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On a class of infinite products occurring in quantum statistical mechanics


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On a class of infinite products occurring in quantum statistical mechanics

by

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ABSTRACT. — We study the class of infinite products

\[ f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(j)^2 \right\}, \quad k_j = j \frac{2\pi}{L} \quad \text{and} \quad L \in \mathbb{R}^+, \]

where \( c(k) = \frac{A}{k^m}, \ k \geq 0, \ A > 0, \ m > \frac{1}{2}, \) which occurs naturally in quantum statistical mechanics. In particular, we compute the limits

\[
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{L} \log a_n^L
\]

which are relevant in the problem of the thermodynamic limit of the BCS superconducting state. By the same way, we get new results concerning the infinite products of the form

\[ g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A}{j^{1/\rho}} \right\}, \quad A > 0, \ 0 < \rho < 1. \]
In particular we are able to compute the limits

\[
\lim_{n \to \infty} n^{1/p} \frac{b_{n+1}}{b_n} \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{2n} \log n^p b_n^{1/4}
\]

RÉSUMÉ. — Nous étudions la classe de produits infinis

\[
f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(k)^2 \right\}, \quad k_j = j \frac{2\pi}{L} \quad \text{et} \quad L \in \mathbb{R}^+
\]

où \( c(k) = \frac{A}{k^m}, \ k \geq 0, \ A > 0, \ m > \frac{1}{2} \), qui se présentent naturellement en mécanique statistique quantique. En particulier nous calculons les limites

\[
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} \quad \text{et} \quad \lim_{n \to \infty} \frac{1}{L} \log a_n^L
\]

qui ont leur importance dans la limite thermodynamique de l'état de la supraconductivité de Bardeen-Cooper-Schriefer. Simultanément nous obtenons des résultats nouveaux relatifs aux produits infinis de la forme

\[
g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{ 1 + \mu c(j^{1/p}) \right\}, \quad A > 0, \ 0 < \rho < 1
\]

En particulier nous sommes en mesure de calculer les limites

\[
\lim_{n \to \infty} n^{1/p} \frac{b_{n+1}}{b_n} \quad \text{et} \quad \lim_{n \to \infty} \frac{1}{2n} \log n^p b_n^{1/4}
\]

I. INTRODUCTION

It is now well known that algebras of observables are useful in the kinematical description, from a quantum mechanical point of view, of systems of interacting particles. For that purpose, we constructed, in a preceeding paper [1], a family of states (positive linear functionals of norm 1) over a Clifford-C*-algebra, each of which characterized by a real function \( c \) of real argument and describing a « condensate » of pairs of fermions with density \( d \).
In that work, we studied a class of entire functions of the complex variables $\lambda$, depending on the function $c$ and on a length $L$, defined by the infinite products

$$f_c^L(\lambda) = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(k_j)^2 \right\}, \quad L \in \mathbb{R}^+, \quad k_j = j \frac{2\pi}{L}$$

More precisely, if we write

$$f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n$$

where

$$a_n^L = \sum_{0 < j_1 < j_2 < \ldots < j_n} c(k_{j_1})^2 c(k_{j_2})^2 \ldots c(k_{j_n})^2$$

the main point was the existence of the limit

$$\gamma(d) = \lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L}, \quad d \in \mathbb{R}^+$$

and we were able to prove the following theorem:

**Theorem.** Let $c$ be a real, bounded, decreasing, square integrable function of a real positive variable $k$, tending to zero, when $k$ tends to infinity, faster than $k^{-1/2}$. Let $d$ and $L$ be positive reals.

Then the limit (4) exists. Moreover, defining the function $g(d)$ as

$$g(d) = \lim_{n \to \infty} \frac{1}{L} \log a_n^L,$$  

then $g(d)$ exists, is convex and differentiable, and

$$\gamma(d) = e^{2\pi g(d)}$$

Finally one has the following integral formula

$$d = \frac{1}{\pi} \int_0^{\infty} \frac{c^2(k)}{\gamma(d) + c^2(k)} \, dk$$

The methods employed are rather involved and do not fully exploit the analyticity properties of $f_c^L$. On the other hand, the theorem quoted
above is only an existence theorem, so that except for very special choices for \( c ([7], (109)) \), it does not allow to compute explicitly the quantities \( g(d) \) and \( \gamma(d) \) as functions of \( d \).

Our purpose is to restate that theorem, using now entire functions technics. To that end, we have to make a different choice for our class of functions \( c \), giving up definiteness at the origin (and so boundedness and square integrability) but requiring an homogeneity condition. The quantities \( a_n^L \), \( g(d) \) and \( \gamma(d) \) are now explicitly given.

By the same way, we study the class of infinite products

\[
\left(8\right) \quad g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{1/\rho}} \right\}, \quad A > 0, \quad 0 < \rho < 1
\]

obtaining some new results for \( \rho \neq \frac{1}{2} \).

II. CHOICE OF THE CLASS OF FUNCTIONS \( c \)

Let us consider, for the moment, positive functions \( c(k) \), defined for \( k > 0 \), and vanishing at infinity faster than \( k^{-1/2} \). The infinite products (1) are then convergent and define entire functions \( f_c^L \) whose zeros lie on the real negative axis and are given by

\[
\left(9\right) \quad \lambda_j = -\frac{1}{c(k_j)^2}
\]

By definition ([4], I, 4), ([5], 2.5.2), the convergence exponent of the sequence (9) is the greatest lower bound \( \rho \) of the reals \( \alpha \) such that

\[
\left(10\right) \quad \sum_{j=1}^{\infty} \left\{ \frac{1}{c(k_j)^2} \right\}^{-\alpha} = \sum_{j=1}^{\infty} c(k_j)^{2\alpha} < + \infty
\]

If, by now, we restrict ourselves to functions such that

\[
\left(11\right) \quad c(k) \sim \frac{A}{k^m}, \quad k \to \infty, \quad m > \frac{1}{2}, \quad A > 0
\]

a necessary and sufficient condition for the convergence of (10) is

\[
\left(12\right) \quad 2m\alpha = 1 + \epsilon, \quad \epsilon > 0
\]
so that

\[ \rho = \inf_{\varepsilon > 0} \left\{ \alpha : \alpha = \frac{1}{2m} + \frac{\varepsilon}{2m} \right\} = \frac{1}{2m} < 1 \]

On the other hand, the \( f_L^e \) are infinite canonical products of order \( \rho \) and genus \( p \) ([4], I, Th. 7), ([5], 2.6.5), the genus \( p \) being related to the order \( \rho \) according to

\[ p \leq \rho \leq p + 1, \quad p \text{ integer.} \]

In the case of (11), we can then conclude that the entire functions \( f_L^e \) are characterised by

\[ p = 0, \quad 0 < \rho = \frac{1}{2m} < 1, \quad \frac{1}{2} < m + \infty \]

Further more, let \( n(r), r \in \mathbb{R}^+ \), the function giving the number of zeros of \( f_L^e \) with modulus less than or equal to \( r \). One can show ([4], I, lemma 1), ([5], 2.5.8) that

\[ \rho = \lim_{r \to \infty} \frac{\log n(r)}{\log r} \]

and one defines the density \( \Delta \) of the sequence (9) as

\[ \Delta = \lim_{r \to \infty} \frac{n(r)}{r^\rho} \]

We are going to estimate these quantities, restricting now to functions \( c \) such that (11) holds, which are monotonic and differentiable at least for \( k \) large and such that

\[ c'(k) \sim -\frac{B}{k^{m+1}}, \quad k \to \infty, \quad B > 0 \]

Turning back to the definition of \( n(r) \), we can write that

\[ n(r) = \max \left\{ j : \frac{1}{c(k_j)^2} \leq r \right\} = \max \left\{ j : k_j \leq \nu \left( \frac{1}{\sqrt{r}} \right) \right\} = \left[ \frac{L}{2\pi} \nu \left( \frac{1}{\sqrt{r}} \right) \right] \]

where \( \nu \) is the inverse function of \( c \) (defined at least for \( r \) large enough) and where the squared brackets mean « the largest integer contained in ». Thanks to our hypothesis (18), it is an easy task to show that

\[ n(r) \sim \frac{L}{2\pi} A^{2\rho^2}, \quad r \to \infty \]
(a result which agrees with (16)) and that

\[ \Delta = \frac{L}{2\pi} A^{2\rho} \]

From these estimations, it is actually possible to deduce that ([4], I, Th. 25), ([5], 4.1.1)

\[ \log f_c^L(re^{i\theta}) \sim e^{i\theta} \frac{L}{2} A^{2\rho} \left( \csc \pi \rho \right) r^\rho, \quad -\pi < \theta < +\pi, \quad r \to \infty \]

or, taking the real part, that

\[ \log |f_c^L(re^{i\theta})| \sim \frac{L}{2} A^{2\rho} \left( \csc \pi \rho \right) \cos \rho \theta \cdot r^\rho, \quad -\pi < \theta < +\pi, \quad r \to \infty \]

uniformly with respect to \( \theta \) if \(-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\).

Then, the indicator function \( h_c^L(\theta) \) of \( f_c^L \) is given by ([4], I, 15), ([5], 2.1.8)

\[ h_c^L(\theta) = \lim_{n \to \infty} \log |f_c^L(re^{i\theta})| = \frac{L}{2} A^{2\rho} \left( \csc \pi \rho \right) \cos \rho \theta, \quad -\pi < \theta < +\pi \]

and also for \(-\pi \leq \theta \leq +\pi\) as \( h_c^L \) is a continuous function, defined by periodicity for other values of \( \theta \).

From now on, we are able to compute the type of \( f_c^L \) according to the formula ([4], I, 1 and Th. 29), ([5], 2.1.4)

\[ \tau(L) = \lim_{r \to \infty} \log \max_{|\lambda| = r} \frac{|f_c^L(\lambda)|}{r^\rho} = \max_{\theta} |h(\theta)| = \frac{L}{2} A^{2\rho} \csc \pi \rho \]

as well as, by ([4], I, Th. 2), ([5], 2.2.10)

\[ \tau(L) = \frac{1}{e^\rho} \lim_{n \to \infty} n(a_n^L)^{\rho/n} \]

Comparing formulas (20) and (16), we see that the \( \lim \) occurring in (16) is in fact a limit, as well as the ones in (17), (24), (25) and consequently in (26).

Unfortunately, the comparison of formulas (25) and (26) does not allow to estimate the limit (4) because of a lack of uniformity with respect to \( L \). A simple case where uniformity can be recovered is given by

\[ a_n^L = \left( \frac{L}{2\pi} \right)^{n/\rho} b_n \quad b_n \text{ independent of } L \]
The following proposition, the proof of which is trivial, shows that (27) holds if and only if $c(k)$ is a homogeneous function of degree $-\frac{1}{2\rho}$:

**PROPOSITION.** — If

$$f_c^L(\lambda) = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(k_j)^2 \right\} = \sum_{n=0}^{\infty} a_n^L \lambda^n$$

where $k_j = j \frac{2\pi}{L}$, then $a_n^L = \left( \frac{L}{2\pi} \right)^{n/\rho} b_m b_n$ independent of $L$, if and only if

$$c(k) = \frac{A}{k^{1/2\rho}} = \frac{A}{k^m},$$

$\rho = \frac{1}{2m}$ being the order of $f_c^L$.

So our conclusion is that entire functions technics can be easily applied to our problem if we restrict ourselves to the class of functions

$$c(k) = \frac{A}{k^m}, \quad \frac{1}{2} < m < \infty, \quad A > 0, \quad 0 < k < \infty$$

for which conditions (11) and (18) are evidently fulfilled.

### III. THE MAIN THEOREM

From now on, we can write, thanks to (25),

$$f_c^L(\lambda) = \sum_{n=0}^{\infty} \left( \frac{L}{2\pi} \right)^{n/\rho} b_n \lambda^n = \sum_{n=0}^{\infty} \frac{(\tau(L)^{1/\rho} \lambda)^n}{(A^{2\rho\pi}/\sin \pi \rho)^{n/\rho} b_n}$$

and

$$g_{\nu}(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{1/\rho}} \right\} = \sum_{n=0}^{\infty} b_n \mu^n$$

Moreover, the expressions

$$\frac{a_n^L}{(\tau(L)e^{\rho n})^{n/\rho} \frac{n!}{(\sin \pi \rho)^{n/\rho}}} = \frac{b_n}{(A^{2\rho\pi}/\sin \pi \rho)^{n/\rho} (e^{\rho n})^{n/\rho}}$$
and (where \( \Gamma \) means the usual gamma function)

\[
\frac{a_{n+1}^L}{a_n^L} = \frac{b_{n+1}}{b_n} = \frac{\left(\frac{\tau(L)e^\rho}{n+1}\right)^{n+1}}{\left(\frac{\tau(L)e^\rho}{n}\right)^{n^\rho}} = \frac{\left(\frac{A^{2\rho\pi}}{\sin \pi \rho}\right)^{n+1}}{\left(\frac{A^{2\rho\pi}}{\sin \pi \rho}\right)^n} \left(\frac{e^\rho}{n+1}\right)^{n+1} \left(\frac{e^\rho}{n}\right)^n
\]

\[
\frac{1}{\left(\frac{\sin \pi \rho}{\Gamma\left(\frac{n+1}{\rho} + 1\right)}\right)}
\]

are now independent of \( L \). Therefore the result we are aiming at is equivalent to

\[
\lim_{n \to \infty} \frac{\chi(n + 1; \rho)}{\chi(n; \rho)} = 1
\]

or

\[
b_n = \left(\frac{A^{2\rho\pi}}{\sin \pi \rho}\right)^{n^\rho} \frac{\chi(n; \rho)}{\Gamma\left(\frac{n}{\rho} + 1\right)}
\]

where

\[
\lim_{n \to \infty} \frac{\chi(n + 1; \rho)}{\chi(n; \rho)} = 1
\]

Incidentally it is interesting to remark that

\[
g_\rho(\mu) = \sum_{n=0}^{\infty} \frac{\left(\frac{A^{2\rho\pi}}{\sin \pi \rho}\right)^{n^\rho} \chi(n; \rho)}{\Gamma\left(\frac{n}{\rho} + 1\right)} \mu^n
\]

is of order \( \rho \) and type

\[
\tau = \frac{A^{2\rho\pi}}{\sin \pi \rho} = \tau(2\pi)
\]

as is also the Mittag-Leffler function \( [6] \)

\[
E_\rho(\mu) = \sum_{n=0}^{\infty} \frac{\left(\frac{A^{2\rho\pi}}{\sin \pi \rho}\right)^{n^\rho}}{\Gamma\left(\frac{n}{\rho} + 1\right)} \mu^n
\]

which shows the close relation between \( E_\rho \) and \( g_\rho \) and asserts the well
known fact that $E_{\rho}$ is, in some sense [7], the simplest entire function of a
given order and type.

Another way of writing (33) is the following (provided the limits exists):

\begin{equation}
\lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} = \lim_{n \to \infty} \frac{(\tau(L)e^{\rho})^{n+1}}{(n+1)^{\rho}} \left\{ \left( \frac{\tau(L)e^{\rho}}{n} \right) \right\}
\end{equation}

\begin{equation}
= \lim_{n \to \infty} \left( \frac{L A^{2\rho} e^{\rho} \csc \pi \rho}{2(n+1)} \right)^{\frac{n+1}{\rho}} \left\{ \left( \frac{L A^{2\rho} e^{\rho} \csc \pi \rho}{2n} \right) \right\}^{n/\rho}
\end{equation}

\begin{equation}
= \lim_{n \to \infty} \left( \frac{A^{2\rho} e^{\rho} \csc \pi \rho}{\mu} \right)^{1/\rho} \left( \frac{1}{1 + \frac{1}{n}} \right)^{n/\rho} \left( \frac{1}{1 + \frac{1}{n}} \right)^{1/\rho} = \left( \frac{A^{2\rho} e^{\rho} \csc \pi \rho}{\mu} \right)^{1/\rho}
\end{equation}

So, once the existence of the limit is proved, then necessarily

\begin{equation}
\gamma(d) = \left( \frac{A^{2\rho} e^{\rho} \csc \pi \rho}{\mu} \right)^{1/\rho}
\end{equation}

and hence

\begin{equation}
\lim_{n \to \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} = (\pi A^{2\rho} e^{\rho} \csc \pi \rho)^{1/\rho}
\end{equation}

Before proving our main theorem, let us give a lemma:

**Lemma.** — We have the inequalities:

\begin{equation}
\frac{a_n^L}{a_{n+1}^L} > \frac{a_n^L}{a_{n+1}^L}
\end{equation}

and

\begin{equation}
\frac{b_n}{b_{n+1}} > \frac{b_n}{b_{n+1}}
\end{equation}

These inequalities are well known in the theory of entire functions
([5], 2.8.2) but we restate the proof by sake of completeness.

One has:

\begin{equation}
f'/(\lambda) = \sum_{j=1}^{\infty} \frac{c(k_j)^2}{1 + \lambda c(k_j)^2}
\end{equation}

and

\begin{equation}
\left\{ \frac{f''(\lambda)}{f(\lambda)} \right\}' = \sum_{j=1}^{\infty} \frac{-c(k_j)^4}{(1 + \lambda c(k_j)^2)^2}
\end{equation}

So $f(\lambda)f''(\lambda) < f'(\lambda)^2$ if $\lambda \in \mathbb{R}$ or, else, $f^{(n-1)}(\lambda)f^{(n+1)}(\lambda) < f^{(n)}(\lambda)^2$ by appli-
cation of the same inequality to \( f^{(n-1)}(\lambda) \), which is also an entire function of the same order, type and genus as \( f \). It follows that

\[
\frac{a_{n+1}}{a_n} \leq \frac{a_n}{a_{n-1}} \leq \frac{a_{n+1}}{a_{n-1}}.
\]

It is interesting to remark that, by adapting to our case the proof of ([1 bis], lemma 3, (80)), one has conversely

\[
\frac{n}{b_{n-1}} \leq (n+1) \frac{b_{n+1}}{b_n} + A^2
\]

(43)

\[
\frac{a_n}{a_{n-1}} \leq \frac{n}{b_n} \frac{a_{n+1}}{a_n} + \frac{1}{\pi d} A^2
\]

We are now able to prove our theorem.

**THEOREM.** — Let \( c(k) = \frac{A}{k^m} \) with \( k > 0, A > 0, m > \frac{1}{2}, L \) and \( d \) be positive, and

\[
\begin{array}{l}
\left\{\begin{array}{l}
f_c^{\ell}(\lambda) = \sum_{n=0}^{\infty} a_n^{\ell+1} \lambda^n = \prod_{j=1}^{\infty} \{1 + \lambda c(k_j)^2\}, \quad k_j = j \frac{2\pi}{L}
\end{array}\right.
\end{array}
\]

(44)

\[
\begin{array}{l}
g_\rho(\mu) = \sum_{n=0}^{\infty} b_n\mu^n = \prod_{j=1}^{\infty} \{1 + \mu \frac{A^2}{j^{1/\rho}}\}, \quad \rho = \frac{1}{2m}, \quad 0 < \rho < 1
\end{array}
\]

Then

\[
\lim_{n \to \infty} \frac{1}{2\pi} \Log n^{1/\rho} b_n^{1/n} = \frac{1}{2\pi \rho} \Log (\pi A^{2\rho} \cosec \pi \rho) = g\left(\frac{1}{\pi}\right)
\]

(46)

Moreover

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \gamma(d) = \left(\frac{A^{2\rho} \cosec \pi \rho}{d}\right)^{1/\rho}
\]

(47)

\[
\lim_n n^{1/\rho} \frac{b_{n+1}}{b_n} = \gamma = (\pi A^{2\rho} \cosec \pi \rho)^{1/\rho} = \gamma\left(\frac{1}{\pi}\right)
\]

and

\[
\left\{\begin{array}{l}
\gamma(d) = e^{2g(d)}
\end{array}\right.
\]

(48)

\[
\gamma = e^{2g\left(\frac{1}{\pi}\right)}
\]

(49)
Finally one has the following integral formulas

\[
\begin{align*}
\left\{ \begin{array}{l}
d = \frac{1}{\pi} \int_{0}^{\infty} \frac{c(k)^2}{c(k)^2 + \gamma(d)} \, dk = \frac{1}{\pi} \int_{0}^{\infty} \frac{dk}{1 + \frac{\gamma(d)}{A^2} k^{2m}} \\
1 = \int_{0}^{\infty} \frac{c(k)^2}{c(k)^2 + \gamma} \, dk = \int_{0}^{\infty} \frac{dk}{1 + \frac{\gamma}{A^2} k^{2m}}
\end{array} \right. 
\end{align*}
\]

(50)

**Proof** (our proof is similar to that used recently by Dobrushin and Minlos [9]). — Formulas (26) and (36) tell us that

\[
\forall \varepsilon > 0, \exists N(\varepsilon): n > N(\varepsilon) \Rightarrow \left(1 - \frac{\varepsilon}{\tau \rho}\right)^{n/\rho} < \frac{b_n}{(\tau \rho)^{n/\rho}} < \left(1 + \frac{\varepsilon}{\tau \rho}\right)^{n/\rho}
\]

which gives rise to the equivalent formulas

\[
\left| \frac{1}{n} \log b_n - \frac{1}{\rho} \log \frac{\tau \rho}{n} \right| < \frac{\varepsilon}{\tau \rho^2}
\]

or

\[
\left| \frac{\log b_n - 1}{\rho} \log \frac{\tau \rho}{n} \right| < \frac{\varepsilon}{\tau \rho} \left| \frac{1}{\log \frac{\tau \rho}{n}} \right| < \varepsilon' \quad \text{for } n \text{ large enough}
\]

Consequently,

\[
\lim_{n \to \infty} \frac{2n}{2n} \lim_{L \to \infty} \frac{1}{2n} \left\{ \log \frac{L}{2\pi} + \frac{\rho}{n} \log b_n \right\} = \lim_{n \to \infty} \frac{2n}{2n} \lim_{L \to \infty} \frac{1}{2n} \left\{ \log \frac{L}{2\pi} + \log \frac{\tau \rho}{n} \right\} = \frac{d}{2\rho} \log \frac{\tau \rho}{\pi d} = g(d)
\]

and we get (45) or, in the same way, (46).

On the other hand, we have, from (27),

\[
\log \frac{a_{n+1}}{a_n} = - \frac{1}{\rho} \log \frac{L}{2\pi} + \log \frac{b_{n+1}}{b_n}
\]

It is sufficient to prove (47) to get also (48) and (49). We shall proceed in two steps, proving successively that

\[ a) \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \geq e^{2g(d)} \]
b) Inequality (41) allows to write, for m positive integer,
\[
\frac{a_{n+1}^L}{a_n^L} = \frac{a_{n+1+m}^L}{a_{n+m}^L} \cdot \frac{a_{n+m-1}^L}{a_{n+m-2}^L} \cdots \frac{a_{n+1}^L}{a_n^L} \leq \left( \frac{a_{n+1}^L}{a_b^L} \right)^{m+1}
\]
or else
\[
\frac{a_{n+1}^L}{a_n^L} \geq \left( \frac{a_{n+1+m}^L}{a_n^L} \right)^{\frac{1}{m+1}} = e^{\frac{1}{m+1} \log(a_{n+1+m}^L - a_n^L)}
\]

Taking the limit of both sides for \( n \to \infty, L \to \infty, \frac{2n}{L} = d, m \to \infty, \frac{2(m+1)}{L} = \varepsilon \), we get
\[
\lim_{n \to \infty, L \to \infty} \frac{a_{n+1}^L}{a_n^L} \geq e^{\frac{2}{m+1} \log(\varepsilon^d - d)}
\]
which proves the desired result thanks to the arbitrariness of \( \varepsilon \).

b) Inequality (41) allows to write, for m positive integer,
\[
\frac{a_{n}^L}{a_{n-m}^L} = \frac{a_{n-1}^L}{a_{n-2}^L} \cdots \frac{a_{n-m+1}^L}{a_{n-m}^L} \geq \left( \frac{a_{n+1}^L}{a_n^L} \right)^m
\]
or else
\[
\frac{a_{n+1}^L}{a_n^L} \leq \left( \frac{a_n^L}{a_{n-m}^L} \right)^{\frac{1}{m}} = e^{\frac{1}{m} \log(a_n^L - a_{n-m}^L)}
\]

Taking the limit of both sides for \( n \to \infty, L \to \infty, \frac{2n}{L} = d, m \to \infty, \frac{2m}{L} = \varepsilon \), we get
\[
\lim_{n \to \infty, L \to \infty} \frac{a_{n+1}^L}{a_n^L} \leq e^{\frac{2}{m} \log(\varepsilon^d - d)}
\]
which again proves the desired result (b) thanks to the arbitrariness of \( \varepsilon \).

Finally, formulas (50) can be easily deduced from the identity ([10], 3.241, 2):
\[
\int_0^\infty \frac{dk}{1 + k^2m} = \frac{\pi}{2m} \csc \frac{\pi}{2m}, \quad m > \frac{1}{2}
\]

To end this part, we want to mention that it is possible to compute the
quantities $b_n$ and consequently $a_n$, with the help of the $\zeta$ function of Riemann, using methods patterned from the Fredholm theory of integral equations [11]. This result cannot be considered new. It is contained, for instance, into the formulas ([10], 8.334, 1, 8.321, 2)

\begin{equation}
(51) \quad g_{\rho}(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{2m}} \right\} = \frac{1}{\mu A^2} \prod_{k=1}^{2m} \frac{1}{\Gamma \left[ -\frac{1}{\mu A^2} \right] \exp \frac{2nki}{2m}}
\end{equation}

and

\begin{equation}
(52) \quad \frac{1}{\Gamma(z + 1)} = \sum_{k=0}^{\infty} d_k z^k
\end{equation}

where

\begin{equation}
1 = 1, \quad d_{n+1} = \frac{(n+1)^{s_{n+1}}}{n+1}, \quad s_1 = C, \quad s_n = \zeta(n) \quad \text{for} \quad n \geq 2
\end{equation}

and $C = \text{Euler's constant} = 0.57721 \ldots$

However, for sake of completeness, we give here a direct proof of it. From (30) we can deduce that

\begin{equation}
(53) \quad \frac{g'(\mu)}{g_{\rho}(\mu)} = \sum_{j=1}^{\infty} \frac{c(j)^2}{1 + \mu c(j)^2} = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} (-1)^n c(j)^{2n+2} \mu^n = \sum_{n=0}^{\infty} (-1)^n \sigma_{n+1} \mu^n
\end{equation}

provided that

\begin{equation}
(54) \quad |\mu c(j)^2| < 1, \quad j = 1, 2, \ldots \quad \text{i.e.} \quad |\mu| < \frac{1}{c(1)^2} = \frac{1}{A^2}
\end{equation}

if we define

\begin{equation}
(55) \quad \sigma_p = \sum_{j=1}^{\infty} c(j)^{2p} = \sum_{j=1}^{\infty} A^{2p \frac{1}{j^{p/r}}} = A^{2p} \zeta(p/r)
\end{equation}

But, on the other hand,

\begin{equation}
(56) \quad g_{\rho}(\mu) = \sum_{n=0}^{\infty} b_n \mu^n, \quad g'_{\rho}(\mu) = \sum_{n=0}^{\infty} (n+1) b_{n+1} \mu^n
\end{equation}

so that formula (53) gives rise to the relation

\begin{equation}
(57) \quad \sum_{n=0}^{\infty} (n+1) b_{n+1} \mu^n = \left\{ \sum_{m=0}^{\infty} b_m \mu^m \right\} \left\{ \sum_{q=0}^{\infty} (-1)^q \sigma_{q+1} \mu^q \right\}
\end{equation}
from which we deduce that

\[ b_{n+1} = \sum_{m+q=n}^{\infty} (-1)^q b_m \sigma_{q+1}; \quad b_0 = 1 \]  

Solving this system of equations, we get the following formula, which is quite familiar in the theory of integral equations (see for instance [11])

\[ b_n = \frac{1}{n!} \begin{vmatrix} \sigma_1 & 1 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & 2 & \cdots & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \cdots & \sigma_1 \end{vmatrix} \]  

(59)

For instance

\[ \begin{cases} b_1 = A^2 \zeta\left(\frac{1}{\rho}\right) \\ b_2 = \frac{A^2}{2} \left[ \zeta\left(\frac{1}{\rho}\right)^2 - \zeta\left(\frac{2}{\rho}\right) \right] \end{cases} \]  

(60)

etc.

IV. AN EXAMPLE

As an illustration of the preceedings results, let us now study the well known case where

\[ c(k) = \frac{1}{k}, \quad A = 1, \quad m = 1, \quad \tau(L) = \frac{L}{2}, \quad \tau = \pi, \quad \rho = \frac{1}{2} \]  

(61)

We have then:

\[ \begin{cases} f_{1nk}(\lambda) = \prod_{j=1}^{\infty} \left\{ 1 + \lambda \left(\frac{L}{2\pi}\right)^2 \frac{1}{j^2} \right\} = \frac{\sin \frac{iL}{2} \sqrt{\lambda}}{iL \sqrt{\lambda}} \\ g_{1/2}(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \frac{\mu}{j^2} \right\} = \frac{\sin i\pi \sqrt{\lambda}}{i\pi \sqrt{\lambda}} \end{cases} \]  

(62)
and consequently,

\[ a_n^L = \left( \frac{L}{2\pi} \right)^{2n} \frac{\pi^{2n}}{(2n + 1)!}; \quad b_n^L = \frac{\pi^{2n}}{(2n + 1)!} \]

We can then immediately see that the ratio (31) is independent of L and that

\[ \chi\left(n; \frac{1}{2}\right) = \frac{1}{2n + 1}; \quad \lim_{n \to \infty} \frac{\chi\left(n + 1; \frac{1}{2}\right)}{\chi\left(n; \frac{1}{2}\right)} = 1 \]

Moreover one get directly that

\[ \lim_{n \to \infty} \frac{a_{n+1}^L}{a_n^L} = \left( \frac{1}{2d} \right)^2 \quad \text{and} \quad \lim_{n \to \infty} n^2 \frac{b_{n+1}^L}{b_n^L} = \frac{\pi^2}{4} \]

in agreement with formulas (47) and (48).

In the same way, we can compute the limits

\[ \lim_{n \to \infty} \frac{1}{L} \log a_n^L = d(1 - \log 2d) \]

\[ \lim_{n \to \infty} \frac{1}{2\pi} \log n^2 b_n^L = \frac{1}{\pi} \left( 1 - \log \frac{2}{\pi} \right) \]

These results are consistent with formulas (45) and (46).

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