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<http://www.numdam.org/item?id=AIHPA_1972__16_2_87_0>
The theorem on Unitary Equivalence of Fock Representations

by

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ABSTRACT. — We prove that two Fock states \( \omega_J \) and \( \omega_K \) (not necessarily gauge invariant) on the CAR-algebra are unitarily equivalent if and only if \( | J - K | \) is a Hilbert-Schmidt operator. We calculate explicitly the norm difference \( \| \omega_J - \omega_K \| \).

Let \( (H, s) \) be a separable Euclidean space and \( J \) and \( K \) complex structures on \( (H, s) \), i.e.

\[
J^+ = - J; \quad J^2 = -1, \quad K^+ = - K; \quad K^2 = -1.
\]

Consider the operators

\[
P = [J, K]^+; \quad Q = [J, K]^- \]

and let \( P = U \upharpoonright P \), \( Q = V \upharpoonright Q \) be their polar decompositions, \( | Q | \), \( | P | \) and \( U \) commute with \( J \) and \( K \); consequently the dimension of \( \text{Ker } P \) is even or infinite; \( Q \) is a normal operator, therefore \( V \) can be chosen such that \( V^+ = - V, \ V^2 = -1 \). The same notations as in [1] are used : \( \mathfrak{a} = \overline{\mathfrak{a}} (H, s) \) is the CAR-algebra and \( \omega_J \) is any pure quasi-free state on \( \mathfrak{a} \); \( J \) satisfies : \( J^+ = - J, \ J^2 = -1 \).

THEOREM 1. — Let the operator \( P \) be diagonalizable [i.e. \( (\psi_i)_{i \in \mathbf{N}} \) orthonormal basis of \( H \) such that \( P \psi_i = \mu_i \psi_i, \ \mu_i \in \mathbf{R} \) (reals)], then there exists a family of subspaces \( (H_n)_{n \in \mathbf{N}} \) of \( H \) invariant under \( J \) and \( K \) such that :

(i) \( H = \bigoplus_{n=0}^{\infty} H_n; \)
(ii) \( \dim H_0 \) and \( \dim H_1 \) is even or infinite, \( \dim H_n = 4 \) for \( n \geq 2 \);
(iii) \( P = \sum_n \lambda_n p_n \), where \( P_n H = H_n \); \( \lambda_0 = -2, \lambda_1 = 2 \) and \(-2 < \lambda_n < 2\) for \( n \geq 2 \).

**Proof.** — Let \( F = \ker Q \); \( F \) and \( F^\perp \) (orthogonal complement of \( F \) for \( s \)) are invariant for \( J \) and \( K \).

(a) Suppose \( F^\perp = \{ 0 \} \); then \( JK = \frac{P}{2} \) is unitary and Hermitian, there exists a decomposition \( F = H_0 + H_1 \) such that \( P = -P_0 + P_1 \), where \( P_0 \) and \( P_1 \) are the orthogonal projection operators on \( H_0 \) respectively \( H_1 \), which are invariant under \( J \) and \( K \) and therefore \( \dim H_0 \) and \( \dim H_1 \) is even or infinite.

(b) Suppose \( F = \{ 0 \} \), let \( H_\alpha \) be subspaces of \( H \) such that \( \Phi H_\alpha = \lambda_\alpha H_\alpha \). Because \( [P, J]_- = [P, K]_- = 0 \), the subspaces \( H_\alpha \) are invariant for \( J \) and \( K \). Remark that \( P^2 + Q^+ Q = 4 \), \( Q^+ Q = |Q|^2 \); therefore \( |Q| \) has the same proper subspaces \( H_\alpha \) as \( |P| \). Let \( |Q| H_\alpha = \mu_\alpha H_\alpha \), then \( \lambda_\alpha^2 + \mu_\alpha^2 = 4 \) for all \( \alpha \). Take any \( \psi_\alpha \in H_\alpha \) and consider the subspaces \( H_{\psi_\alpha} \) generated by the real orthogonal set \( \{ \psi_\alpha, V \psi_\alpha, J \psi_\alpha, JV \psi_\alpha \} \). It is clear that \( H_{\psi_\alpha} \) is a real subspace invariant under \( J \) and \( K \) of dimension four.

In general \( H = F + F^\perp \) the results of (a) and (b) prove the theorem.

Q. E. D.

**Lemma.** — Let \( \pi_J \) and \( \pi_K \) be the Fock representations associated with \( J \) respectively \( K \). If \( \pi_J \) and \( \pi_K \) are unitarily equivalent then \( [J, K]_+ \) has \(-2\) as the only accumulation point of its spectrum.

**Proof.** — Let \( \{ \psi_j \}_{j \in \mathbb{N}} \) be any infinite orthonormal set of \( H \) and

\[
L_n = -\frac{i}{n} \sum_{j=1}^n ( \langle \psi_j | B (J \psi_j) \rangle )
\]

then

\[
(\Omega_J, \pi_J (L_n) \Omega_J) = \omega_J (L_n) = 1.
\]

Using Schwartz's inequality, we have

\[
\| \pi_J (L_n) \Omega_J \| = 1 \quad \text{furthermore} \quad \left\| \prod_{i=1}^k B (\psi_i) L_n \right\| \leq \frac{k}{n}
\]

proving

\[
1 - \frac{k}{n} \leq \left\| \prod_{i=1}^k \pi_J (\psi_i) \Omega_J \right\| \leq 1 + \frac{k}{n}
\]
i.e. $\pi_j(L_n)$ tends strongly to one for $n$ tending to infinity. Because $\pi_j$ and $\pi_K$ are unitarily equivalent $\pi_K(L_n)$ tends strongly to one on $\mathcal{H}_K$ and therefore weakly.

Further the expression

$$\omega_K(L_n) = (\Omega_K, \pi_K(L_n) \Omega_K) = -\frac{1}{2n} \sum_{i=1}^{n} s \left( P \psi_i, \psi_i \right)$$

must tend to one for all orthonormal sets $(\psi_i)_{i \in \mathbb{N}}$ which is possible if $P$ has no accumulation points in its spectrum different from $-2$.

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**Theorem 2.** — If $\omega_j$ and $\omega_K$ are pure quasi-free states, then $\pi_j$ and $\pi_K$ are unitarily equivalent iff $| J - K |$ is a Hilbert-Schmidt operator.

**Proof.** — By Theorem 1,

$$H = \bigoplus_{n=0}^{\infty} H_n; \quad P = \sum_{n=0}^{\infty} \lambda_n P_n; \quad P_n H = H_n,$$

where $\dim H_n = 4$ for $n \geq 2$. By the lemma, $\dim H_1 < \infty$. Let $\{ \Phi_1, \ldots, \Phi_r; J \Phi_1, \ldots, J \Phi_r \}$ be an orthonormal basis of $H_1$ and

$$u_i = \prod_{k=1}^{r} B(\Phi_k).$$

In each $H_n (n \geq 2)$ we choose the following orthonormal basis $(\psi_n, V \psi_n, J \psi_n, JV \psi_n)$, where $\psi_n$ is any normalized vector of $H_n$ and let

$$u_n = B(J \psi_n) B(\psi_n'),$$

where

$$\psi_n' = \frac{1}{(2 - \lambda_n)^{\frac{1}{2}}} (J \psi_n + K \psi_n).$$

If $u_0$ is the unit of $\mathcal{A}(H_0, s)$, then for all $n \geq 0$ and all $x \in \mathcal{A}(H_n, s)$,

$$\omega_K(x) = \omega_j(u_0^* x u_0).$$

In order that $U = \bigotimes_{n=0}^{\infty} \pi_{J_n}(u_n)$ is an unitary operator on $\mathcal{A}_j = \bigotimes_{n=0}^{\infty} \mathcal{A}_{J_n}$ ($J_n$ is the restriction of $J$ to $H_n$) it is necessary and sufficient that

$$U \Omega_j \in \mathcal{A}_j$$

i.e.

$$\prod_{n=0}^{\infty} (\Omega_{J_n}, \pi_{J_n}(u_n) \Omega_{J_n}) = \prod_{n=0}^{\infty} \frac{1}{2} (2 - \lambda_n)^{\frac{1}{2}}$$
does not vanish. But
\[ \prod_{n=1}^{\infty} \left( 2 - \lambda_n \right)^{\frac{1}{2}} = 0 \iff \prod_{n=1}^{\infty} \left( 1 - \lambda_n \frac{4}{4} \right) = 0 \]
\[ \iff \frac{1}{4} \sum_{n=2}^{\infty} (2 + \lambda_n) < \infty \iff \text{Tr} (2 + P) < \infty. \]

Otherwise \( (J - K)^+ (J - K) = 2 + P, \) therefore \( \pi_J \) and \( \pi_K \) are unitarily equivalent if \( J - K \) is a Hilbert-Schmidt operator.

Conversely, suppose that \( |J - K| \) is not a Hilbert-Schmidt operator, hence
\[ \prod_{i=1}^{m} \left( 1 - \lambda_i \frac{4}{4} \right) = 0. \]
Let \( E_{n,m} = \bigoplus_{i=1}^{m} H_i; \) the restrictions of \( \omega_J \) and \( \omega_K \) to \( \alpha (E_{n,m}, s) \) remain pure states unitarily equivalent because if \( U_{n,m} = \prod_{i=1}^{m} u_i, \) then
\[ \forall x \in \alpha (E_{n,m}, s), \quad \omega_J (x) = \omega_K (u_{n,m} x u_{n,m}^*) \]

Hence by Lemma 2.4 of [2]
\[ || (\omega_J - \omega_K) | \alpha (E_{n,m}, s) || = 2 \left( 1 - | \omega_J (u_{n,m}) |^2 \right)^{\frac{1}{2}} \]
\[ = 2 \left( 1 - \prod_{i=1}^{m} \left( \frac{1}{2} - \frac{\lambda_i}{4} \right) \right)^{\frac{1}{2}}. \]

Denote by \( \alpha (E_n, s) \) the commutant of \( \alpha (E_n, s) \) in \( \alpha. \) By lemma 2.3 of [2],
\[ || (\omega_J - \omega_K) | \alpha (E_n, s)^c || = || (\omega_J - \omega_K) | \alpha (E_n, s)^c ||. \]
Since \( \alpha (E_n, s) \) is the inductive limit of \( \alpha (E_{n,m}, s) \) when \( m \to \infty, \) we have
\[ || (\omega_J - \omega_K) | \alpha (E_n, s)^c || = \lim_{m \to \infty} || (\omega_J - \omega_K) | \alpha (E_{n,m}, s) || = 2. \]

By lemma 2.1 of [2] \( \pi_J \) and \( \pi_K \) are not unitarily equivalent.

\textbf{Corollary.} — \textit{The representations} \( \pi_J \) \textit{and} \( \pi_K \) \textit{are unitarily equivalent if} \( || \omega_J - \omega_K || < 2, \) \textit{and}
\[ || \omega_J - \omega_K || = 2 \left( 1 - \prod_{i=1}^{m} \left( \frac{1}{2} - \frac{\lambda_i}{4} \right) \right)^{\frac{1}{2}}. \]
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Proof. — Lemma 2.1 of [2] proves that if $\pi_j$ is not unitarily equivalent with $\pi_K$, then $\|\omega_j - \omega_K\| = 2$. Otherwise if $\pi_j$ and $\pi_K$ are equivalent, it follows from the calculations done in Theorem 2, that

$$\|\omega_j - \omega_K\| = 2 \left( 1 - \prod_{i=1}^{\infty} \left( \frac{1}{2} - \frac{\lambda_i}{4} \right)^{\frac{1}{2}} \right).$$

Q. E. D.

REFERENCES


(Manuscrit reçu le 6 juillet 1971.)