. The 4-electromagnetic potential A_α takes account of the Maxwell's equations $(32)_1$. $\tilde{f}$ is the modified index of the fluid. $\mathcal{A}$ is a scalar fluid current potential. $\mathcal{B}$ is linked in some way to the variable Θ introduced earlier in other treatments. As to $\mathcal{C}_K$, it is not possible to determine its value because of its symmetric appearance in the two equations (96) and (97). As Seliger and Whitham (1968) point out, Lin's device is somewhat artificial and remains somewhat mysterious from a strictly mathematical point of view although its necessity seems to be firmly established. The only thing we can say about the $\mathcal{C}_K$'s is that they are constant along streamlines, i. e.,

$$\frac{\partial \mathcal{C}_K}{\partial \tau} = 0. \tag{127}$$

This follows from equations (78) if we consider an *independent* variation of X^K (7). Let us remark that the $\mathcal{C}_K$'s always appear in a summation with X^K. The fact that we introduced three multipliers $\mathcal{C}_K$ is therefore

(7) This is why we used the suggestive symbol $\mathcal{C}_K$ for these multipliers, i. e., a notation very close to the streamline notation $(\mathcal{C}_{X^K})$.

irrelevant. Only one scalar quantity that we could call " name " of the particle, so one multiplier $\mathcal{C}_{(\text{name})}$ need be introduced to obtain a representation of the form of (97) in order to assure of a sufficiently large class of flows.

7. EULER EQUATIONS OF MOTION

Let us show that these equations follow from equations (96)-(97). First define the commutators

$$\left\{\begin{array}{l}[A_\alpha,\ B_\beta] \equiv 2\,A_{[\alpha}B_{\beta]} = A_\alpha B_\beta - A_\beta B_\alpha, \\ \text{hence}\quad [\nabla_\alpha, \nabla_\beta] \equiv \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha.\end{array}\right. \tag{128}$$

For a scalar $\mathcal{A}$, the Riemann curvature tensor is not involved and

$$[\nabla_\alpha, \nabla_\beta]\,\mathcal{A} = 0. \tag{129}$$

Now, define the *modified* vorticity tensor $\tilde{\Omega}_{\alpha\beta}$ by

$$\tilde{\Omega}_{\alpha\beta} = \partial_\alpha \tilde{C}_\beta - \partial_\beta \tilde{C}_\alpha \equiv \nabla_\alpha \tilde{C}_\beta - \nabla_\beta \tilde{C}_\alpha. \tag{130}$$

From this definition and equation $(109)_1$, we get

$$\tilde{\Omega}_{\alpha\beta} = [\nabla_\alpha \eta, \nabla_\beta \mathcal{B}] + \sum_{K=1}^{3} [\nabla_\alpha X^K, \nabla_\beta \mathcal{C}_K]. \tag{131}$$

Contract this expression with $C^\alpha \equiv \tilde{f} u^\alpha$ and use the results (98), $(69)_2$ and (127) to get

$$C^\alpha\,\tilde{\Omega}_{\alpha\beta} = \tilde{f}\,\theta \nabla_\beta \eta. \tag{132}$$

This equation of the Crocco-Varsonyi type is similar to the streamline equation of Lichnerowicz [1967, eq. (21.4)]. If we contract with the modified current $\tilde{C}^\alpha$ instead of C^α, we should get after the definition $(109)_1$

$$\tilde{C}^\alpha\,\tilde{\Omega}_{\alpha\beta} = \tilde{f}\,\theta\nabla_\beta \eta - \frac{k}{c} A^\alpha\,\tilde{\Omega}_{\alpha\beta} \tag{133}$$

but this form yields no interesting consequences.

We now transform both sides of equation (132). From equation (60) we get

$$\theta\nabla_\beta \eta = \nabla_\beta\, i - \frac{1}{\rho}\nabla_\beta\, p + \frac{1}{2} F^{\mu\nu}\,\nabla_\beta\,\tilde{\pi}_{\nu\mu}. \tag{134}$$

The transformation of the left-hand-side is more lengthy. We have

$$(135)\qquad \begin{aligned} u^\alpha \tilde{\Omega}_{\alpha\beta} &= u^\alpha (\nabla_\alpha \tilde{C}_\beta - \tilde{C}_\alpha) \\ &= \frac{\partial}{\partial\tau}\tilde{C}_\beta - u^\alpha \nabla_\beta \left[\tilde{f} u_\alpha - \frac{k}{c} A_\alpha\right], \end{aligned}$$

$$(136)\qquad u^\alpha \tilde{\Omega}_{\alpha\beta} = \frac{\partial}{\partial\tau}\tilde{C}_\beta + c^2 \nabla_\beta \tilde{f} + \frac{1}{\rho c} J^\alpha \nabla_\beta A_\alpha$$

in which we used equations $(3)_2$ and $(3)_1$, have assumed the homogeneity from the point of view of convection, i. e., $\nabla_\beta k = 0$ [a condition stronger than (117)] and finally made use of equation $(71)_1$. But,

$$(137)\qquad \frac{\partial}{\partial\tau}\tilde{C}_\beta = \frac{\partial}{\partial\tau}\left[\tilde{f} u_\beta - \frac{k}{c} A_\beta\right] = \frac{\partial}{\partial\tau} C_\beta - \frac{1}{\rho c} J^\gamma \nabla_\gamma A_\beta.$$

Thus, collecting the two last terms of equations (136) and (137) and using (36), we get

$$(138)\qquad u^\alpha \tilde{\Omega}_{\alpha\beta} = \frac{\partial}{\partial\tau} C_\beta + c^2 \nabla_\beta \tilde{f} + \frac{1}{\rho c} J^\alpha F_{\beta\alpha}.$$

According to equation (111),

$$(139)\qquad \begin{aligned} \nabla_\beta \tilde{f} = \frac{1}{c^2}\nabla_\beta i &+ \frac{1}{2c^2}(\nabla_\beta \tilde{\pi}_{\mu\nu}) F^{\nu\mu} \\ &+ \frac{1}{2c^2}\tilde{\pi}_{\mu\nu}\nabla_\beta F^{\nu\mu} - \nabla_\beta\left[\left(\frac{1}{\rho c^3}\right) J^\alpha A_\alpha\right]. \end{aligned}$$

Then, carrying the result (139) into equation (138) and combining the results (138) and (134) into equation (132), we obtain

$$(140)\qquad \begin{aligned} \frac{\partial\tilde{f}}{\partial\tau} u_\beta + \tilde{f}\frac{\partial u_\beta}{\partial\tau} = &- \frac{1}{\rho}\nabla_\beta p + \frac{1}{\rho c} J^\alpha F_{\beta\alpha} \\ &+ \frac{1}{2\rho}\pi_{\mu\nu}\nabla_\beta F^{\mu\nu} + \nabla_\beta\left(\frac{1}{\rho c} J^\alpha A_\alpha\right) \end{aligned}$$

in which we used the definition $C_\beta = \tilde{f} u_\beta$. The Euler equations of the motion are the three independent equations obtained by projection of equation (140) onto $V_\perp^3$ with the help of the projector $(5)_1$. Using the property $(5)_4$ and the fact that

$$P^{\gamma\beta}\frac{\partial}{\partial\tau} u_\beta \equiv \frac{\partial u^\gamma}{\partial\tau}$$

which follows from $(5)_4$, we get

$$(141)\qquad \begin{aligned} \rho\tilde{f}\frac{\partial u^\gamma}{\partial\tau} = &-\left(g^{\gamma\beta} + \frac{1}{c^2}u^\gamma u^\beta\right)\nabla_\beta p + \frac{q}{c}\mathcal{E}^\gamma \\ &+ \frac{1}{2}\pi_{\mu\nu} P^{\gamma\beta}\nabla_\beta F^{\mu\nu} + \rho P^{\gamma\beta}\nabla_\beta\left[\frac{1}{\rho c} J^\alpha A_\alpha\right] \end{aligned}$$

in which we have defined the 4-electric field $\mathcal{E}^\gamma$ by

$$(142)\qquad \mathcal{E}^\gamma \equiv F^{\alpha\gamma} u_\alpha, \qquad \mathcal{E}^\gamma u_\gamma = 0, \qquad P^{\gamma\beta} \mathcal{E}_\beta \equiv \mathcal{E}^\gamma.$$

The different terms of equation (141) have the following meaning. The left-hand-side is the acceleration term $\Big[$ which, in an inertial frame, at the limit $c \mapsto \infty$, yields the three-dimensional term $\rho \frac{D\mathbf{v}}{Dt} \Big]$. In the right-hand-side, we have the pressure term, the Lorentz term $f^\gamma_{(L)}$ and the Stern-Gerlach term $f^\gamma_{(S-G)}$ (*cf.* Maugin, 1971 *g*) defined as

$$(143)\qquad f^\gamma_{(L)} \equiv \frac{q}{c} \mathcal{E}^\gamma, \qquad f^\gamma_{(S-G)} \equiv \frac{1}{2} \pi_{\mu\nu} P^{\gamma\beta} \nabla_\beta F^{\mu\nu}.$$

The last term which is of the " pressure gradient " type and involves the invariant $\langle \mathbf{A}, \mathbf{J} \rangle$ is peculiar. It is however consistent with the general treatment since, as can be easily checked after some algebra, equation (141) is nothing but

$$(144)\qquad P^\gamma_{.\alpha} \nabla_\beta T^{\alpha\beta} = 0,$$

where $T^{\alpha\beta}$ is given by equation (103) or equation (112).

The projection of equation (140) along the time direction, i. e., the equation obtained by contraction with u^β would yield what we called the conservation of energy equation (*cf.* Maugin, 1971 *g*). It is none other than equation (57) ([8]).

([8]) Indeed take $d \mapsto \frac{\partial}{\partial\tau}$ in equation (57). We note that $\frac{\partial \eta}{\partial \tau} = 0$ along a streamline. Thus equation (57) reads :

$$(a)\qquad \frac{\partial \epsilon}{\partial \tau} = \frac{p}{\rho^2} \frac{\partial \rho}{\partial \tau} + \frac{1}{2} \tilde{\pi}_{\alpha\beta} \frac{\partial}{\partial \tau} F^{\beta\alpha}.$$

More generally, this equation should read [*cf.* Maugin, 1971 *g*, eq. (4.20)]

$$(b)\qquad \frac{\partial \epsilon}{\partial \tau} = \frac{1}{\rho} t^{\alpha\beta} \sigma_{\beta\alpha} + \frac{1}{2} \tilde{\pi}_{\alpha\beta} \frac{\partial}{\partial \tau} F^{\beta\alpha},$$

where $t^{\alpha\beta}$ is the relativistic stress tensor (electromagnetic and heat conduction effects excluded) and $\sigma_{\beta\alpha}$ is the relativistic rate of strain tensor (*cf.* Maugin, 1971 *b*)

$$(c)\qquad t^{\alpha\beta} u_\beta = 0, \qquad t^{[\alpha\beta]} = 0, \qquad \sigma_{\beta\alpha} \equiv P_{(\beta\mu} \nabla^\mu u_\nu P^\nu_{.\alpha)}.$$

For a perfect compressible fluid, take

$$(d)\qquad t^{\alpha\beta} = -p P^{\alpha\beta}, \qquad \text{then} \quad t^{\alpha\beta} \sigma_{\beta\alpha} = -p \nabla_\alpha u^\alpha$$

but, from the continuity equation (100), we get $\nabla_\alpha u^\alpha = -\frac{\partial}{\partial \tau} (\ln \rho)$, thus equation (*b*) yields equation (*a*).

Q. E. D.

8. CASE OF PERFECT MAGNETOHYDRODYNAMICS

In the case perfect magnetohydrodynamics, we assume that the perfect fluid has an *infinite conductivity*. A consequence of this fact is that the 4-electric current $\mathcal{E}^\alpha$ must be taken equal to zero in order to assure of the finiteness of the total electric current (convection + conduction), *cf.* Lichnerowicz, 1967, p. 93. Although we did not take account of conduction, we shall assume that the latter condition is realized, i. e., (9)

$$\mathcal{E}^\alpha \equiv 0. \tag{145}$$

The Lorentz term disappears from equation (141). Moreover, we shall take (10)

$$u^\alpha \mathrm{A}_\alpha = 0, \tag{146}$$

hence,

$$\mathrm{J}^\alpha \mathrm{A}_\alpha = q u^\alpha \mathrm{A}_\alpha = 0$$

---

(9) In an *inertial* frome, the 4-vector $\mathcal{E}^\alpha$ assumes the space-time decomposition (*cf.* Grot and Eringen, 1966 *a*)

$$(a) \qquad \mathcal{E}^\alpha \stackrel{*}{=} \left[\frac{\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}}{\sqrt{1-\beta^2}}, \frac{i}{c}\frac{\mathbf{v}.\mathbf{E}}{\sqrt{1-\beta^2}} \right], \qquad \beta = \left|\frac{\mathbf{v}}{c}\right|, \qquad i = \sqrt{-1},$$

with obvious notations. Equation (145) requires in an inertial frame for small β that

$$(b) \qquad \mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B} = 0, \qquad \mathbf{v}.\mathbf{E} = 0.$$

The first of these is the classical Ohm's law of three-dimensional magnetohydrodynamics (*cf.* Cabannes, 1970, p. 12) for infinite conductivity. The second of equations (*b*) asserts for the case of *pure* convection that the electric dissipation vanishes since

$$\text{Joule term} \propto \mathbf{v}.\mathbf{E} = 0.$$

Thus, equation (145) does not require in general that $\mathbf{E}$ vanishes *but in a rest frame.* The electric field is only due to convected magnetic field.

(10) The covariant equation (146) says that the time-like component of the 4-electromagnetic potential vanishes, i. e., for small β,

$$u^\alpha \mathrm{A}_\alpha \propto ic.i\,\Phi = -c\,\Phi = 0,$$

then

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi = -\frac{\partial \mathbf{A}}{\partial t},$$

thus, $\mathbf{E}$ is determined only by the 3-magnetic potential vector $\mathbf{A}$. This is consistent with equation $(b)_1$ above.

and equation (141) reduces to

$$
(147)\quad \begin{cases} \rho \tilde{f} \dfrac{\partial u^\gamma}{\partial \tau} = -\left(g^{\gamma\beta} + \dfrac{1}{c^2} u^\gamma u^\beta\right) \nabla_\beta p + \dfrac{1}{2} \pi_{\mu\nu} \mathrm{P}^{\gamma\beta} \nabla_\beta \mathrm{F}^{\mu\nu}, \\ \qquad \text{with} \quad \tilde{f} = 1 + \dfrac{\mathbf{i}}{c^2} (\rho, \eta, \mathrm{F}^{\alpha\beta}). \end{cases}
$$

Now introduce the general decompositions of $\pi_{\mu\nu}$ and $\mathrm{F}_{\alpha\beta}$ by (*cf.* Grot and Eringen, 1966 *a*)

$$
(148)\quad \begin{cases} \mathrm{F}_{\alpha\beta} = \dfrac{1}{c} (\mathcal{E}_\beta u_\alpha - \mathcal{E}_\alpha u_\beta) + \dfrac{1}{ic} \varepsilon_{\alpha\beta\gamma\delta} \sqrt{-g}\, \mathcal{B}^\gamma u^\delta, \qquad \mathcal{B}^\alpha u_\alpha = 0, \\ \pi^{\alpha\beta} = \dfrac{1}{c} (\mathcal{P}^\alpha u^\beta - \mathcal{P}^\beta u^\alpha) + \dfrac{1}{ic} \dfrac{\varepsilon^{\alpha\beta\sigma\rho}}{\sqrt{-g}} \mathcal{M}_\sigma u_\rho, \qquad \mathcal{P}^\alpha u_\alpha = \mathcal{M}_\sigma u^\sigma = 0, \\ \mathrm{G}^{\alpha\beta} = \dfrac{1}{c} (\mathcal{D}^\beta u^\alpha - \mathcal{D}^\alpha u^\beta) + \dfrac{1}{ic} \dfrac{\varepsilon^{\alpha\beta\gamma\delta}}{\sqrt{-g}} \mathcal{H}_\gamma u^\delta, \qquad \mathcal{D}^\beta u_\beta = \mathcal{H}_\gamma u^\gamma = 0, \\ \qquad \text{with} \quad \mathrm{G}^{\alpha\beta} = \mathrm{F}^{\alpha\beta} - \pi^{\alpha\beta} \quad \text{or} \quad \mathcal{D}^\alpha = \mathcal{E}^\alpha + \mathcal{P}^\alpha, \\ \qquad\qquad \mathcal{H}^\alpha = \mathcal{B}^\alpha - \mathcal{M}^\alpha, \end{cases}
$$

which reduce to

$$
(149)\quad \begin{cases} \mathrm{F}_{\alpha\beta} = \dfrac{1}{ic} \varepsilon_{\alpha\beta\gamma\delta} \sqrt{-g}\, \mathcal{B}^\gamma u^\delta, \qquad \pi^{\alpha\beta} = \dfrac{1}{ic} \dfrac{\varepsilon^{\alpha\beta\sigma\rho}}{\sqrt{-g}} \mathcal{M}_\sigma u_\rho, \\ \mathrm{G}^{\alpha\beta} = \dfrac{1}{ic} \dfrac{\varepsilon^{\alpha\beta\gamma\delta}}{\sqrt{-g}} \mathcal{H}_\gamma u_\delta, \qquad \mathcal{E}_\alpha = \mathcal{P}_\alpha = \mathcal{D}_\alpha = 0 \end{cases}
$$

for the case of perfect magnetohydrodynamics. After some lengthy algebra, we get [11] :

$$
(150)\quad \begin{cases} \varphi = \dfrac{1}{2} \mathrm{F}_{\mu\nu} \mathrm{F}^{\nu\mu} = -\mathcal{B}_\gamma \mathcal{B}^\gamma, \\ \mathrm{T}^{\alpha\beta}_{(\mathrm{em})} = -\mathcal{H}^\alpha \mathcal{B}^\beta + \mathcal{B}_\sigma \mathcal{H}^\sigma \mathrm{P}^{\alpha\beta} - \dfrac{1}{2} \mathcal{B}_\sigma \mathcal{B}^\sigma g^{\alpha\beta}, \\ f^\gamma_{(\mathrm{S-G})} = \dfrac{1}{2} \pi_{\mu\nu} \mathrm{P}^{\gamma\beta} \nabla_\beta \mathrm{F}^{\mu\nu} = \mathcal{M}^\lambda \nabla^\gamma \mathcal{B}_\lambda + \dfrac{1}{c^2} \mathcal{M}^\lambda u^\gamma \dfrac{\partial \mathcal{B}_\lambda}{\partial \tau}. \end{cases}
$$

---

(11) The general form of φ and $\mathrm{T}^{\alpha\beta}_{(\mathrm{em})}$ without term involving $\langle \mathbf{A}, \mathbf{J} \rangle$, computed from equations (148) would be

$$
\varphi = \frac{1}{2} (\mathcal{E}_\gamma \mathcal{E}^\gamma - \mathcal{B}_\gamma \mathcal{B}^\gamma),
$$

$$
\mathrm{T}^{\alpha\beta}_{(\mathrm{em})} = \varphi g^{\alpha\beta} - (\mathcal{E}^\alpha \mathcal{D}^\beta + \mathcal{H}^\alpha \mathcal{B}^\beta) + \mathcal{B}^\sigma \mathcal{H}_\sigma \mathrm{P}^{\alpha\beta} + \mathcal{E}_\sigma \mathcal{D}^\sigma \frac{u^\alpha u^\beta}{c^2} + u^\alpha \mathrm{V}^\beta + \mathrm{W}^\alpha u^\beta,
$$

where

$$
\mathrm{V}^\beta \overset{\text{def}}{=} \frac{1}{ic^2} \frac{\varepsilon^{\beta\gamma\mu\rho}}{\sqrt{-g}} \mathcal{E}_\gamma \mathcal{H}_\mu u_\rho,
$$

$$
\mathrm{W}^\alpha \overset{\text{def}}{=} \frac{1}{ic^2} \frac{\varepsilon^{\alpha\gamma\mu\rho}}{\sqrt{-g}} \mathcal{D}_\gamma \mathcal{B}_\mu u_\rho.
$$

Finally we must make some hypotheses in regard with the electromagnetic constitutive equation. The simplest case is that of a linear constitutive equation. The inductions depend linearly on the fields. For the *isotropic* case and for perfect magnetohydrodynamics, we shall set

$$\mathcal{B}^\alpha = \mu\, \mathcal{H}^\alpha. \tag{151}$$

Here μ is the *constant* magnetic permeability. According to the last definition of (148), we then have

$$\mathcal{M}^\alpha = (1 - \mu^{-1})\, \mathcal{B}^\alpha = (\mu - 1)\, \mathcal{H}^\alpha. \tag{152}$$

But instead of the function dependence $\in(\rho, \eta, F^{\alpha\beta})$, we can consider $\in(\rho, \eta, \mathcal{B}^\alpha)$, Then equation (95) reduces to

$$\mathcal{M}^\alpha = -\rho \frac{\partial \in(\rho, \eta, \mathcal{B}_\gamma)}{\partial \mathcal{B}_\alpha} \tag{153}$$

(*cf.* Grot, 1970). For instance, we may take ([12])

$$\begin{aligned} \in(\rho, \eta, \mathcal{B}_\gamma) &= \tilde{\in}(\rho, \eta) - \frac{1}{\rho}(1 - \mu^{-1})\, \mathcal{B}_\alpha \mathcal{B}^\alpha \\ &= \tilde{\in}(\rho, \eta) - \frac{1}{\rho}\mu(\mu - 1)\, \mathcal{H}_\alpha \mathcal{H}^\alpha \end{aligned} \tag{154}$$

from which equation (153) follows. On account of (146) and (152), the modified index $\tilde{f}$ can be written in that case as

$$\tilde{f} = 1 + \frac{p}{\rho c^2} + \frac{\tilde{\in}}{c^2} - \frac{1}{\rho c^2}\mathcal{M}_\alpha \mathcal{B}^\alpha \tag{155}$$

of which the last term is the energy density of a magnetic doublet. With equations (151)-(155) valid, we obtain from equation (150) :

$$\begin{cases} T^{\alpha\beta}_{(em)} = \mu \left\{ \left[\left(1 - \frac{\mu}{2}\right) g^{\alpha\beta} + \frac{1}{c^2} u^\alpha u^\beta \right] \boldsymbol{\mathcal{H}}^2 - \mathcal{H}^\alpha \mathcal{H}^\beta \right\}, \\ f^\gamma_{(S-G)} = \mu(\mu - 1)\, \mathcal{H}^\lambda \overset{*}{\nabla}{}^\gamma \mathcal{H}_\lambda, \qquad \boldsymbol{\mathcal{H}}^2 \overset{\text{def}}{=} \mathcal{H}_\sigma \mathcal{H}^\sigma \end{cases} \tag{156}$$

in which we have defined the *projected operator* of covariant differentiation $\overset{*}{\nabla}{}^\gamma$ by

$$\overset{*}{\nabla}{}^\gamma \equiv P^{\gamma\alpha} \nabla_\alpha = \nabla^\gamma + \frac{u^\gamma}{c^2} \frac{\partial}{\partial \tau}. \tag{157}$$

For small velocities, this reduces to the classical spatial differential operator ∇^k $(k = 1, 2, 3)$ since $u_\gamma \overset{*}{\nabla}{}^\gamma \mapsto 0$ from equation $(5)_1$.

([12]) Equation (154) is nothing but a special contact transformation.

Of course, equation (156) differs from that given by Lichnerowicz (1967, p. 93) since the invariants in front of $g^{\alpha\beta}$ in the original definitions of $T^{\alpha\beta}_{(em)}$ are different. Moreover the metric signature is taken to be $(+, +, +, -)$ hence a plus sign in front of $u^\alpha\, u^\beta$.

Within the frame of this simplified model of magnetohydrodynamics, the total stress-energy-momentum tensor (112) takes the form

$$T^{\alpha\beta} = \rho\, \tilde{f}\, u^\alpha\, u^\beta + p g^{\alpha\beta} + T^{\alpha\beta}_{(em)}, \tag{158}$$

where $\tilde{f}$ and $T^{\alpha\beta}_{(em)}$ are given by equations (155) and $(156)_1$ respectively. Or [13]

$$T^{\alpha\beta} = \left(\rho\, \tilde{f} + \frac{\mu}{c^2}\mathcal{H}^2\right) u^\alpha\, u^\beta + \left[p + \mu\left(1 - \frac{\mu}{2}\right)\mathcal{H}^2\right] g^{\alpha\beta} - \mu\, \mathcal{H}^\alpha\, \mathcal{H}^\beta. \tag{159}$$

The 4-velocity is an eigenvector of this energy-momentum tensor since, using (155) and the equation $\mathcal{H}^\beta\, u_\beta = 0$, one finds

$$\left\{\begin{aligned} &T^{\alpha\beta}\, u_\beta = \omega\, u^\alpha, \\ &\omega = -\rho\left[c^2 + \tilde{\epsilon} + \rho^{-1}\left(\frac{\mathcal{B}^2}{2} - \mathcal{M}_\sigma\, \mathcal{B}^\sigma\right)\right]. \end{aligned}\right. \tag{160}$$

The eigenvalue ω is clearly minus the sum of the rest energy, the internal energy $\tilde{\epsilon}(\rho, \eta)$ and the magnetic energy.

Remark now that with the foregoing assumptions, the Maxwell's equations $(32)_1$ are reduced to

$$\nabla_\beta\, (u^\alpha\, \mathcal{H}^\beta - u^\beta\, \mathcal{H}^\alpha) = 0, \tag{161}$$

which imply some interesting consequences (*cf.* Lichnerowicz, 1967, p. 94).

In conclusion of this section, we have obtained the complete set of equations for the interior problem in perfect general relativistic magnetohydrodynamics. They are the Einstein's equations, the Euler equations of motion, the conservation of energy equation and the Maxwell's equations (161). For this scheme, the total energy-momentum tensor

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[13] Authors interested in astrophysical applications of magnetohydrodynamics take $\mu = 1$ since the magnetic permeability differs slightly from unity (*cf.* Landau and Lifshitz, 1960, p. 213). With this assumption, equation (159) can be written in the form used by almost every astrophysicist

$$T^{\alpha\beta} = (\tilde{\rho} + p + \mathcal{H}^2)\, u^\alpha\, u^\beta + \left(p + \frac{\mathcal{H}^2}{2}\right) g^{\alpha\beta} - \mathcal{H}^\alpha\, \mathcal{H}^\beta,$$

with

$$\tilde{\rho} = \rho\,(1 + \tilde{\epsilon}), \qquad c = 1.$$

is given by equation (159) and the fluid index by equation (155). The corresponding boundary equations are obtained by letting the discontinuity hypersurface (Σ) coincide with the boundary of $(\mathcal{B})$.

In the last two sections of this article, we shall return to the general case treated in the foregoing sections.

9. A BERNOULLIAN THEOREM

In classical hydrodynamics, a straightforward consequence of the Clebsch's representation of the fluid velocity is the existence of a *Bernoullian* theorem (*cf.* Lamb, 1932; Seliger and Whitham, 1968), i. e., a theorem *à la* Bernoulli and *not* a theorem of Bernoulli since the conditions of validity of the so-called theorem along a streamline are quite general : neither stationarity nor irrotationality are implied. Only insentropy along streamlines is required. The same holds true in relativistic hydrodynamics. Indeed with equations (4) and $(3)_1$, we recall that equation (97) when contracted with u^β yields the time-like component of the Clebsch's representation

$$\rho\, c^2\, \tilde{f} + \frac{q}{c} A_\alpha\, u^\alpha = -\rho \left(\frac{\partial \mathcal{A}}{\partial \tau} + \eta \frac{\partial \mathcal{B}}{\partial \tau} + X^K \frac{\partial \mathcal{C}_K}{\partial \tau} \right) \tag{162}$$

along a streamline. This is the Bernoullian theorem looked for. To materialize this assumption, it is sufficient : (*a*) to go to flat space-time and thus make appear the Newtonian gravitational potential; (*b*) to take the slow motion limit $c \mapsto \infty$. For the flat space-time approximation, we take

$$g_{\alpha\beta} \begin{cases} g_{kl} = \delta_{kl} + \mathcal{O}(c^{-2}); \qquad k, l = 1, 2, 3, \\ g_{k4} = 0, \\ g_{44} = -1 - \dfrac{2\,U}{c^2} + \mathcal{O}(c^{-4}), \end{cases}$$

hence,

$$\sqrt{-g} = 1 + \frac{U}{c^2} + \mathcal{O}(c^{-4}),$$

where U is the Newtonian gravitational potential. Then, note that

$$\frac{\partial}{\partial \tau} \mapsto (1 - \beta^2)^{-\frac{1}{2}} \left(\frac{\partial}{\partial t} + v^k \nabla_k \right), \qquad \beta \stackrel{\text{def}}{=} \left| \frac{\mathbf{v}}{c} \right|, \tag{163}$$

wherre t is the Newtonian absolute time and $\mathbf{v}$ is the three-dimensional velocity of the fluid. Also,

$$\frac{q}{c} A_\alpha\, u^\alpha \mapsto \frac{1}{c} (i\, \Phi) . \left[ic \left(1 - \beta^2\right)^{-\frac{1}{2}} \right] \simeq -q\, \Phi,$$

where Φ is the electric potential. We denote by ϵ^*, p^*, η^*, ρ^*, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{M}$, the internal energy density, the pressure, the entropy density and the matter density of classical continuum mechanics and the three-dimensional electric field, magnetic field and magnetization vector : $\mathcal{A}^*$ and $\mathcal{B}^*$ the corresponding values of $\mathcal{A}$ and $\mathcal{B}$. ρ is a scalar density. ρ^* is the classical matter density measured in a laboratory, e. g., wind tunnel (*cf.* Landau and Lifshitz, 1959, chapter 15). Thus,

$$(164)\qquad \rho \mapsto \rho^* \left(\frac{\sqrt{-g}}{\sqrt{1-\beta^2}}\right)_{limit} \simeq \rho^* \left(1 + \frac{\mathrm{U}}{c^2}\right)\left(1 + \frac{\mathbf{v}^2}{2\,c^2}\right).$$

Finally $\tilde{f}$ is given by equation (111). Collecting these approximations, we can write equation (162) as

$$(165)\qquad c^2 + \frac{\mathbf{v}^2}{2} + \mathrm{U} + \frac{p^*}{\rho^*} + \epsilon^* (\rho^*, \eta^*, \mathbf{B}, \mathbf{E}) = -\left(\frac{\partial \mathcal{A}^*}{\partial t} + \eta^* \frac{\partial \mathcal{B}^*}{\partial t} + \mathrm{X}^{\mathrm{K}} \frac{\partial \mathcal{C}_{\mathrm{K}}}{\partial t}\right)$$

along a streamline. This equation, except for the constant c^2 (rest energy) inherent to the limiting process of relativistic theories, is similar to the result quoted by Lamb (1932). For the case of perfect magnetohydrodynamics, we may use the decomposition (154) for the internal energy ϵ^* and equation (165) reads, dropping the purely relativistic term c^2

$$(166)\qquad \frac{\mathbf{v}^2}{2} + \mathrm{U} + \tilde{\epsilon}^* (\rho^*, \eta^*) + \frac{p^*}{\rho^*} - \frac{1}{\rho^*} \mathbf{M}.\mathbf{B} = -\left(\frac{\partial \mathcal{A}^*}{\partial t} + \eta^* \frac{\partial \mathcal{B}^*}{\partial t} + \mathrm{X}^{\mathrm{K}} \frac{\partial \mathcal{C}_{\mathrm{K}}}{\partial t}\right).$$

10. ANOTHER FORM OF THE ACTION PRINCIPLE

We shall derive here a form of the action principle given in the foregoing sections, which is closely related to that given by Bateman (1929, 1944) for isentropic flows in classical hydrodynamics.

Let us consider equation (78). On account of the variational result (96), we can write

$$(167)\qquad \overset{*}{\Lambda} = \rho \left[\mathbf{i}\,(\rho, \eta, \mathrm{F}^{\alpha\beta}) + \frac{1}{2} \tilde{\pi}_{\alpha\beta}\, \mathrm{F}^{\beta\alpha} + c^2\right] \sqrt{-g}.$$

Hence,

$$(168)\qquad \Lambda_m + \overset{*}{\Lambda} = \rho \left[\mathbf{i}\,(\rho, \eta, \mathrm{F}^{\alpha\beta}) - \epsilon\,(\rho, \eta, \mathrm{F}^{\alpha\beta}) + \frac{1}{2} \tilde{\pi}_{\alpha\beta}\, \mathrm{F}^{\beta\alpha}\right] \sqrt{-g} = p \sqrt{-g}.$$

The last relation follows from equation (59). Of course, according to equation (60), we can consider p to be a function of $\mathbf{i}$, η and $\tilde{\pi}^{\alpha\beta}$ since equation (60) is nothing but

$$dp = \rho\, d\mathbf{i} - \rho\theta\, d\eta + \frac{\rho}{2} F^{\alpha\beta}\, d\tilde{\pi}_{\beta\alpha} \tag{169}$$

with

$$\frac{\partial p}{\partial \mathbf{i}} = \rho, \qquad \frac{\partial p}{\partial \eta} = -\rho\theta, \qquad \frac{\partial p}{\partial \tilde{\pi}_{\alpha\beta}} = -\frac{\rho}{2} F^{\alpha\beta}. \tag{170}$$

The total Lagrangian density is then written as

$$\begin{aligned} \Lambda_{\text{total}} &= \Lambda_G + \Lambda_{(\text{em})} + p\,(\mathbf{i}, \eta, \tilde{\pi}^{\alpha\beta})\sqrt{-g} \\ &= \left[(2\varkappa)^{-1}\mathbf{R} + \varphi - \frac{1}{c} A_\alpha J^\alpha + A_\alpha \nabla_\beta G^{\alpha\beta}\right]\sqrt{-g} \\ &\quad + p\,(\mathbf{i}, \eta, \tilde{\pi}^{\alpha\beta})\sqrt{-g}. \end{aligned} \tag{171}$$

We now consider the case for which there is no discontinuity hypersurface (Σ), the variations of all arguments defined according to equation $(8)_2$, vanishing on the boundary $(\partial\mathcal{B})$. The term involving $G^{\alpha\beta}$ in equation $(171)_2$ can be integrated by parts. Using equation (36), we have

$$\begin{aligned} \Lambda_{\text{total}} = \Big[&(2\varkappa)^{-1}\mathbf{R} + \varphi - \frac{1}{c} A_\alpha J^\alpha \\ &- \frac{1}{2} F_{\beta\alpha}(F^{\alpha\beta} - \rho\tilde{\pi}^{\alpha\beta})\Big]\sqrt{-g} + p\sqrt{-g} \end{aligned} \tag{172}$$

in which we used the relation $\pi^{\alpha\beta} = \rho\tilde{\pi}^{\alpha\beta} = F^{\alpha\beta} - G^{\alpha\beta}$. Equations $(171)_2$ and (172) are equivalent. If A_α is varied independently in equation $(171)_2$, the Maxwell's equations $(32)_1$ follow while, if $\tilde{\pi}^{\alpha\beta}$ is varied independently in equation (172), equations $(32)_1$ and (36) being now assumed, we get in $(\mathcal{B})$, setting equal to zero the coefficient of $\delta\tilde{\pi}^{\alpha\beta}$

$$\delta\tilde{\pi}^{\alpha\beta} : \quad \frac{\partial p}{\partial \tilde{\pi}_{\alpha\beta}} + \frac{\rho}{2} F^{\alpha\beta} = 0, \tag{173}$$

or,

$$\frac{\partial p}{\partial \tilde{\pi}_{\alpha\beta}} = -\frac{\rho}{2} F^{\alpha\beta} \tag{174}$$

that does agree with equation $(170)_3$. So much for the electromagnetic variations. The variations of the metric and those with respect to $\mathbf{i}$ and η are more involved. We shall be satisfied with indicating the method. We need use the thermodynamical relations $(170)_{1-2}$ and the representations (*cf.* Maugin, 1971 *e*) for u_β and i :

$$(175)\quad \begin{cases} u_\beta = \tilde{f}^{-1}\left[\frac{q}{c}\mathrm{A}_\beta + \nabla_\beta \mathcal{A} + \eta\, \nabla_\beta \mathcal{B} + \mathrm{X}^{\mathrm{K}}\, \nabla_\beta\, \mathcal{C}_{\mathrm{K}}\right], \\ \mathbf{i} = -\, u^\beta\, (\nabla_\beta\, \mathcal{A} + \eta\, \nabla_\beta\, \mathcal{B} + \mathrm{X}^{\mathrm{K}}\, \nabla_\beta\, \mathcal{C}_{\mathrm{K}}) \\ \qquad + \frac{1}{2}\tilde{f}\,(g_{\alpha\beta}\, u^\alpha\, u^\beta + c^2) - c^2 + \frac{1}{2}\tilde{\pi}_{\alpha\beta}\, \mathrm{F}^{\alpha\beta}. \end{cases}$$

In the second of these relations, u^β is given by the first one. Then one can show that independant variations of η, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}_{\mathrm{K}}$ would yield a system of variational equations equivalent ot the system (96)-(102). In varying the metric $g_{\alpha\beta}$, one must take account of the effect on p of this variation (*cf.* the variation of ρ in section 4).

In absence of electromagnetic fields, the expression $(171)_2$ reduces to

$$(176)\qquad \Lambda_{\text{total}} = [(2\,\varkappa)^{-1}\,\mathbf{R} + p\,(\mathbf{i}, \eta)]\,\sqrt{-\,g},$$

A_β disappears from equation $(175)_1$ and $\tilde{f}$ is replaced by $f = 1 + \frac{\mathbf{i}}{c^2}\cdot$ For the special relativistic case, this reduces to the result enunciated in Maugin (1971 *e*), i. e., we have the action principle in M^4

$$(177)\qquad \delta \int_{(\mathcal{B})} p\,(\mathbf{i}, \eta)\, d^4\,\mathbf{x} = 0.$$

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