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Section A:
Physique théorique.

On a class of infinite products occurring in quantum statistical mechanics

by

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ABSTRACT. — We study the class of infinite products

$$f_c^{\mathrm{L}}(\lambda) = \sum_{n=0}^{\infty} a_n^{\mathrm{L}} \lambda^n = \prod_{j=1}^{\infty} \{ 1 + \lambda c(k_j)^2 \}, \quad k_j = j\frac{2\pi}{\mathrm{L}} \quad \text{and} \quad \mathrm{L} \in \mathbb{R}^+,$$

where $c(k) = \frac{\mathrm{A}}{k^m}$, $k \geq 0$, $\mathrm{A} > 0$, $m > \frac{1}{2}$, which occurs naturaly in quantum statistical mechanics. In particular, we compute the limits

$$\underset{\substack{n\to\infty \\ \mathrm{L}\to\infty}\left.\right\} \frac{2n}{\mathrm{L}}=d}{\text{limit}} \frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}} \quad \text{and} \quad \underset{\substack{n\to\infty \\ \mathrm{L}\to\infty}\left.\right\} \frac{2n}{\mathrm{L}}=d}{\text{limit}} \frac{1}{\mathrm{L}} \operatorname{Log} a_n^{\mathrm{L}}$$

which are relevant in the problem of the thermodynamic limit of the BCS superconducting state. By the same way, we get new results concerning the infinite products of the form

$$g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{\mathrm{A}}{j^{1/\rho}} \right\}, \qquad \mathrm{A} > 0,\ 0 < \rho < 1.$$

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In particular we are able to compute the limits

$$\lim_{n\to\infty} n^{1/\rho}\frac{b_{n+1}}{b_n} \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{2n}\operatorname{Log} n^{\frac{1}{\rho}}b_n^{\frac{1}{n}}$$

RÉSUMÉ. — Nous étudions la classe de produits infinis

$$f_c^{\mathrm{L}}(\lambda) = \sum_{n=0}^{\infty} a_n^{\mathrm{L}}\lambda^n = \prod_{j=1}^{\infty}\{\,1 + \lambda c(k_j)^2\,\}, \quad k_j = j\frac{2\pi}{\mathrm{L}} \quad \text{et} \quad \mathrm{L}\in\mathbb{R}^+,$$

où $c(k) = \frac{\mathrm{A}}{k^m}$, $k \geqslant 0$, $\mathrm{A} > 0$, $m > \frac{1}{2}$, qui se présentent naturellement en mécanique statistique quantique. En particulier nous calculons les limites

$$\operatorname*{limite}_{\substack{n\to\infty\\ \mathrm{L}\to\infty}\left\}\frac{2n}{\mathrm{L}}=d\right.} \frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}} \quad \text{et} \quad \operatorname*{limite}_{\substack{n\to\infty\\ \mathrm{L}\to\infty}\left\}\frac{2n}{\mathrm{L}}=d\right.} \frac{1}{\mathrm{L}}\operatorname{Log} a_n^{\mathrm{L}}$$

qui ont leur importance dans la limite thermodynamique de l'état de la supraconductivité de Bardeen-Cooper-Schriefer. Simultanément nous obtenons des résultats nouveaux relatifs aux produits infinis de la forme

$$g_\rho(\mu) = \sum_{n=0}^{\infty} b_n\mu^n = \prod_{j=1}^{\infty}\left\{1 + \mu\frac{\mathrm{A}}{j^{1/\rho}}\right\}, \qquad \mathrm{A} > 0,\ 0 < \rho < 1$$

En particulier nous sommes en mesure de calculer les limites

$$\operatorname*{limite}_{n\to\infty} n^{1/\rho}\frac{b_{n+1}}{b_n} \quad \text{et} \quad \operatorname*{limite}_{n\to\infty}\frac{1}{2n}\operatorname{Log} n^{\frac{1}{\rho}}b_n^{\frac{1}{n}}$$

I. INTRODUCTION

It is now well known that algebras of observables are useful in the kinematical description, from a quantum mechanical point of view, of systems of interacting particles. For that purpose, we constructed, in a preceeding paper [*1*], a family of states (positive linear functionals of norm 1) over a Clifford-C*-algebra, each of which characterized by a real function c of real argument and describing a « condensate » of pairs of fermions with density d.

In that work, we studied a class of entire functions of the complex variables λ, depending on the function c and on a length L, defined by the infinite products

$$f_c^{\mathrm{L}}(\lambda) = \prod_{j=1}^{\infty} \{ 1 + \lambda c(k_j)^2 \}, \quad \mathrm{L} \in \mathbb{R}^+, \quad k_j = j\frac{2\pi}{\mathrm{L}} \tag{1}$$

More precisely, if we write

$$f_c^{\mathrm{L}}(\lambda) = \sum_{n=0}^{\infty} a_n^{\mathrm{L}} \lambda^n \tag{2}$$

where

$$a_n^{\mathrm{L}} = \sum_{0<j_1<j_2<\ldots<j_n} c(k_{j_1})^2 c(k_{j_2})^2 \ldots c(k_{j_n})^2 \tag{3}$$

the main point was the existence of the limit

$$\gamma(d) = \lim_{\substack{n\to\infty \\ \mathrm{L}\to\infty}\left\} \frac{2n}{\mathrm{L}} = d\right.} \frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}}, \qquad d \in \mathbb{R}^+ \tag{4}$$

and we were able to prove the following theorem:

THEOREM. — Let c be a real, bounded, decreasing, square integrable function of a real positive variable k, tending to zero, when k tends to infinity, faster than $k^{-1/2}$. Let d and L be positive reals.

Then the limit (4) exists. Moreover, defining the function $g(d)$ as

$$g(d) = \lim_{\substack{n\to\infty \\ \mathrm{L}\to\infty}\left\} \frac{2n}{\mathrm{L}} = d\right.} \frac{1}{\mathrm{L}} \operatorname{Log} a_n^{\mathrm{L}}, \tag{5}$$

then $g(d)$ exists, is convex and differentiable, and

$$\gamma(d) = e^{2g'(d)} \tag{6}$$

Finaly one has the following integral formula

$$d = \frac{1}{\pi} \int_0^\infty \frac{c^2(k)}{\gamma(d) + c^2(k)} dk \tag{7}$$

The methods employed are rather involved and do not fully exploit the analyticity properties of f_c^{L}. On the other hand, the theorem quoted above is only an existence theorem, so that except for very special choices for c ([1], (109)), it does not allow to compute explicitely the quantities $g(d)$ and $\gamma(d)$ as functions of d.

Our purpose is to restate that theorem, using now entire functions technics. To that end, we have to make a different choice for our class of functions c, giving up definiteness at the origin (and so boundedness and square integrability) but requiring an homogeneity condition. The quantities a_n^L, $g(d)$ and $\gamma(d)$ are now explicitely given.

By the same way, we study the class of infinite products

$$g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{1/\rho}} \right\}, \quad A > 0, \quad 0 < \rho < 1 \tag{8}$$

obtaining some new results for $\rho \neq \frac{1}{2}$.

II. CHOICE OF THE CLASS OF FUNCTIONS c

Let us consider, for the moment, positive functions $c(k)$, defined for $k > 0$, and vanishing at infinity faster than $k^{-1/2}$. The infinite products (1) are then convergent and define entire functions f_c^L whose zeros lie on the real negative axis and are given by

$$\lambda_j = -\frac{1}{c(k_j)^2} \tag{9}$$

By definition ([4], I, 4), ([5], 2.5.2), the convergence exponent of the sequence (9) is the greatest lower bound ρ of the reals α such that

$$\sum_{j=1}^{\infty} \left\{ \frac{1}{c(k_j)^2} \right\}^{-\alpha} = \sum_{j=1}^{\infty} c(k_j)^{2\alpha} < +\infty \tag{10}$$

If, by now, we restrict ourselves to functions such that

$$c(k) \sim \frac{A}{k^m}, \quad k \to \infty, \quad m > \frac{1}{2}, \quad A > 0 \tag{11}$$

a necessary and sufficient condition for the convergence of (10) is

$$2m\alpha = 1 + \varepsilon, \qquad \varepsilon > 0 \tag{12}$$

so that

$$\rho = \operatorname*{Inf}_{\varepsilon > 0} \left\{ \alpha : \alpha = \frac{1}{2m} + \frac{\varepsilon}{2m} \right\} = \frac{1}{2m} < 1 \tag{13}$$

On the other hand, the f_c^L are infinite canonical products of order ρ and genus p ([4], I, Th. 7), ([5], 2.6.5), the genus p beeing related to the order ρ according to

$$p \leq \rho \leq p + 1, \qquad p \text{ integer.} \tag{14}$$

In the case of (11), we can then conclude that the entire functions f_c^L are characterised by

$$p = 0, \quad 0 < \rho = \frac{1}{2m} < 1, \quad \frac{1}{2} < m + \infty \tag{15}$$

Further more, let $n(r)$, $r \in \mathbb{R}^+$, the function giving the number of zeros of f_c^L with modulus less than or equal to r. One can show ([4], I, lemma 1), ([5], 2.5.8) that

$$\rho = \overline{\lim_{r \to \infty}} \frac{\operatorname{Log} n(r)}{\operatorname{Log} r} \tag{16}$$

and one defines the density Δ of the sequence (9) as

$$\Delta = \overline{\lim_{r \to \infty}} \frac{n(r)}{r^\rho} \tag{17}$$

We are going to estimate these quantities, restricting now to functions c such that (11) holds, which are monotonic and differentiable at least for k large and such that

$$c'(k) \sim \frac{-B}{k^{m+1}}, \qquad k \to \infty, \quad B > 0 \tag{18}$$

Turning back to the definition of $n(r)$, we can write that

$$n(r) = \operatorname{Max} \left\{ j : \frac{1}{c(k_j)^2} \leq r \right\} = \operatorname{Max} \left\{ j : k_j \leq v\left(\frac{1}{\sqrt{r}}\right) \right\} = \left[\frac{L}{2\pi} v\left(\frac{1}{\sqrt{r}}\right) \right] \tag{19}$$

where v is the inverse function of c (defined at least for r large enough) and where the squared brackets mean « the largest integer contained in ». Thanks to our hypothesis (18), it is an easy task to show that

$$n(r) \sim \frac{L}{2\pi} A^{2\rho} r^\rho, \qquad r \to \infty \tag{20}$$

(a result which agrees with (16)) and that

$$\Delta = \frac{L}{2\pi} A^{2\rho} \tag{21}$$

From these estimations, it is actually possible to deduce that ([*4*], I, Th. 25), ([*5*], 4.1.1)

$$\operatorname{Log} f_c^L(re^{i\theta}) \sim e^{i\rho\theta} \frac{L}{2} A^{2\rho} (\operatorname{cosec} \pi\rho) r^\rho, \quad -\pi < \theta < +\pi, \quad r \to \infty \tag{22}$$

or, taking the real part, that

$$\operatorname{Log} | f_c^L(re^{i\theta}) | \sim \frac{L}{2} A^{2\rho} (\operatorname{cosec} \pi\rho) \cos \rho\theta \cdot r^\rho, \quad -\pi < \theta < +\pi, \quad r \to \infty \tag{23}$$

uniformly with respect to θ if $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$.

Then, the indicator function $h_c^L(\theta)$ of f_c^L is given by ([*4*], I, 15), ([*5*], 2.1.8)

$$h_c^L(\theta) = \overline{\lim_{n\to\infty}} \frac{\operatorname{Log} | f_c^L(re^{i\theta}) |}{r^\rho} = \frac{L}{2} A^{2\rho} (\operatorname{cosec} \pi\rho) \cos \rho\theta, \quad -\pi < \theta < +\pi \tag{24}$$

and also for $-\pi \leq \theta \leq +\pi$ as h_c^L is a continuous function, defined by periodicity for other values of θ.

From now on, we are able to compute the type of f_c^L according to the formula ([*4*], I, 1 and Th. 29), ([*5*], 2.1.4)

$$\tau(L) = \overline{\lim_{r\to\infty}} \frac{\operatorname{Log} \operatorname{Max}_{|\lambda|=r} | f_c^L(\lambda) |}{r^\rho} = \operatorname{Max}_\theta | h(\theta) | = \frac{L}{2} A^{2\rho} \operatorname{cosec} \pi\rho \tag{25}$$

as well as, by ([*4*], I, Th. 2), ([*5*], 2.2.10)

$$\tau(L) = \frac{1}{e\rho} \overline{\lim_{n\to\infty}} \, n (a_n^L)^{\rho/n} \tag{26}$$

Comparing formulas (20) and (16), we see that the $\overline{\lim}$ occuring in (16) is in fact a limit, as well as the ones in (17), (24), (25) and consequently in (26).

Unfortunately, the comparison of formulas (25) and (26) does not allow to estimate the limit (4) because of a lack of uniformity with respect to L. A simple case where uniformity can be recovered is given by

$$a_n^L = \left(\frac{L}{2\pi}\right)^{n/\rho} b_n, \quad b_n \text{ independent of } L \tag{27}$$

The following proposition, the proof of which is trivial, shows that (27) holds if and only if $c(k)$ is a homogeneous function of degree $-\frac{1}{2\rho}$:

PROPOSITION. — If

$$f_c^{\mathrm{L}}(\lambda) = \prod_{j=1}^{\infty}\{ 1 + \lambda c(k_j)^2 \} = \sum_{n=0}^{\infty} a_n^{\mathrm{L}} \lambda^n$$

where $k_j = j\frac{2\pi}{\mathrm{L}}$, then $a_n^{\mathrm{L}} = \left(\frac{\mathrm{L}}{2\pi}\right)^{n/\rho} b_n$, b_n independent of L, if and only if

$$c(k) = \frac{\mathrm{A}}{k^{1/2\rho}} = \frac{\mathrm{A}}{k^m},$$

$\rho = \frac{1}{2m}$ beeing the order of f_c^{L}.

So our conclusion is that entire functions technics can be easily applied to our problem if we restrict ourselves to the class of functions

$$c(k) = \frac{\mathrm{A}}{k^m}, \quad \frac{1}{2} < m < \infty, \quad \mathrm{A} > 0, \quad 0 < k < \infty \tag{28}$$

for which conditions (11) and (18) are evidently fulfilled.

III. THE MAIN THEOREM

From now on, we can write, thanks to (25),

$$f_c^{\mathrm{L}}(\lambda) = \sum_{n=0}^{\infty} \left(\frac{\mathrm{L}}{2\pi}\right)^{n/\rho} b_n \lambda^n = \sum_{n=0}^{\infty} \frac{(\tau(\mathrm{L})^{1/\rho}\lambda)^n}{\left(\frac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{n/\rho} \frac{1}{b_n}} \tag{29}$$

and

$$g_\rho(\mu) = \prod_{j=1}^{\infty}\left\{ 1 + \mu\frac{\mathrm{A}^2}{j^{1/\rho}} \right\} = \sum_{n=0}^{\infty} b_n \mu^n \tag{30}$$

Moreover, the expressions

$$\frac{a_n^{\mathrm{L}}}{\left(\frac{\tau(\mathrm{L})e\rho}{n}\right)^{n/\rho}} = \frac{b_n}{\left(\frac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{n/\rho}\left(\frac{e\rho}{n}\right)^{n/\rho}} \tag{31}$$

and (where Γ means the usual gamma function)

$$\text{(32)}\quad \frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}} \Bigg/ \frac{\left(\dfrac{\tau(\mathrm{L})e\rho}{n+1}\right)^{\frac{n+1}{\rho}}}{\left(\dfrac{\tau(\mathrm{L})e\rho}{n}\right)^{n/\rho}} = \frac{b_{n+1}}{\left(\dfrac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{\frac{n+1}{\rho}}\left(\dfrac{e\rho}{n+1}\right)^{\frac{n+1}{\rho}}} \frac{\left(\dfrac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{\frac{n}{\rho}}\left(\dfrac{e\rho}{n}\right)^{\frac{n}{\rho}}}{b_n}$$

$$\widetilde{_{n\to\infty}}\ \frac{b_{n+1}}{\left(\dfrac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{\frac{n+1}{\rho}} \dfrac{1}{\Gamma\left(\dfrac{n+1}{\rho}+1\right)}} \frac{\left(\dfrac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{\frac{n}{\rho}} \dfrac{1}{\Gamma\left(\dfrac{n}{\rho}+1\right)}}{b_n} = \frac{\chi(n+1;\,\rho)}{\chi(n;\,\rho)}$$

are now independent of L. Therefore the result we are aiming at is equivalent to

$$\text{(33)}\qquad \lim_{n\to\infty} \frac{\chi(n+1;\,\rho)}{\chi(n;\,\rho)} = 1$$

or

$$\text{(34)}\qquad b_n = \left(\frac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{n/\rho} \frac{\chi(n;\,\rho)}{\Gamma\left(\dfrac{n}{\rho}+1\right)} \quad \text{where} \quad \lim_{n\to\infty} \frac{\chi(n+1;\,\rho)}{\chi(n;\,\rho)} = 1$$

Incidentally it is interesting to remark that

$$\text{(35)}\qquad g_\rho(\mu) = \sum_{n=0}^{\infty} \frac{\left(\dfrac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{n/\rho} \chi(n;\,\rho)}{\Gamma\left(\dfrac{n}{\rho}+1\right)} \mu^n$$

is of order ρ and type

$$\text{(36)}\qquad \tau = \frac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho} = \tau(2\pi)$$

as is also the Mittag-Leffler function [6]

$$\text{(37)}\qquad \mathrm{E}_\rho(\mu) = \sum_{n=0}^{\infty} \frac{\left(\dfrac{\mathrm{A}^{2\rho}\pi}{\sin \pi\rho}\right)^{n/\rho}}{\Gamma\left(\dfrac{n}{\rho}+1\right)} \mu^n$$

which shows the close relation between E_ρ and g_ρ and asserts the well known fact that E_ρ is, in some sense [7], the simplest entire function of a given order and type.

Another way of writing (33) is the following (provided the limits exists):

$$\text{(38)} \quad \lim_{\substack{n\to\infty \\ L\to\infty}\big\}\frac{2n}{L}=d} \frac{a_{n+1}^L}{a_n^L} = \lim_{\substack{n\to\infty \\ L\to\infty}\big\}\frac{2n}{L}=d} \left(\frac{\tau(L)e\rho}{n+1}\right)^{\frac{n+1}{\rho}} \Big/ \left(\frac{\tau(L)e\rho}{n}\right)^{n/\rho}$$

$$= \lim_{\substack{n\to\infty \\ L\to\infty}\big\}\frac{2n}{L}=d} \left(\frac{LA^{2\rho}e\rho \operatorname{cosec} \pi\rho}{2(n+1)}\right)^{\frac{n+1}{\rho}} \Big/ \left(\frac{LA^{2\rho}e\rho \operatorname{cosec} \pi\rho}{2n}\right)^{n/\rho}$$

$$= \lim_{n\to\infty} \left(\frac{A^{2\rho}e\rho \operatorname{cosec} \pi\rho}{d}\right)^{1/\rho} \frac{1}{\left(1+\frac{1}{n}\right)^{n/\rho}} \frac{1}{\left(1+\frac{1}{n}\right)^{1/\rho}} = \left(\frac{A^{2\rho}\rho \operatorname{cosec} \pi\rho}{d}\right)^{\frac{1}{\rho}}$$

So, once the existence of the limit is proved, then necessarily

$$\text{(39)} \qquad \gamma(d) = \left(\frac{A^{2\rho}\rho \operatorname{cosec} \pi\rho}{d}\right)^{1/\rho}$$

and hence

$$\text{(40)} \qquad \lim_{n\to\infty} n^{1/\rho} \frac{b_{n+1}}{b_n} = (\pi A^{2\rho}\rho \operatorname{cosec} \pi\rho)^{1/\rho}$$

Before proving our main theorem, let us give a lemma:

LEMMA. — We have the inequalities:

$$\text{(41)} \qquad \frac{a_n^L}{a_{n-1}^L} > \frac{a_{n+1}^L}{a_n^L}$$

and

$$\text{(42)} \qquad \frac{b_n}{b_{n-1}} > \frac{b_{n+1}}{b_n}$$

These inequalities are well known in the theory of entire functions ([5], 2.8.2) but we restate the proof by sake of completeness.

One has:

$$\frac{f'(\lambda)}{f(\lambda)} = \sum_{j=1}^{\infty} \frac{c(k_j)^2}{1+\lambda c(k_j)^2}$$

and

$$\left\{\frac{f'(\lambda)}{f(\lambda)}\right\}' = \sum_{j=1}^{\infty} \frac{-c(k_j)^4}{[1+\lambda c(k_j)^2]^2}$$

So $f(\lambda)f''(\lambda) < f'(\lambda)^2$ if $\lambda \in \mathbb{R}$ or, else, $f^{(n-1)}(\lambda)f^{(n+1)}(\lambda) < f^{(n)}(\lambda)^2$ by application of the same inequality to $f^{(n-1)}(\lambda)$, which is also an entire function of the same order, type and genus as f. It follows that

$$\frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}} < \frac{n}{n+1}\frac{a_n^{\mathrm{L}}}{a_{n-1}^{\mathrm{L}}} < \frac{a_n^{\mathrm{L}}}{a_{n-1}^{\mathrm{L}}}.$$

It is interesting to remark that, by adapting to our case the proof of ([*1 bis*], lemma 3, (80)), one has conversely

$$\text{(43)} \qquad \begin{aligned} n\frac{b_n}{b_{n-1}} &\leq (n+1)\frac{b_{n+1}}{b_n} + \mathrm{A}^2 \\ \frac{a_n^{\mathrm{L}}}{a_{n-1}^{\mathrm{L}}} &\leq \frac{n+1}{n}\frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}} + \frac{1}{\pi d}\mathrm{A}^2 \end{aligned}$$

We are now able to prove our theorem.

THEOREM. — Let $c(k) = \dfrac{\mathrm{A}}{k^m}$ with $k > 0$, $\mathrm{A} > 0$, $m > \dfrac{1}{2}$, L and d be positive, and

$$\text{(44)} \qquad \begin{cases} f_c^{\mathrm{L}}(\lambda) = \displaystyle\sum_{n=0}^{\infty} a_n^{\mathrm{L}}\lambda^n = \prod_{j=1}^{\infty}\{1 + \lambda c(k_j)^2\}, & k_j = j\dfrac{2\pi}{\mathrm{L}} \\ g_\rho(\mu) = \displaystyle\sum_{n=0}^{\infty} b_n\mu^n = \prod_{j=1}^{\infty}\left\{1 + \mu\dfrac{\mathrm{A}^2}{j^{1/\rho}}\right., & \rho = \dfrac{1}{2m}, \quad 0 < \rho < 1 \end{cases}$$

Then

$$\text{(45)} \qquad \underset{\substack{n\to\infty \\ \mathrm{L}\to\infty}\big\}\frac{2n}{\mathrm{L}}=d}{\text{limit}} \frac{1}{\mathrm{L}}\operatorname{Log} a_n^{\mathrm{L}} = g(d) = \frac{d}{2\rho}\operatorname{Log}\left(\frac{\mathrm{A}^{2\rho}e\rho\operatorname{cosec}\pi\rho}{d}\right)$$

and

$$\text{(46)} \qquad \lim_{n\to\infty}\frac{1}{2\pi}\operatorname{Log} n^{1/\rho}b_n^{1/n} = \frac{1}{2\pi\rho}\operatorname{Log}(\pi\mathrm{A}^{2\rho}e\rho\operatorname{cosec}\pi\rho) = g\left(\frac{1}{\pi}\right)$$

Moreover

$$\text{(47)} \qquad \underset{\substack{n\to\infty \\ \mathrm{L}\to\infty}\big\}\frac{2n}{\mathrm{L}}=d}{\text{limit}} \frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}} = \gamma(d) = \left(\frac{\mathrm{A}^{2\rho}\rho\operatorname{cosec}\pi\rho}{d}\right)^{1/\rho}$$

$$\text{(48)} \qquad \lim_{n\to\infty} n^{1/\rho}\frac{b_{n+1}}{b_n} = \gamma = (\pi\mathrm{A}^{2\rho}\rho\operatorname{cosec}\pi\rho)^{1/\rho} = \gamma\left(\frac{1}{\pi}\right)$$

and

$$\text{(49)} \qquad \begin{cases} \gamma(d) = e^{2g'(d)} \\ \gamma = e^{2g'\left(\frac{1}{\pi}\right)} \end{cases}$$

Finally one has the following integral formulas

$$
(50)\qquad \begin{cases} d = \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{c(k)^2}{c(k)^2+\gamma(d)}\,dk = \frac{1}{\pi}\int_0^\infty \frac{dk}{1+\dfrac{\gamma(d)}{A^2}k^{2m}} \\ 1 = \displaystyle\int_0^\infty \frac{c(k)^2}{c(k)^2+\gamma}\,dk = \int_0^\infty \frac{dk}{1+\dfrac{\gamma}{A^2}k^{2m}} \end{cases}
$$

Proof (our proof is similar to that used recently by Dobrushin and Minlos [9]). — Formulas (26) and (36) tell us that

$$
\forall\varepsilon>0,\ \exists N(\varepsilon):\ n>N(\varepsilon) \Rightarrow \left(1-\frac{\varepsilon}{\tau e\rho}\right)^{n/\rho} < \frac{b_n}{\left(\dfrac{\tau e\rho}{n}\right)^{n/\rho}} < \left(1+\frac{\varepsilon}{\tau e\rho}\right)^{n/\rho}
$$

which gives rise to the equivalent formulas

$$
\left|\frac{1}{n}\operatorname{Log} b_n - \frac{1}{\rho}\operatorname{Log}\frac{\tau e\rho}{n}\right| < \frac{\varepsilon}{\tau e\rho^2}
$$

or

$$
\left|\frac{\operatorname{Log} b_n}{\dfrac{n}{\rho}\operatorname{Log}\dfrac{\tau e\rho}{n}} - 1\right| < \frac{\varepsilon}{\tau e\rho}\left|\frac{1}{\operatorname{Log}\dfrac{\tau e\rho}{n}}\right| < \varepsilon' \quad \text{for } n \text{ large enough}
$$

Consequently,

$$
\begin{aligned}
\lim_{\substack{n\to\infty \\ L\to\infty}\big\}\frac{2n}{L}=d} \frac{1}{L}\operatorname{Log} a_n^L &= \lim_{\substack{n\to\infty \\ L\to\infty}\big\}\frac{2n}{L}=d} \frac{d}{2\rho}\left\{\operatorname{Log}\frac{L}{2\pi} + \frac{\rho}{n}\operatorname{Log} b_n\right\} \\
&= \lim_{\substack{n\to\infty \\ L\to\infty}\big\}\frac{2n}{L}=d} \frac{d}{2\rho}\left\{\operatorname{Log}\frac{L}{2\pi} + \operatorname{Log}\frac{\tau e\rho}{n}\right\} = \frac{d}{2\rho}\operatorname{Log}\frac{\tau e\rho}{\pi d} = g(d)
\end{aligned}
$$

and we get (45) or, in the same way, (46).

On the other hand, we have, from (27),

$$
\operatorname{Log}\frac{a_{n+1}^L}{a_n^L} = \frac{1}{\rho}\operatorname{Log}\frac{L}{2\pi} + \operatorname{Log}\frac{b_{n+1}}{b_n}
$$

It is sufficient to prove (47) to get also (48) and (49). We shall proceed in two steps, proving successively that

$$
a)\qquad \underline{\lim}_{\substack{n\to\infty \\ L\to\infty}\big\}\frac{2n}{L}=d} \frac{a_{n+1}^L}{a_n^L} \geq e^{2g'(d)}
$$

b)
$$\overline{\lim_{\substack{n\to\infty \\ \mathrm{L}\to\infty}}}\Big\}_{\frac{2n}{\mathrm{L}}=d} \frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{n}} \leq e^{2g'(d)}$$

a) Inequality (41) allows to write, for m positive integer,

$$\frac{a^{\mathrm{L}}_{n+1+m}}{a^{\mathrm{L}}_{n}} = \frac{a^{\mathrm{L}}_{n+1+m}}{a^{\mathrm{L}}_{n+m}} \cdot \frac{a^{\mathrm{L}}_{n+m}}{a^{\mathrm{L}}_{n+m-1}} \cdot \frac{a^{\mathrm{L}}_{n+m-1}}{a^{\mathrm{L}}_{n+m-2}} \cdots \frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{n}} \leq \left(\frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{b}}\right)^{m+1}$$

or else

$$\frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{n}} \geq \left(\frac{a^{\mathrm{L}}_{n+1+m}}{a^{\mathrm{L}}_{n}}\right)^{\frac{1}{m+1}} = e^{\frac{\mathrm{L}}{m+1}\frac{1}{\mathrm{L}}\mathrm{Log}(a^{\mathrm{L}}_{n+m+1}-a^{\mathrm{L}}_{n})}$$

Taking the limit of both sides for $n \to \infty$, $\mathrm{L} \to \infty$, $\frac{2n}{\mathrm{L}} = d$, $m \to \infty$, $\frac{2(m+1)}{\mathrm{L}} = \varepsilon$, we get

$$\underline{\lim_{\substack{n\to\infty \\ \mathrm{L}\to\infty}}}\Big\}_{\frac{2n}{\mathrm{L}}=d} \frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{n}} \geq e^{\frac{2}{\varepsilon}[g(d+\varepsilon)-g(d)]}$$

which proves the desired result thanks to the arbitrariness of ε.

b) Inequality (41) allows to write, for m positive integer,

$$\frac{a^{\mathrm{L}}_{n}}{a^{\mathrm{L}}_{n-m}} = \frac{a^{\mathrm{L}}_{n}}{a^{\mathrm{L}}_{n-1}}\,\frac{a^{\mathrm{L}}_{n-1}}{a^{\mathrm{L}}_{n-2}} \cdots \frac{a^{\mathrm{L}}_{n-m+1}}{a^{\mathrm{L}}_{n-m}} \geq \left(\frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{n}}\right)^{m}$$

or else

$$\frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{n}} \leqslant \left(\frac{a^{\mathrm{L}}_{n}}{a^{\mathrm{L}}_{n-m}}\right)^{\frac{1}{m}} = e^{\frac{\mathrm{L}}{m}\frac{1}{\mathrm{L}}\mathrm{Log}(a^{\mathrm{L}}_{n}-a^{\mathrm{L}}_{n-m})}$$

Taking the limit of both sides for $n \to \infty$, $\mathrm{L} \to \infty$, $\frac{2n}{\mathrm{L}} = d$, $m \to \infty$, $\frac{2m}{\mathrm{L}} = \varepsilon$, we get

$$\overline{\lim_{\substack{n\to\infty \\ \mathrm{L}\to\infty}}}\Big\}_{\frac{2n}{\mathrm{L}}=d} \frac{a^{\mathrm{L}}_{n+1}}{a^{\mathrm{L}}_{n}} \leq e^{\frac{2}{\varepsilon}[g(d)-g(d-\varepsilon)]}$$

which again proves the desired result (*b*) thanks to the arbitrariness of ε.

Finaly, formulas (50) can be easily deduced from the identity ([*10*], 3.241, 2):

$$\int_0^\infty \frac{dk}{1+k^{2m}} = \frac{\pi}{2m}\,\mathrm{cosec}\,\frac{\pi}{2m}, \qquad m > \frac{1}{2}$$

To end this part, we want to mention that it is possible to compute the quantities b_n, and consequently a_n^L, with the help of the ζ function of Riemann, using methods patterned from the Fredholm theory of integral equations [*11*]. This result cannot be considered new. It is contained, for instance, into the formulas ([*10*], 8.334, 1, 8.321, 2)

$$g_\rho(\mu) = \prod_{j=1}^{\infty}\left\{1 + \mu\frac{A^2}{j^{2m}}\right\} = \frac{1}{\mu A^2}\prod_{k=1}^{2m}\frac{1}{\Gamma\left[-(-\mu A^2)^{\frac{1}{2m}}\exp\frac{2\pi k i}{2m}\right]} \tag{51}$$

and

$$\frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k \tag{52}$$

where

$$d_0 = 1, \quad d_{n+1} = \frac{\sum_{k=0}^{n}(-1)^k s_{k+1} d_{n-k}}{n+1}, \quad s_1 = C, \quad s_n = \zeta(n) \quad \text{for} \quad n \geq 2$$

and C = Euler's constant = 0,57721 ...

However, for sake of completeness, we give here a direct proof of it. From (30) we can deduce that

$$\frac{g_\rho'(\mu)}{g_\rho(\mu)} = \sum_{j=1}^{\infty}\frac{c(j)^2}{1+\mu c(j)^2} = \sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n c(j)^{2n+2}\mu^n = \sum_{n=0}^{\infty}(-1)^n\sigma_{n+1}\mu^n \tag{53}$$

provided that

$$|\mu c(j)^2| < 1, \quad j = 1, 2, \ldots \quad \text{i.e.} \quad |\mu| < \frac{1}{c(1)^2} = \frac{1}{A^2} \tag{54}$$

if we define

$$\sigma_p = \sum_{j=1}^{\infty} c(j)^{2p} = \sum_{j=1}^{\infty} A^{2p}\frac{1}{j^{p/\rho}} = A^{2p}\zeta(p/\rho) \tag{55}$$

But, on the other hand,

$$g_\rho(\mu) = \sum_{n=0}^{\infty} b_n\mu^n, \quad g_\rho'(\mu) = \sum_{n=0}^{\infty}(n+1)b_{n+1}\mu^n \tag{56}$$

so that formula (53) gives rise to the relation

$$\sum_{n=0}^{\infty}(n+1)b_{n+1}\mu^n = \left\{\sum_{m=0}^{\infty} b_m\mu^m\right\}\left\{\sum_{q=0}^{\infty}(-1)^q\sigma_{q+1}\mu^q\right\} \tag{57}$$

from which we deduce that

$$b_{n+1} = \sum_{m+q=n}^{\infty} (-1)^q b_m \sigma_{q+1}; \qquad b_0 = 1 \tag{58}$$

Solving this system of equations, we get the following formula, which is quite familiar in the theory of integral equations (see for instance [11])

$$b_n = \frac{1}{n!} \begin{vmatrix} \sigma_1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \sigma_2 & \sigma_1 & 2 & 0 & \cdot & \cdot & \cdot & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n-1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdot & \cdot & \cdot & \cdot & \sigma_1 \end{vmatrix} \tag{59}$$

For instance

$$\begin{cases} b_1 = \mathrm{A}^2 \zeta\left(\dfrac{1}{\rho}\right) \\ b_2 = \dfrac{\mathrm{A}^2}{2}\left[\zeta\left(\dfrac{1}{\rho}\right)^2 - \zeta\left(\dfrac{2}{\rho}\right)\right] \\ \text{etc.} \end{cases} \tag{60}$$

IV. AN EXAMPLE

As an illustration of the preceedings results, let us now study the well known case where

$$c(k) = \frac{1}{k}, \quad \mathrm{A} = 1, \quad m = 1, \quad \tau(\mathrm{L}) = \frac{\mathrm{L}}{2}, \quad \tau = \pi, \quad \rho = \frac{1}{2} \tag{61}$$

We have then:

$$\begin{cases} f_{1/k}^{\mathrm{L}}(\lambda) = \displaystyle\prod_{j=1}^{\infty} \left\{ 1 + \lambda\left(\frac{\mathrm{L}}{2\pi}\right)^2 \frac{1}{j^2} \right\} = \frac{\sin \dfrac{i\mathrm{L}}{2}\sqrt{\lambda}}{\dfrac{i\mathrm{L}}{2}\sqrt{\lambda}} \\ g_{1/2}(\mu) = \displaystyle\prod_{j=1}^{\infty} \left\{ 1 + \frac{\mu}{j^2} \right\} = \frac{\sin i\pi\sqrt{\lambda}}{i\pi\sqrt{\lambda}} \end{cases} \tag{62}$$

and consequently,

$$a_n^{\mathrm{L}} = \left(\frac{\mathrm{L}}{2\pi}\right)^{2n} \frac{\pi^{2n}}{(2n+1)!}\,; \qquad b_n = \frac{\pi^{2n}}{(2n+1)!} \tag{63}$$

We can then immediately see that the ratio (31) is independent of L and that

$$\chi\left(n; \frac{1}{2}\right) = \frac{1}{2n+1}\,; \quad \lim_{n\to\infty} \frac{\chi\left(n+1; \frac{1}{2}\right)}{\chi\left(n; \frac{1}{2}\right)} = 1 \tag{64}$$

Moreover one get directly that

$$\underset{\substack{n\to\infty \\ \mathrm{L}\to\infty}\left.\right\}\frac{2n}{\mathrm{L}}=d}{\text{limit}} \frac{a_{n+1}^{\mathrm{L}}}{a_n^{\mathrm{L}}} = \left(\frac{1}{2d}\right)^2 \quad \text{and} \quad \lim_{n\to\infty} n^2 \frac{b_{n+1}}{b_n} = \frac{\pi^2}{4} \tag{65}$$

in agreement with formulas (47) and (48).

In the same way, we can compute the limits

$$\begin{aligned} &\underset{\substack{n\to\infty \\ \mathrm{L}\to\infty}\left.\right\}\frac{2n}{\mathrm{L}}=d}{\text{limit}} \frac{1}{\mathrm{L}} \operatorname{Log} a_n^{\mathrm{L}} = d(1 - \operatorname{Log} 2d) \\ &\lim_{n\to\infty} \frac{1}{2\pi} \operatorname{Log} n^2 b_n^{\frac{1}{n}} = \frac{1}{\pi}\left(1 - \operatorname{Log} \frac{2}{\pi}\right) \end{aligned} \tag{66}$$

These results are consistent with formulas (45) and (46).

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